An introduction to Bayesian inference for material parameter identification

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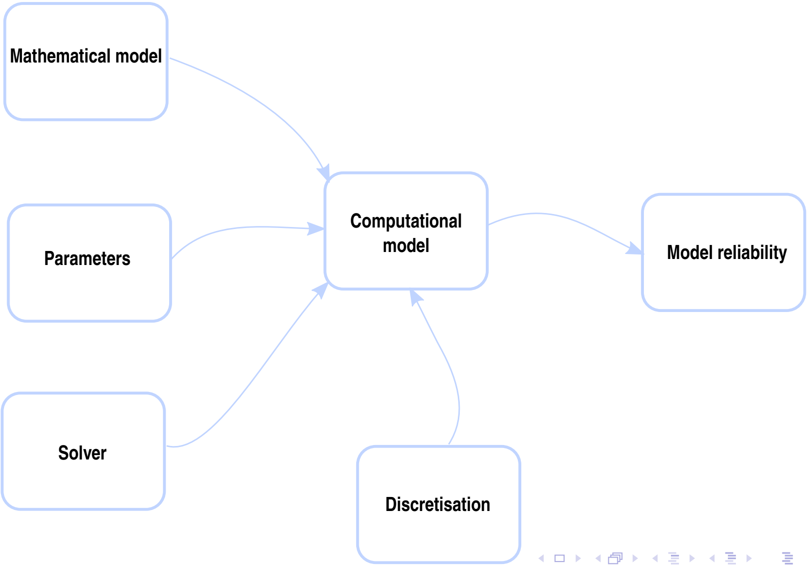
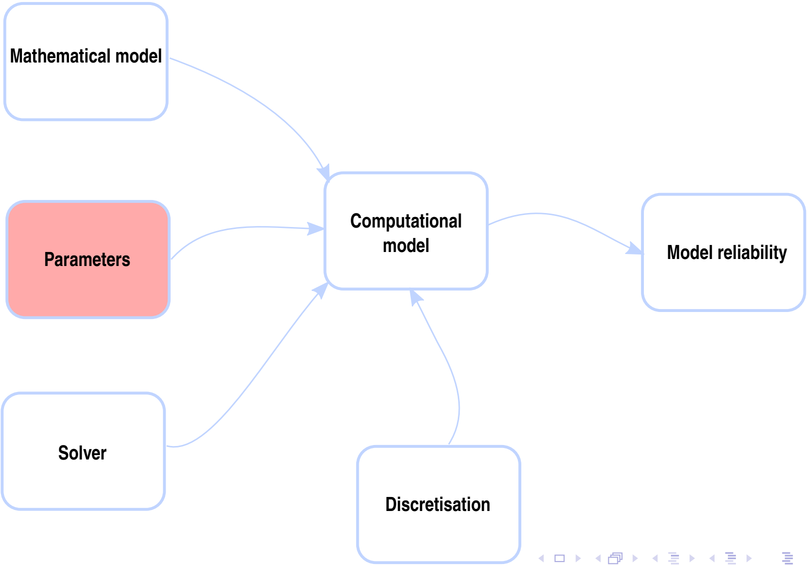
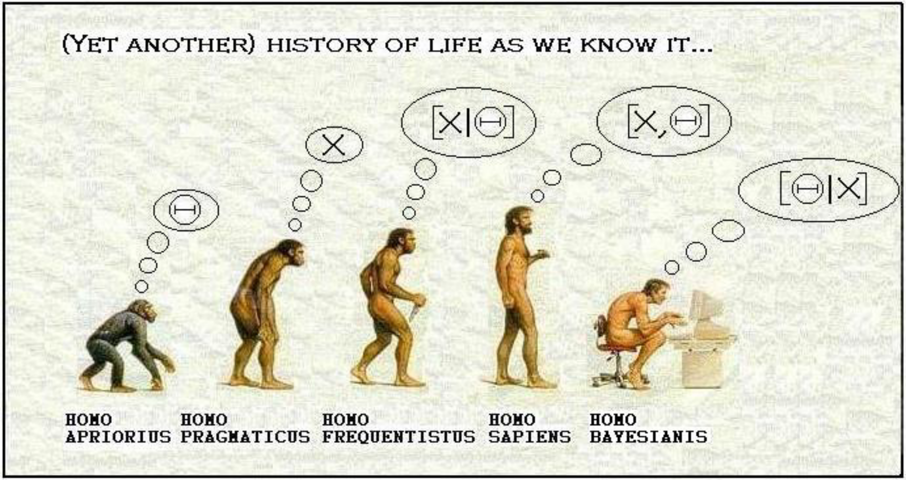
Introduction



Introduction



Young's modulus identification



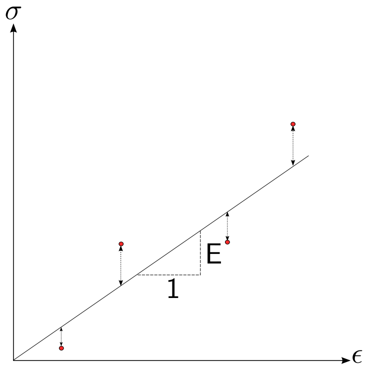
Error minimisation

Least squares method(conventional approach):

$$\sigma = E\epsilon$$

$$J = \frac{1}{2}\sum_{i=1}^{N}(\sigma_i - E\epsilon_i)^2$$

$$\overline{E} = \underset{E}{\operatorname{argmin}} J(E)$$



Frequentist inference

Example

Chance that a specific coin lands heads or tails

Frequentist inference

10

6

Frequentist inference

$$Pr(head) = \frac{10}{16}$$

Frequentist inference

$Pr(head) = \frac{10}{16}$

$Pr(tail) = \frac{6}{16}$

Frequentist inference: Young's modulus identification

Method of maximum likelihood (ML):

$\sigma = E\epsilon$

$\sigma_i = E\epsilon_i + \Omega$

Ω : noise in stress measurement, it is a random variable

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Calibrate $\pi_{noise}(\omega)$:

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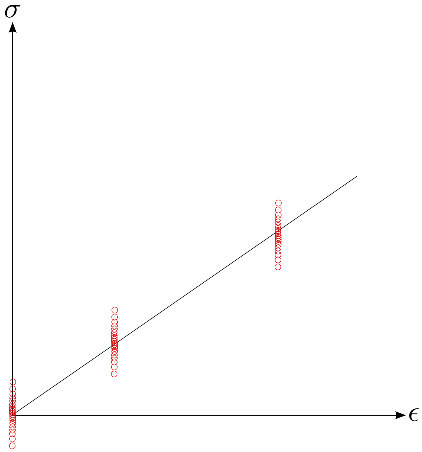
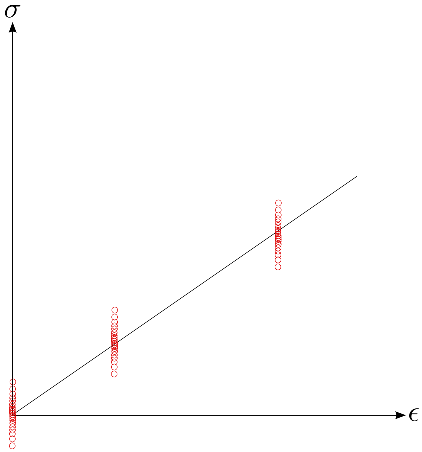


Frequentist inference: Young's modulus identification

Method of maximum likelihood (ML):

Calibrate $\pi_{noise}(\omega)$:

$$\pi_{noise}(\omega) = \frac{1}{\sqrt{2\pi}S_{noise}} \exp\Big(- \frac{\omega^2}{2S_{noise}^2}\Big)$$



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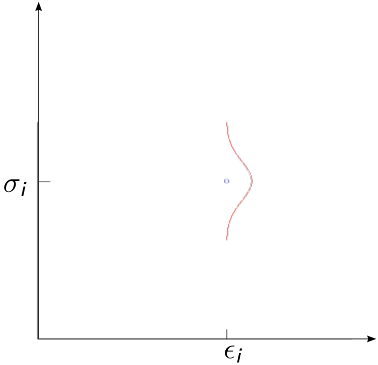

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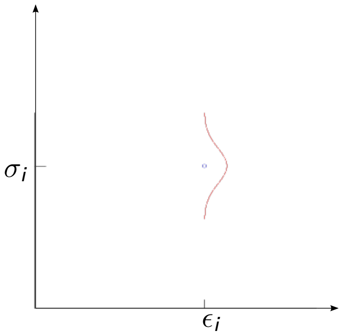
$$\boxed{\pi(\sigma_i|E, S_{noise}) = \frac{1}{\sqrt{2\pi}S_{noise}} \exp\Big(-\frac{(\sigma_i - E\epsilon_i)^2}{2S_{noise}^2}\Big)}$$



ML for M measurements:

$$\boldsymbol{\sigma}^m = \Big[\sigma_1, \sigma_2, \cdots, \sigma_M\Big]$$

$$\boldsymbol{\epsilon}^m = \Big[\epsilon_1, \epsilon_2, \cdots, \epsilon_M\Big]$$

$$\boxed{\pi(\boldsymbol{\sigma}^m | E, S_{noise}) = \frac{1}{(2\pi S_{noise}^2)^{\frac{M}{2}}} \exp\Bigg(-\frac{\sum\limits_{i=1}^{M}(\sigma_i - E\epsilon_i)^2}{2S_{noise}^2}\Bigg)}$$

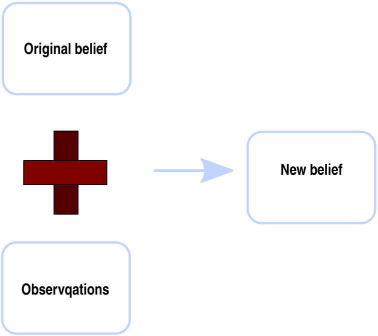
Bayesian inference

Coin example:

Bayesian inference

- Given a coin is it biased or not?
- If two rolls turn up head, do we have biased coin?

Bayesian inference



$$\pi(cause|effect) = \frac{\overbrace{\pi(cause)}^{prior}\times\overbrace{\pi(effect|cause)}^{likelihood}}{\underbrace{\pi(effect)}_{evidence}}$$

Bayesian inference: Young's modulus identification

Bayes' formula:

$\sigma_i = E\epsilon_i + \Omega$

Ω : noise in stress measurement

$\pi(E|\sigma_i) = \frac{\pi(E)\pi(\sigma_i|E)}{\pi(\sigma_i)}$

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$$\boxed{\pi(E|\sigma_i) \propto \pi(E)\pi(\sigma_i|E)}$$

Bayesian inference: Young's modulus identification

BI for M measurment:

$$\text{prior}: \pi(E)\pi(\sigma_1|E)\pi(\sigma_2|E)\cdots\pi(\sigma_{M-1}|E)$$

$$\text{likelihood}: \pi(\sigma_M|E)$$

$$\boxed{\pi(E|\sigma_M) \propto \pi(E)\pi(\sigma_1|E)\pi(\sigma_2|E)\cdots\pi(\sigma_{M-1}|E)\pi(\sigma_M|E)}$$

Bayesian inference: Young's modulus identification

$$\boxed{\pi(E|\sigma_M) \propto \prod_{i=1}^{M} \exp\Big(-\frac{(\sigma_i - E\epsilon_i)^2}{2S_{noise}^2}\Big)\exp\Big(-\frac{(E-\overline{E})^2}{2S_E^2}\Big)}$$

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$$\boxed{\pi(E|\sigma_M) \propto \exp\Big(-\frac{(E-\mu)^2}{2S_{post}^2}\Big)}$$

with $\mu = f(\sigma_i, \overline{E}, S_{nosie}, S_E, \epsilon_i)$
$S_{post} = f(\sigma_i, \overline{E}, S_{nosie}, S_E, \epsilon_i)$

Frequentist vs Bayesian

Frequentist:

- Data are a repeatable random sample there is a frequency
- Underlying parameters remain constant during this repeatable process
- Parameters are fixed

Bayesian:

- Data are observed from the realised sample
- Parameters are unknown and described probabilistically
- Data are fixed

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- Error minimisation cannot take statistical info of measurement device into account
- You probably will not test hundreds of specimens and then the prior (π_{prior}) may have a positive influence
- For inverse problems, the prior (π_{prior}) regularises the system (avoids ill-posedness)

What have we accomplished?

- Closed form expression of the posterior for:
  - elastoplasticity with perfect plasticity
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If heterogeneity is to be incorporated, we have to search for that as well, hence

search for parameters **and** the distribution thereof $\rightarrow$ future work...

The End