An extension of the concept of distance to functions of several variables

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A pair (X, d) is called a *metric space*, if X is a nonempty set and d is a distance on X, that is a function $d\colon X^2 \to \mathbb{R}_+$ such that:

(i) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$,

(ii) $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$,

(iii) $d(x_1, x_2) \leqslant d(x_1, z) + d(z, x_2)$ for all $x_1, x_2, z \in X$.

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Multidistance: A generalization of a distance by Martín and Mayor. We say that $d\colon \cup_{n\geqslant 1} X^n \to \mathbb{R}_+$ is a *multidistance* if:

(i) $d(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n$,

(ii) $d(x_1, \ldots, x_n) = d(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $x_1, \ldots, x_n \in X$ and all $\pi \in S_n$,

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We denote by K^* the smallest constant K for which (iii) holds. For $n = 2$, we assume that $K^* = 1$.

Example (Drastic n-distance)

The function $d\colon X^n \to \mathbb{R}_+$ defined by $d(x_1, \ldots, x_n) = 0$, if $x_1 = \cdots = x_n$, and $d(x_1, \ldots, x_n) = 1$, otherwise.

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Proposition

Let d and d' be n-distances on X and let $\lambda > 0$. The following assertions hold.

(1) *$d + d'$ and $\lambda\, d$ are n-distance on X.*

(2) *$\frac{d}{1+d}$ is an n-distance on X, with value in $[0, 1]$.*

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Lemma

Let $a, a_1, \ldots, a_n$ be nonnegative real numbers such that $\sum_{i=1}^n a_i \geq a$. Then

$$\frac{a}{1+a} \leq \frac{a_1}{1+a_1} + \cdots + \frac{a_n}{1+a_n}.$$

A generalization of n-distance

Condition (iii) in Definition 1 can be generalized as follows.

Definition

Let $g\colon \mathbb{R}_+^n \to \mathbb{R}_+$ be a symmetric function. We say that a function $d\colon X^n \to \mathbb{R}^+$ is a *g-distance* if it satisfies conditions (i), (ii) and

$$d(x_1,\ldots,x_n) \;\leqslant\; g\big(d(x_1,\ldots,x_n)|_{x_1=z}\,,\ldots,\, d(x_1,\ldots,x_n)|_{x_n=z}\big)$$

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It is natural to ask that $d+d'$, $\lambda\, d$, and $\frac{d}{1+d}$ be g-distances whenever so are d and d'.

Proposition

Let $g : \mathbb{R}_+^n \to \mathbb{R}_+$ *be a (symmetric) function,* d *and* d' *be* g*-distances. The following assertions hold.*

(1) *If* g *is positively homogeneous, i.e.,* $g(\lambda\,\mathbf{r}) = \lambda\,g(\mathbf{r})$ *for all* $\mathbf{r} \in \mathbb{R}_+^n$ *and all* $\lambda > 0$*, then for every* $\lambda > 0$*,* $\lambda\,d$ *is a* g*-distance.*

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(3) *If g is both positively homogeneous and superadditive, then it is concave.*

(4) *If g is bounded below (at least on a measurable set) and additive, that is, $g(\mathbf{r}+\mathbf{s}) = g(\mathbf{r}) + g(\mathbf{s})$ for all $\mathbf{r},\mathbf{s} \in \mathbb{R}_+^n$, then and only then there exist $\lambda_1, \ldots, \lambda_n \geqslant 0$ such that*

$$g(\mathbf{r}) \;=\; \sum_{i=1}^{n} \lambda_i\, r_i \tag{1}$$

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Example (Basic examples)

Given a metric space (X, d) and $n \geqslant 2$, the maps $d_{\max}\colon X^n \to \mathbb{R}_+$ and $d_\Sigma\colon X^n \to \mathbb{R}_+$ defined by

$$\begin{aligned} d_{\max}(x_1, \ldots, x_n) &= \max_{1 \leqslant i < j \leqslant n} d(x_i, x_j) \\ d_\Sigma(x_1, \ldots, x_n) &= \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j) \end{aligned}$$

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are n-distances for which the best constants are given by $K^* = \frac{1}{n-1}$.

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Theorem

Let (X, d) be a metric space and $n \geqslant 2$. Then for any nonempty class $\mathcal{P}$ the map $d_{Gr} : X^n \to \mathbb{R}_+$ defined by

$$d_{Gr}(x_1, \ldots, x_n) = \max_{G \in \mathcal{P}} \sum_{(x_i, x_j) \in E(G)} d(x_i, x_j)$$

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3. For any $1 \leq s \leq n$ let $\mathcal{P} = \{G \simeq K_s\}$. Then

$$d_{K_s}(x_1, \dots, x_n) = \max_{G \in \mathcal{P}} \sum_{(x_i, x_j) \in E(G)} d(x_i, x_j)$$

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4. If $\mathcal{P}$ is the class of Hamiltonian cycles of K_n. Then

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5. $\mathcal{P}$ is a class of circles of given size, or the class of spanning trees, etc.

Examples II.

Example (Geometric constructions)

Let $x_1, \ldots, x_n$ be $n \geqslant 2$ arbitrary points in $\mathbb{R}^k$ ($k \geqslant 2$) and denote by $B(x_1, \ldots, x_n)$ the smallest closed ball containing $x_1, \ldots, x_n$. It can be shown that this ball always exist, is unique, and can be determined in linear time.

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(3) The k-dimensional volume of $B(x_1, \ldots, x_n)$ is an n-distance and we conjecture that the best constant K^* is given by $K^* = \frac{1}{n-2+(1/2)^{k-1}}$. This is correct for $k = 1$ or 2.

Examples III.

Example (Fermat point based *n*-distances)

Given a metric space (X, d), and an integer $n \geq 2$, the *Fermat set* F_Y of any element subset $Y = \{x_1, \ldots, x_n\}$ of X, is defined as

$$F_Y = \left\{ x \in X : \sum_{i=1}^{n} d(x_i, x) \leq \sum_{i=1}^{n} d(x_i, z) \text{ for all } z \in X \right\}.$$

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Since $h(x) = \sum_{i=1}^{n} d(x_i, x)$ is continuous and bounded from below by 0, F_Y is non-empty **but** usually not a singleton.
We can define $d_F : X^n \to \mathbb{R}_+$ by

$$d_F(x_1, \ldots, x_n) = min\Big\{\sum_{i=1}^{n} d(x_i, x) : x \in X\Big\}.$$

Proposition

d_F is an n-distance and $K^* \leq \frac{1}{\lceil \frac{n-1}{2} \rceil}$.

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We can define $d_m : V^3 \to \mathbb{R}_+$ by

$$d_m(u, v, w) = \min_{s \in V} \left\{ d(u, s) + d(v, s) + d(w, s) \right\}.$$

Proposition

d_m is a 3-distance, $d_m(u, v, w)$ is realized by $s = m(u, v, w)$ and $K^* = \frac{1}{2}$.

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Theorem

d_{gm} is a n-distance, $d_{gm}(x_1,\ldots,x_n)$ is realized by (any) $m = Maj(x_1,\ldots,x_n)$ and $K^* = \frac{1}{n-1}$.

$K^* = 1$, Example IV.

For all of the previous examples $\frac{1}{n-1} \leq K^* \leq \frac{1}{n-2}$ (when we know the exact value).

Question

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Yes. In $\mathbb{R}$ we can define

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and $d_n(\mathbf{x}) = A_n(\mathbf{x}) - \min_n(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

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d_n is an n-distance for every $n \geq 2$ and $K^* = 1$.

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Proposition

d_n is an n-distance for every $n \geq 2$ and $K^* = 1$.
But it is not realized. (For every $\varepsilon > 0$ it can be shown that $K^* > 1 - \varepsilon$.)

Summary

Table: Critical values

n-distance	space X	K^*	nb. of var.
$d_{Gr}, d_{max}, d_{\sum}$	arbitrary metric	$\frac{1}{n-1}$	$n > 1$
$d_{diameter}$	$\mathbb{R}^m$ $(m \geq 1)$	$\frac{1}{n-1}$	$n > 1$
d_{area}	$\mathbb{R}^m$ $(m \geq 2)$	$\frac{1}{n-3/2}$	$n > 1$
$d_{volume(k)}$	$\mathbb{R}^m$ $(m \geq k)$	$? = \frac{1}{n-1-(1/2)^{k-1}}$	$n > 1$
d_{Fermat}	arbitrary metric	$? \leq \frac{1}{\lceil \frac{n-1}{2} \rceil}$	$n > 1$
d_{median}	median graph G	$\frac{1}{2}$	$n = 3$
$d_{hypercube}$	$\{0,1\}^n$	$\frac{1}{n-1}$	$n > 1$
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Conjecture

$$\frac{1}{n-1} \leq K^* \leq 1.$$

Question

1. *Are there any n-distance such that* $K^* < \frac{1}{n-1}$*?*
2. *Can we characterize the n-distances for which* $K^* = \frac{1}{n-1}$*?*
3. *Can we characterize the n-distances for which* $K^* = 1$*?*
4. *Can we show an example where* $K^* = 1$ *is realized?*

Thank you for your kind attention!