Generalized Information Theory for Hints

Marc Pouly

marc.pouly@uni.lu
Interdisciplinary Centre for Security, Reliability and Trust
University of Luxembourg

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securityandtrust.lu

Hartley's Measure (1928)

Given a set $S = \{s_1, \ldots, s_n\}$ how can we measure its uncertainty $u(S)$

1. uncertainty is a non-negative value
2. monotone: $|S_1| \leq |S_2| \Rightarrow u(S_1) \leq u(S_2)$
3. additive: $u(S_1 \times S_2) = u(S_1) + u(S_2)$

Theorem: There is only one function that satisfies these requirements

$$\boxed{u(S) \;=\; \log|S|}$$

Shannon's Measure (1948)

Given a set $S = \{s_1, \ldots, s_n\}$ with probabilities $p_i = p(s_i)$ how can we measure its uncertainty $u(S)$

$$\boxed{S(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \log p_i}$$

- We have similar uniqueness results for specific requirements
- Shannon generalizes Hartley: $S(\frac{1}{n}, \ldots, \frac{1}{n}) = \log n$

Uncertainty Measure for Dempster-Shafer Theory

What is the uncertainty of a mass function $\rightsquigarrow$ Requirements

1. **Generalization** of the Shannon and Hartley measure
2. **Additivity:** for non-interactive mass functions

$$GS(m_1 \otimes m_2) \quad = \quad GS(m_1) + GS(m_2)$$

3. **Subadditivity:** if m is defined over partition $s \cup t$

$$GS(m) \quad \leq \quad GS(m^{\downarrow s}) + GS(m^{\downarrow t})$$

4. **Expansibility, Symmetry, Continuity, Normalization**

The Aggregate Uncertainty

In 1994 several authors independently proposed

$$AU(m) \quad = \quad \max_{\mathcal{P}_{bel}} \left[-\sum_{\theta \in \Theta} p(\theta) \log p(\theta) \right]$$

- This measure is called Aggregate Uncertainty
- It satisfies all the requirements above
- The max is taken over all probability distributions that dominate *bel*

Shortcomings of the Aggregate Uncertainty

- Computing uncertainty requires to solve a non-linear optimization problem. Although an algorithm exists, computing the uncertainty of even simplest mass functions by hand is impossible (for me).
- It has been shown that the Aggregate Uncertainty is highly insensitive to changes in evidence.
- Anyway, it does (!) satisfy the requirements.

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Uniqueness of the Aggregate Uncertainty

Citation from (Harmanec, 1999) page 6:

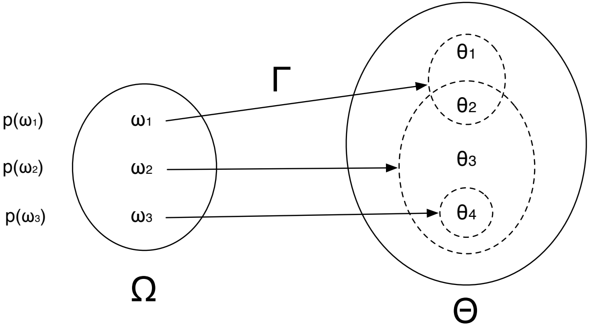
> *An interesting question is whether the measure AU is the only measure satisfying the requirements. It is still an open question ...*

Citation from (Klir, 2005) page 228:

> *It is thus a well-justified measure [...] in Dempster-Shafer Theory. Although its uniqueness is still an open problem ...*

The Theory of Hints (Kohlas & Monney, 1995)

- A particular approach to Dempster-Shafer theory
- Do hints give a new perspective on uncertainty ?



The Theory of Hints

1. A hint refers to a question with possible answers $\Theta = \{\theta_1, \dots, \theta_n\}$
2. Set of configurations (states) of the actual world $\Omega = \{\omega_1, \dots, \omega_m\}$
3. If $\omega \in \Omega$ holds, the true answer belongs to $\Gamma(\omega) \subseteq \Theta$ (causal relation)
4. Each $\omega \in \Omega$ has a probability $p(\omega)$
5. Hint: $\mathcal{H} = (\Theta, \Omega, \Gamma, p)$ where $\Gamma : \Omega \rightarrow \mathcal{P}(\Theta)$

Hints and Dempster-Shafer Theory

- Hints give information over $\Omega \times \Theta$, mass functions over Θ
- Hints induce mass functions, i.e. for $A \subseteq \Theta$ we define

$$m(A) \quad = \quad \sum_{\omega \in \Omega : \Gamma(\omega) = A} p(\omega)$$

- Hints are equivalent if they induce the same mass function
- Hints therefore provide more fine-grained information, i.e. one mass function stands for an equivalence class of hints

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Uncertainty of Hints

- Given $\omega \in \Omega$ the true answer is in $\Gamma(\omega)$
- There is not more evidence $\leadsto$ Hartley is the only justified measure: $H(\mathcal{H}|\omega) = \log|\Gamma(\omega)|$
- Hartley equals Shannon on uniform distributions, we set

$$p(\theta|\omega) = \frac{1}{|\Gamma(\omega)|} \quad \text{for all } \theta \in \Gamma(\omega)$$

- Therefore $p(\omega,\theta) = p(\theta|\omega) \cdot p(\omega) = \frac{p(\omega)}{|\Gamma(\omega)|}$ for $\theta \in \Gamma(\omega)$

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The Hints Entropy

We measure the uncertainty of a hint $\mathcal{H} = (\Theta, \Omega, \Gamma, p)$ by Shannon's entropy applied to the joint distribution $p(\omega, \theta)$:

$$\boxed{H(\mathcal{H}) = -\sum_{\omega \in \Omega} \sum_{\theta \in \Gamma(\omega)} p(\omega, \theta) \log p(\omega, \theta)}$$

Track Record of the Hints Entropy

1. **Generalization** of the Shannon and Hartley measure $\checkmark$
2. **Additivity:** for non-interactive hints

$$H(\mathcal{H}_1 \otimes \mathcal{H}_2) \quad = \quad H(\mathcal{H}_1) + H(\mathcal{H}_2) \;\checkmark$$

3. **Subadditivity:** if $\mathcal{H}$ is defined over partition $s \cup t$

$$H(\mathcal{H}) \quad \leq \quad H(\mathcal{H}^{\downarrow s}) + H(\mathcal{H}^{\downarrow t}) \;\checkmark$$

4. **Expansibility, Symmetry, Continuity, Normalization** $\checkmark$

Hints Entropy in the Literature

How is the hints entropy related to other functionals ?

- From $p(\omega,\theta) = \frac{p(\omega)}{|\Gamma(\omega)|}$ for $\theta \in \Gamma(\omega)$ follows

$$H(\mathcal{H}) = -\sum_{\omega\in\Omega} p(\omega) \log p(\omega) + \sum_{\omega\in\Omega} p(\omega) \log |\Gamma(\omega)|$$

- 2nd term is known as the Generalized Hartley Measure

- The hints entropy is the sum of two well-studied functionals

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Discussion

The hints entropy ...

- ... satisfies the same properties as the Aggregate Uncertainty
- ... does not suffer from the sensitivity problem
- ... is defined as an explicit formula, which is easy to compute

Does this disprove uniqueness of the Aggregate Uncertainty ?

No, because the two functionals have different range

- Aggregate Uncertainty: $0 \leq AU(m) \leq \log|\Theta|$
- Hints Entropy: $0 \leq H(m) \leq \log|\Omega \times \Theta|$

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Pignistic Probabilities

- Hints entropy defined on $\Omega \times \Theta \rightsquigarrow$ marginalize to Θ
- We had $p(\omega, \theta) = \frac{p(\omega)}{|\Gamma(\omega)|}$ for $\theta \in \Gamma(\omega)$
- The marginal distribution therefore is

$$p(\theta) = \sum_{\omega \in \Omega} p(\omega, \theta) = \sum_{\omega \in \Omega : \theta \in \Gamma(\omega)} \frac{p(\omega)}{|\Gamma(\omega)|}$$

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Pignistic Entropy

- The hints entropy restricted to Θ is the pignistic entropy

$$\boxed{H(\Theta) = -\sum_{\theta \in \Theta} p(\theta) \log p(\theta)}$$

where

$$p(\theta) = \sum_{\omega \in \Omega : \theta \in \Gamma(\omega)} \frac{p(\omega)}{|\Gamma(\omega)|}$$

- Equivalent hints have the same pignistic entropy
- Uncertainty measure for Dempster-Shafer theory

History of the Pignistic Entropy

- In 2006, the pignistic entropy was proposed as an uncertainty measure that satisfies all requirements
- In 2008, a mistake in the proof of subadditivity was pointed out and a counter-example was given
- Flaw: the pignistic distribution of a marginal $m^{\downarrow s}$ is not necessarily the marginal of the pignistic distribution of m

Conclusions

- The hints entropy on $\Omega \times \Theta$ satisfies all properties, the pignistic entropy on Θ violates subadditivity
- But the hints entropy satisfies a weaker form of subadditivity $\rightsquigarrow$ see paper
- DS theory studies equivalence classes of hints
- By looking at equivalence classes we naturally loose information $\rightsquigarrow$ destroys strong subadditivity

Late Credits

- The paper from 2006 on the pignistic entropy is flawed - the pignistic entropy does not satisfy subadditivity
- Eliminating the mistake still proves weak subadditivity
- However, the same ideas applied to hints instead of mass functions gives an uncertainty measure that satisfies all properties including strong subadditivity.


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