53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

H_∞ dynamic observer design for linear time invariant systems

Nan GAO[1], Mohamed DAROUACH[1], Holger VOOS[2] and Marouane ALMA[1]

***Abstract*— This paper proposes an H_∞ dynamic observer (DO) for a class of linear time invariant (LTI) systems in the presence of disturbances. It generalizes the existing results on the proportional observer (PO), the proportional integral observer (PIO) and the DO. The design method is derived from a new formulation of linear matrix inequality (LMI), based on the solutions of the algebraic constraints obtained from the unbiasedness conditions of the estimation error. A numerical example is provided to show the applicability and performances of our observer.**

I. INTRODUCTION

Due to the fact that the state variables of a dynamic system are often not all available but have great importance in real-world systems, such as state feedback control and fault diagnosis, the application of the observer has received considerable attention, since the first work of observer presented in [1] and [2].

In the last several decades, the observer has been developed for systems in the presence of exogenous disturbances. One approach to deal with disturbances is the disturbance observer. A time-varying exponentially stable interval observer was constructed for time-invariant exponentially stable linear systems with additive disturbances in [3], where the systems could be transformed into cooperative and exponentially stable systems. In [4], a reduced-order disturbance observer was proposed to attenuate the after-effects caused by the friction on the output of the traditional PD-type control scheme, instead of compensating the primary friction mechanics. An output-based disturbance observer of reduced order was presented for a class of discrete-time linear systems in [5].

Another method, named the H_∞ observer, which combined the H_∞ theory with the observer, was introduced to deal with disturbances. Instead of estimating disturbances, the H_∞ observer offers a direct method to limit the negative effect of disturbances. In [6], the linear matrix inequality (LMI) was employed to construct the delay-dependent non-fragile H_∞ observer-based feedback control for a class of time-delay systems, extended to observer-based finite-time H_∞ control problem for one family of discrete-time Markovian jump systems with time-varying norm-bounded disturbance [7]. The authors of [8] proposed a robust H_∞ fuzzy observer-based controller design method for uncertain T-S fuzzy systems, by using Finsler's lemma.

[1]CRAN-CNRS UMR7039, University de Lorraine, IUT de Longwy, 186, Rue de Lorraine, Cosnes et Romain 54400, France. gaonanlp@live.cn
[1]Mohamed.Darouach@univ-lorraine.fr
[1]marouane.alma@univ-lorraine.fr
[2]University of Luxembourg, Faculty des Sciences de la Technologie et de la Communication, 6, Rue Richard Coudenhove-Kalergi, L-1359. holger.voos@uni.lu

All the observers introduced before are the kind of proportional observers (PO), which are not capable to handle the static error. Consequently, the proportional integral observer (PIO) has been introduced by duality to the PI controller, which is frequently used to achieve steady-state performance. The first result on the PIO was presented for single-input-single-output (SISO) systems in [9]. The authors of [10] designed a discrete-time PIO for both system states and disturbances for systems with unknown inputs and output disturbances. In [11], through the asymptotic error analysis for PIO, the authors proposed integral observers for unbiased output estimation in the presence of uncertainty. The application of the H_∞ PIO can be found in [12] for Synchronization problem of chaotic systems.

Recently, a new structure of the observer, called dynamic observer (DO), has been developed, which presents an alternative state estimation structure. Different from the PO and the PIO, the DO obtains the observer gain through state space equation. In [13], it was shown that the mechanism of the proposed DO design is the dual of the output feedback controller design for linear time invariant (LTI) systems. The dynamic observer-based H_∞ controller design, based on a new form of change-of-variables, was proposed for linear systems in [14].

In this paper, by combining the H_∞ theory with the DO, we propose an H_∞ DO for LTI systems subject to disturbances. The proposed observer has a more generalized form, of which the popularly used PO and PIO are only particular cases. The observer is derived from the solution of new LMI formulations, based on the transformation of the algebraic constraint.

The paper is organized as follows: In section II, the DO design problem and some preliminaries relevant to this paper are presented. Section III gives some parameterization results. In section IV, the design problem is solved for LTI systems in the presence of disturbances. An illustrative example is provided to show the performance of our observer in Section V. Some conclusions are drawn in Section VI.

The following notations will be used throughout this paper: $\mathfrak{R}^n$ and $\mathfrak{R}^{n\times m}$ denote the set of n dimensional real vectors and the set of all $n\times m$ real matrices, respectively; $\|.\|_\infty$ is the H_∞ norm; A^+ is the generalized inverse of matrix A satifying $AA^+A = A$; A^T denotes the transpose of A; A is symmetric positive definite if and only if $A^T = A$ and $A > 0$; matrices I and 0 denote the identity matrix and zero matrix of appropriate dimensions, respectively; Let V be a vector space over a filed F, for a subset W of V, we define the left orthogonal complement $W^\perp$ to be $W^\perp = \{x \in V : x^T y = 0 \text{ for all } y \in W\}$. There is a corresponding definition of the

978-1-4673-6090-6/14/$31.00 ©2014 IEEE

right orthogonal complement.

II. Problem formulation

Let us consider the following LTI system in the presence of disturbances:

$$\begin{array}{rcl} \dot{x} & = & Ax+Bu+D_1w \\ y & = & Cx+D_2w \end{array} \tag{1}$$

with the initial state $x(0)=x_0$, where $x\in\Re^n, y\in\Re^p, u\in\Re^l$ and $w\in\Re^f$ are the state vector, the measurement output vector, the control input vector and the disturbance vector, respectively. A,B,C,D_1 and D_2 are known constant matrices and of appropriate dimensions.

Next, let us consider the following DO:

$$\begin{array}{rcl} \dot{z} & = & Nz+Jy+Hu+Mv \\ \dot{v} & = & Pz+Qy+Gv \\ \hat{x} & = & Rz+Sy \end{array} \tag{2}$$

where $z\in\Re^q, v\in\Re^q$ and $\hat{x}\in\Re^n$ are the state vector of the observer, the auxiliary state vector and the estimation of x, respectively. Matrices N,J,H,M,P,Q,G,R and S are unknown to be determined and of appropriate dimensions. The auxiliary vector v is similar to the additional term in the PIO, which is proportional to the integral of the output error, aiming to achieve robustness performance.

Remark 1:

1. The observer (2) is in a generalized form. In fact, if $M=0,P=0,Q=0,G=0,S=0$, and $R=I$, we obtain the full order PO:

$$\dot{\hat{x}}=N\hat{x}+Jy+Hu$$

If $G=0,Q=I,P=-C,R=I$ and $S=0$, we obtain the following PIO:

$$\begin{array}{rcl} \dot{\hat{x}} & = & N\hat{x}+Jy+Hu+Mv \\ \dot{v} & = & y-C\hat{x} \end{array}$$

If $R=I,S=0,\Xi=G,\Gamma=Q,\Lambda=M,\Psi=J,A-JC=N$ and $-QC=P$, we obtain the following standard DO:

$$\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x}+Bu+\xi \\ \dot{v} & = & \Xi v+\Gamma\mu \\ \xi & = & \Lambda v+\Psi\mu \\ \mu & = & y-\hat{y} \end{array}$$

2. Assume that *rank* $C=p$, if $q=n-p$, we obtain the reduced-order observer; if $q=n$, we obtain the full-order one.

Our aim is to design a DO to estimate the system state x, and the design problem is to determine all the parameter matrices N,J,H,M,P,Q,G,R and S such that the following two conditions are satisfied:

1 for disturbances $w=0$, the estimation error $e\to 0(e=\hat{x}-x)$ when $t\to\infty$;

2 for disturbances $w\neq 0$, the norm $\|T_{we}\|_\infty<\gamma$.

where T_{we} represents the transfer function from the disturbances w to the estimation error e and γ is a given positive scalar.

III. Parameterization of the observer

In this section, we will present the parameterization of the observer. Firstly, we define a new error variable $\varepsilon=z-Tx$, where the matrix $T\in\Re^{q\times n}$ is an arbitrary matrix. Then we can give the following lemma:

Lemma 1: For $w=0$, the system (2) is a dynamic observer for system (1) if there exits an arbitrary matrix T such that the following constraints are satisfied:

$$NT-TA+JC = 0 \tag{3a}$$
$$H = TB \tag{3b}$$
$$PT+QC = 0 \tag{3c}$$
$$RT+SC = I \tag{3d}$$

and the matrix

$$\mathbb{A}=\begin{pmatrix} N & M \\ P & G \end{pmatrix}$$

is Hurwitz.

Proof: From the definition of the error ε, we obtain the dynamic of the error ε:

$$\begin{array}{rcl} \dot{\varepsilon} & = & \dot{z}-T\dot{x} \\ & = & Nz+Jy+Hu+Mv-TAx-TBu \\ & = & N\varepsilon+(NT-TA+JC)x+(H-TB)u+Mv \end{array}$$

Furthermore, we obtain:

$$\begin{array}{rcl} \dot{v} & = & P\varepsilon+(PT+QC)x+Gv \\ \hat{x} & = & R\varepsilon+(RT+SC)x \end{array}$$

One can see that the dynamics of the error ε and the auxiliary state v are independent of x and u if the following constraints are satisfied:

$$\begin{array}{rcl} NT-TA+JC & = & 0 \\ H & = & TB \\ PT+QC & = & 0 \end{array}$$

On the other hand, if $RT+SC=I$, the estimation error e becomes:

$$e=R\varepsilon$$

In this case, we obtain the following system:

$$\begin{array}{rcl} \dot{\zeta} & = & \mathbb{A}\zeta \\ e & = & \mathbb{C}\zeta \end{array} \tag{4}$$

where $\zeta=\begin{pmatrix}\varepsilon \\ v\end{pmatrix}$, $\mathbb{A}=\begin{pmatrix} N & M \\ P & G \end{pmatrix}$ and $\mathbb{C}=\begin{pmatrix} R & 0\end{pmatrix}$.

Obviously, the estimation error $e\to 0$ if $\varepsilon\to 0$, and $\varepsilon\to 0$ and $v\to 0$ if and only if $\mathbb{A}$ is Hurwitz, which completes the proof of the lemma. ■

Consequently, the observer design problem in the case of $w=0$ is reduced to the stabilization of system (4).

Now, from equations (3c) and (3d) we obtain the following equation:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} T \\ C \end{pmatrix}=\begin{pmatrix} 0 \\ I \end{pmatrix} \tag{5}$$

The necessary and sufficient condition for (5) to have a solution is:

$$rank\begin{pmatrix}T\\C\\0\\I\end{pmatrix} = rank\begin{pmatrix}T\\C\end{pmatrix} = n$$

Assumed the condition above is satisfied, let $E \in \Re^{q\times n}$ be an arbitrary matrix such that:

$$rank\begin{pmatrix}E\\C\end{pmatrix} = rank\begin{pmatrix}T\\C\end{pmatrix} = n \tag{6}$$

since T is unknown, then there always exits parameter matrices T and K such that:

$$\begin{pmatrix}T\\C\end{pmatrix} = \begin{pmatrix}I & -K\\0 & I\end{pmatrix}\begin{pmatrix}E\\C\end{pmatrix}$$

or equivalently $T = E - KC$.

Consequently, equation (5) becomes:

$$\begin{pmatrix}P & Q\\R & S\end{pmatrix}\begin{pmatrix}I & -K\\0 & I\end{pmatrix}\begin{pmatrix}E\\C\end{pmatrix} = \begin{pmatrix}0\\I\end{pmatrix} \tag{7}$$

The general solution to (7) is given by:

$$\begin{pmatrix}P & Q\\R & S\end{pmatrix} = \left\{\begin{pmatrix}0\\I\end{pmatrix}\Sigma^{+} - Z\left[I-\Sigma\Sigma^{+}\right]\right\}\begin{pmatrix}I & K\\0 & I\end{pmatrix}$$

or equivalently:

$$P = -Z_2\beta_1, Q = -Z_2\beta_2, R = \alpha_1 - Z_3\beta_1 \text{ and } S = \alpha_2 - Z_3\beta_2 \tag{8}$$

where

$$\begin{cases}\Sigma = \begin{pmatrix}E\\C\end{pmatrix}, & Z \text{ is an arbitrary matrix,}\\ Z_2 = \begin{pmatrix}I & 0\end{pmatrix}Z, & Z_3 = \begin{pmatrix}0 & I\end{pmatrix}Z,\\ \alpha_1 = \Sigma^{+}\begin{pmatrix}I\\0\end{pmatrix}, & \beta_1 = (I-\Sigma\Sigma^{+})\begin{pmatrix}I\\0\end{pmatrix},\\ \alpha_2 = \Sigma^{+}\begin{pmatrix}K\\I\end{pmatrix} \quad \text{and} & \beta_2 = (I-\Sigma\Sigma^{+})\begin{pmatrix}K\\I\end{pmatrix}.\end{cases} \tag{9}$$

One can see from the expression of estimation error $e = R\varepsilon$ that $e \to 0$ when $\varepsilon \to 0$, i.e. e is independent of matrix R. Then we can take $Z_3 = 0$ and we obtain $R = \alpha_1$ and $S = \alpha_2$.

Notice that $T = E - KC$, we have the equation:

$$\begin{pmatrix}T & K\end{pmatrix}\begin{pmatrix}I\\C\end{pmatrix} = E \tag{10}$$

which has a solution if:

$$rank\begin{pmatrix}I\\C\\E\end{pmatrix} = rank\begin{pmatrix}I\\C\end{pmatrix} \tag{11}$$

In this case, one solution to (10) is given by:

$$T = E\begin{pmatrix}I\\C\end{pmatrix}^{+}\begin{pmatrix}I\\0\end{pmatrix} \text{ and } K = E\begin{pmatrix}I\\C\end{pmatrix}^{+}\begin{pmatrix}0\\I\end{pmatrix}. \tag{12}$$

Furthermore, equation (3a) can be rewritten as:

$$N(R-KC) - (R-KC)A + JC = 0$$

or

$$\begin{pmatrix}N & K_1\end{pmatrix}\Sigma = \Theta \tag{13}$$

where $K_1 = J - NK$ and $\Theta = TA$ and the general solution to (13) is given by:

$$\begin{pmatrix}N & K_1\end{pmatrix} = \Theta\Sigma^{+} - Z_1(I-\Sigma\Sigma^{+})$$

where Z_1 is an arbitrary matrix, or equivalently:

$$N = \alpha_3 - Z_1\beta_1 \text{ and } K_1 = \alpha_4 - Z_1\beta_3 \tag{14}$$

where

$$\alpha_3 = \Theta\Sigma^{+}\begin{pmatrix}I\\0\end{pmatrix}, \alpha_4 = \Theta\Sigma^{+}\begin{pmatrix}0\\I\end{pmatrix} \text{ and } \beta_3 = (I-\Sigma\Sigma^{+})\begin{pmatrix}0\\I\end{pmatrix}. \tag{15}$$

Then we can obtain the equation of J:

$$J = \Theta\alpha_2 - Z_1\beta_2 \tag{16}$$

From these results above, matrix $\mathbb{A}$ becomes:

$$\mathbb{A} = \begin{pmatrix}\alpha_3 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}Z_1 & M\\Z_2 & G\end{pmatrix}\begin{pmatrix}\beta_1 & 0\\0 & -I\end{pmatrix} = \mathbb{A}_1 - \mathbb{Z}\mathbb{A}_2 \tag{17}$$

and we obtain

$$\dot{\zeta} = (\mathbb{A}_1 - \mathbb{Z}\mathbb{A}_2)\,\zeta \tag{18}$$

Then, the design problem without disturbance is reduced to study system (18), i.e. to determine the parameter matrix $\mathbb{Z}$ which can be obtained from the following theorem:

Theorem 1: System (18) is stable if and only if there exists a symmetric positive definite matrix $\mathbb{P}$ and a matrix $\mathbb{Y}$ such that

$$\mathbb{A}_1^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 - \mathbb{A}_2^T\mathbb{Y}^T < 0 \tag{19}$$

in this case, $\mathbb{Z} = \mathbb{P}^{-1}\mathbb{Y}$.

Proof: Choose a Lyapunov function candidate $V(\zeta) = \zeta^T\mathbb{P}\zeta, \mathbb{P} = \mathbb{P}^T > 0$, then the differential of $V(\zeta)$ is along with the solution of (18):

$$\begin{aligned}\dot{V}(\zeta) &= \dot{\zeta}^T\mathbb{P}\zeta + \zeta^T\mathbb{P}\dot{\zeta}\\ &= \zeta^T(\mathbb{A}_1 - \mathbb{Z}\mathbb{A}_2)^T\mathbb{P}\zeta + \zeta^T\mathbb{P}(\mathbb{A}_1 - \mathbb{Z}\mathbb{A}_2)\zeta\\ &= \zeta^T(\mathbb{A}_1^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 - \mathbb{A}_2^T\mathbb{Y}^T)\zeta\end{aligned}$$

where $\mathbb{Y} = \mathbb{P}\mathbb{Z}$.

One can see that system (18) is stable if and only if $\dot{V}(\zeta) < 0$, or equivalently:

$$\mathbb{A}_1^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 - \mathbb{A}_2^T\mathbb{Y}^T < 0$$

This completes the proof. ■

The design procedure of the proposed observer for $w = 0$ is summarized as follows:

1. Choose the matrix E according to the condition (6);
2. Compute T and K from (12);
3. Compute $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ and β_3 from (9) and (15);
4. Compute the parameter matrix $\mathbb{Z}$ from the solution of the LMI (19);
5. Deduce all the matrices N, J, H, M, P, Q, G, R and S from equations (3b), (8), (14) and (16).

IV. H_∞ DYNAMIC OBSERVER DESIGN

In this section, we will investigate the observer design problem in the case that $w \neq 0$. By using constraint equations of lemma 1, we obtain the dynamic of error ε:

$$\dot{\varepsilon} = N\varepsilon + Mv + (JD_2 - TD_1)w$$

the dynamic of v becomes:

$$\dot{v} = P\varepsilon + Gv + QD_2 w$$

and the estimation error e becomes:

$$e = R\varepsilon + SD_2 w$$

In this case, we obtain:

$$\begin{pmatrix} \dot{\varepsilon} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} N & M \\ P & G \end{pmatrix} \begin{pmatrix} \varepsilon \\ v \end{pmatrix} + \begin{pmatrix} JD_2 - TD_1 \\ QD_2 \end{pmatrix} w$$

and estimation error e becomes:

$$e = R\varepsilon + SD_2 v$$

From the above results, we obtain the following system:

$$\begin{array}{rcl} \dot{\zeta} & = & \mathbb{A}\zeta + \mathbb{B}w \\ e & = & \mathbb{C}\zeta + \mathbb{D}w \end{array} \quad (20)$$

where

$\mathbb{A} = \begin{pmatrix} N & M \\ P & G \end{pmatrix}$, $\mathbb{B} = \begin{pmatrix} JD_2 - TD_1 \\ QD_2 \end{pmatrix}$, $\mathbb{C} = \begin{pmatrix} R & 0 \end{pmatrix}$ and $\mathbb{D} = SD_2$.

Remark 2: According to the results of section III, the four matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ can be rewritten as:

$$\begin{cases} \mathbb{A} = \mathbb{A}_1 - \mathbb{Z}\mathbb{A}_2, \mathbb{B} = \mathbb{B}_1 - \mathbb{Z}\mathbb{B}_2, \\ \mathbb{B}_1 = \begin{pmatrix} -TD_1 + \Theta\Sigma^+ \begin{pmatrix} K \\ I \end{pmatrix} D_2 \\ 0 \end{pmatrix}, \\ \mathbb{B}_2 = \begin{pmatrix} -\beta_2 D_2 \\ 0 \end{pmatrix}, \mathbb{C} = \begin{pmatrix} \alpha_1 & 0 \end{pmatrix} \text{and } \mathbb{D} = \alpha_2 D_2. \end{cases} \quad (21)$$

Then, the observer design problem in the presence of disturbances is reduced to determine the parameter matrix $\mathbb{Z}$ such that:

- Matrix $\mathbb{A}$ is a stability matrix for $w = 0$;
- for $w \neq 0$, $\|T_{we}\|_\infty$ is minimized.

Generally, according to the bounded-real lemma, the problem above can be settled by solving one LMI and the solution leads to $M = 0$, $P = 0$ and $Q = 0$. In this case, we can only obtain PO. In order to obtain the more general solution, we propose the following theorem to obtain the parameter matrix $\mathbb{Z}$:

Theorem 2: There exists a parameter matrix $\mathbb{Z}$ such that the system (20) is asymptotically stable for $w = 0$ and $\|T_{we}\|_\infty < \gamma$ for $w \neq 0$, if and only if there exist a symmetric positive definite matrix $\mathscr{X}$ and a positive scalar γ such that the following LMIs are satisfied:

$$\begin{cases} \mathbb{D}^T\mathbb{D} - \gamma^2 I < 0 \\ \mathscr{C}^{T\perp}\mathscr{Q}\mathscr{C}^{T\perp T} < 0 \end{cases} \quad (22)$$

where

$$\mathscr{Q} = \begin{pmatrix} \mathscr{X}\mathbb{A}_1 + \mathbb{A}_1^T\mathscr{X} + \mathbb{C}^T\mathbb{C} & \mathscr{X}\mathbb{B}_1 + \mathbb{C}^T\mathbb{D} \\ \mathbb{B}_1^T\mathscr{X} + \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} - \gamma^2 I \end{pmatrix} \quad (23)$$

Suppose the above statements hold, let $(\mathscr{B}_l, \mathscr{B}_r)$ and $(\mathscr{C}_l, \mathscr{C}_r)$ be any full rank factors of $\mathscr{B}$ and $\mathscr{C}$, i.e. $\mathscr{B} = \mathscr{B}_l\mathscr{B}_r, \mathscr{C} = \mathscr{C}_l\mathscr{C}_r$.

In this case, matrix $\mathbb{Z} = \mathscr{X}^{-1}\mathscr{Y}$ and $\mathscr{Y}$ is given by

$$\mathscr{Y} = \mathscr{B}_r^+\mathscr{K}\mathscr{C}_l^+ + \mathscr{Z} - \mathscr{Z}\mathscr{C}_l\mathscr{C}_l^+ \quad (24)$$

where $\mathscr{Z}$ is an arbitrary matrix and

$$\mathscr{K} \triangleq -\mathscr{R}^{-1}\Phi\mathscr{C}_r^T(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1} + \mathscr{S}^{1/2}\mathscr{L}(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1/2} \quad (25)$$

$$\mathscr{S} \triangleq \mathscr{R}^{-1} - \mathscr{R}^{-1}[\Phi - \Phi\mathscr{C}_r^T(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1}\mathscr{C}_r\Phi]\mathscr{R}^{-1} \quad (26)$$

where $\mathscr{L}$ is an arbitrary matrix such that $\|\mathscr{L}\| < 1$ and $\mathscr{R}$ is an arbitrary positive definite matrix such that

$$\Phi \triangleq (\mathscr{R}^{-1} - \mathscr{Q})^{-1} > 0 \quad (27)$$

Proof: According to the bounded-real lemma, system (20) is asymptotically stable for $w = 0$ and $\|T_{we}\|_\infty < \gamma$ for $w \neq 0$ if and only if there exist a symmetric positive definite matrix $\mathscr{X}$ and a positive scalar γ such that the following LMI is satisfied:

$$\begin{pmatrix} \mathbb{A}^T\mathscr{X} + \mathscr{X}\mathbb{A} + \mathbb{C}^T\mathbb{C} & \mathscr{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} \\ (\mathscr{X}\mathbb{B})^T + \mathbb{D}^T\mathbb{C} & -\gamma^2 I + \mathbb{D}^T\mathbb{D} \end{pmatrix} < 0 \quad (28)$$

By inserting matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ of remark 2 into the LMI (28), we obtain the following LMI:

$$\begin{array}{c} \begin{pmatrix} \Pi & \Omega \\ \Omega^T & -\gamma^2 I + \mathbb{D}^T\mathbb{D} \end{pmatrix} < 0 \\ \Leftrightarrow \quad \mathscr{Q} + \mathscr{B}\mathscr{Y}\mathscr{C} + (\mathscr{B}\mathscr{Y}\mathscr{C})^T < 0 \end{array} \quad (29)$$

where

$$\begin{array}{rcl} \Pi & = & \mathscr{X}\mathbb{A}_1 + \mathbb{A}_1^T\mathscr{X} - \mathscr{Y}\mathbb{A}_2 - (\mathscr{Y}\mathbb{A}_2)^T + \mathbb{C}^T\mathbb{C} \\ \Omega & = & \mathscr{X}\mathbb{B}_1 - \mathscr{Y}\mathbb{B}_2 + \mathbb{C}^T\mathbb{D} \\ \mathscr{Q} & = & \begin{pmatrix} \mathscr{X}\mathbb{A}_1 + \mathbb{A}_1^T\mathscr{X} + \mathbb{C}^T\mathbb{C} & \mathscr{X}\mathbb{B}_1 + \mathbb{C}^T\mathbb{D} \\ \mathbb{B}_1^T\mathscr{X} + \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} - \gamma^2 I \end{pmatrix} \end{array}$$

$\mathscr{B} = \begin{pmatrix} -I \\ 0 \end{pmatrix}$, $\mathscr{C} = \begin{pmatrix} \mathbb{A}_2 & \mathbb{B}_2 \end{pmatrix}$ and $\mathscr{Y} = \mathscr{X}\mathbb{Z}$.

According to [15], the inequality (29) is equivalent to:

$$\mathscr{B}^\perp\mathscr{Q}\mathscr{B}^{\perp T} < 0$$
$$\mathscr{C}^{T\perp}\mathscr{Q}\mathscr{C}^{T\perp T} < 0$$

In this case, matrix $\mathscr{Y}$ can be obtained by:

$$\mathscr{Y} = \mathscr{B}_r^+\mathscr{K}\mathscr{C}_l^+ + \mathscr{Z} - \mathscr{B}_r^+\mathscr{B}_r\mathscr{Z}\mathscr{C}_l\mathscr{C}_l^+ \quad (30)$$

where $\mathscr{Z}$ is an arbitrary matrix and

$$\mathscr{K} \triangleq -\mathscr{R}^{-1}\Phi\mathscr{C}_r^T(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1} + \mathscr{S}^{1/2}\mathscr{L}(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1/2} \quad (31)$$

$$\mathscr{S} \triangleq \mathscr{R}^{-1} - \mathscr{R}^{-1}\mathscr{B}_l^T[\Phi - \Phi\mathscr{C}_r^T(\mathscr{C}_r\Phi\mathscr{C}_r^T)^{-1}\mathscr{C}_r\Phi]\mathscr{B}_l\mathscr{R}^{-1} \quad (32)$$

where $\mathscr{L}$ is an arbitrary matrix such that $\|\mathscr{L}\| < 1$ and $\mathscr{R}$ is an arbitrary positive definite matrix such that

$$\Phi \triangleq (\mathscr{B}_l\mathscr{R}^{-1}\mathscr{B}_l^T - \mathscr{Q})^{-1} > 0 \quad (33)$$

Now, since matrix $\mathscr{B} = \begin{pmatrix} -I \\ 0 \end{pmatrix}$, we obtain:

$$\mathscr{B}^{\perp} = \begin{pmatrix} 0 & -I \end{pmatrix}, \mathscr{B}_l = I, \mathscr{B}_r = \begin{pmatrix} -I \\ 0 \end{pmatrix} \text{ and } \mathscr{B}_r^{+} = \begin{pmatrix} -I & 0 \end{pmatrix}. \tag{34}$$

Inserting these matrices into (31)-(33), we obtain the equations (25)-(27) and the inequality (22), which completes the proof.

■

Finally, the H_∞ dynamic observer design procedure can be summarized as follows:

1. Choose the matrix E according to condition (6);
2. Compute T and K from (12);
3. Compute matrices α_1, α_2, α_3, α_4, β_1, β_2 and β_3 from (9) and (15);
4. Compute matrices $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{C}$ and $\mathbb{D}$ given in remark 2;
5. Compute $\mathbb{Z}$ from the theorem 2;
6. Deduce all the matrices N, J, H, M, P, Q, G, R and S from equations (3b), (8), (14) and (16).

V. Numerical Example

The following numerical example illustrates the proposed observer design procedure . Let us consider a LTI system of the form (1) that:

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.2 \\ 0.5 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1.5 & 0 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0.1 \end{bmatrix}.$$

The initial conditions are $x_1(0) = 1$, $x_2(0) = 2$ and $x_3(0) = 3$.

By applying our design approach to design a DO, we obtain the following results, by choosing the matrix $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$: $\gamma = 0.5$,

$$X = 10^4 \begin{bmatrix} 1.0155 & -0.4445 & -0.4815 & 0.0001 & 0.0001 & 0.0001 \\ -0.4445 & 0.2482 & 0.1736 & 0.0001 & 0.0001 & 0.0001 \\ -0.4815 & 0.1736 & 0.2573 & 0.0001 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.0001 & 0.4826 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.4826 & 0.0001 \\ 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.4826 \end{bmatrix},$$

and $$\mathbb{Z} = \begin{bmatrix} -87.2562 & 43.7778 & 43.8220 & -29.1456 & 0.5363 & 0.5363 & 0.5363 \\ -82.6827 & 41.4775 & 41.5175 & -27.6154 & 0.5254 & 0.5254 & 0.5254 \\ -107.5952 & 53.9871 & 54.0437 & -35.9417 & 0.7135 & 0.7135 & 0.7135 \\ -0.0100 & 0.0049 & 0.0049 & -0.0033 & -0.5003 & -0.0003 & -0.0003 \\ -0.0100 & 0.0049 & 0.0049 & -0.0033 & -0.0003 & -0.5003 & -0.0003 \\ -0.0100 & 0.0049 & 0.0049 & -0.0033 & -0.0003 & -0.0003 & -0.5003 \end{bmatrix}.$$

Finally, the obtained DO is then given by:

$$\begin{aligned} \dot{z}_g &= \begin{bmatrix} 86.9615 & -42.4807 & -44.4807 \\ 82.4805 & -42.2403 & -41.2403 \\ 107.6003 & -49.8771 & -55.8002 \end{bmatrix} z_g + \begin{bmatrix} 10.4576 \\ 8.4595 \\ 13.7026 \end{bmatrix} y \\ &+ \begin{bmatrix} 2 \\ 1 \\ 3.6 \end{bmatrix} u + \begin{bmatrix} 0.5363 & 0.5363 & 0.5363 \\ 0.5254 & 0.5254 & 0.5254 \\ 0.7135 & 0.7135 & 0.7135 \end{bmatrix} v_g \\ \dot{v}_g &= \begin{bmatrix} 0.0099 & -0.0050 & -0.0050 \\ 0.0099 & -0.0050 & -0.0050 \\ 0.0099 & -0.0050 & -0.0050 \end{bmatrix} z_g + \begin{bmatrix} 0.0010 \\ 0.0010 \\ 0.0010 \end{bmatrix} y \\ &+ \begin{bmatrix} -0.5003 & -0.0003 & -0.0003 \\ -0.0003 & -0.5003 & -0.0003 \\ -0.0003 & -0.0003 & -0.5003 \end{bmatrix} v_g \\ \hat{x}_g &= \begin{bmatrix} 0.0690 & -0.5345 & 0.4655 \\ -0.1379 & 0.0690 & 0.0690 \\ 0.3103 & 0.8448 & -0.1552 \end{bmatrix} z_g + \begin{bmatrix} -0.0955 \\ 0.6525 \\ 0.0318 \end{bmatrix} y \end{aligned}$$

We also design a PO and a PIO to provide comparisons with our DO. These observers are designed under the same condition with $\gamma = 0.5$. The obtained PO is given by:

$$\dot{\hat{x}}_p = A\hat{x}_p + Bu + \begin{bmatrix} 1.4976 \\ 0.6177 \\ 4.0444 \end{bmatrix} (y - C\hat{x}_p)$$

The obtained PIO is given by:

$$\begin{aligned} \dot{\hat{x}}_{pi} &= \begin{bmatrix} -2 & -1.1716 & 1 \\ 0 & -4.0686 & 3 \\ 0 & 1.0865 & -1 \end{bmatrix} \hat{x}_{pi} + Bu \\ &+ \begin{bmatrix} 2.1144 \\ 1.3791 \\ -0.7243 \end{bmatrix} y + \begin{bmatrix} 2.6206 \\ 21.4783 \\ -2.3894 \end{bmatrix} v_{pi} \\ \dot{v}_{pi} &= -C\hat{x}_{pi} + y \end{aligned}$$

In the simulation, we add an uncertain term ΔA in the system $(A + \Delta A)$, where $\Delta A = 0.1\Delta \sin 10\pi t$ and $\Delta = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 1 & 0.1 & 0.5 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}$.

The simulation results are shown in the following figures. Figure 1 shows the disturbance and the uncertainty.

Figures 2, 4 and 6 present the state estimations, obtained from DO, PO and PIO, respectively.

The solid line represents the original system state. The dashed line represents the state estimation obtained from the DO. The dot-dashed line represents the state estimation obtained from the PO. The dotted line represents the state estimation obtained for the PIO.

Figures 3, 5 and 7 show the estimation errors ($e = \hat{x} - x$) of DO, PO and PIO, respectively.

From these simulations, one can see that our observer presents the best performances, compared with the PO and PIO.

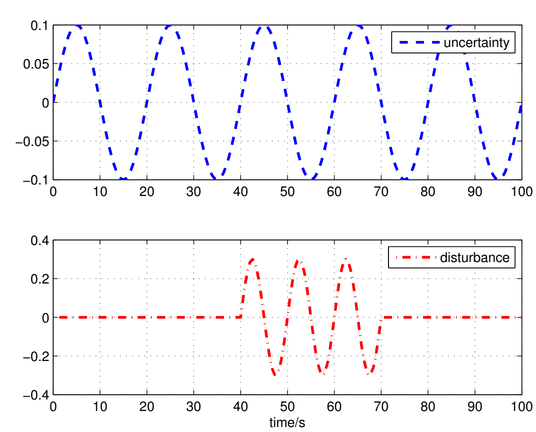

Fig. 1. uncertainty and disturbance (dashed line: uncertainty; dot-dashed line: disturbance)

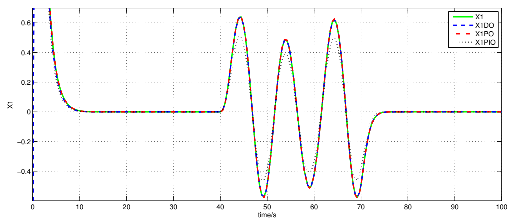

Fig. 2. estimation x1 (solid line: original state; dashed line: DO; dot-dashed line: PO; dotted line: PIO)

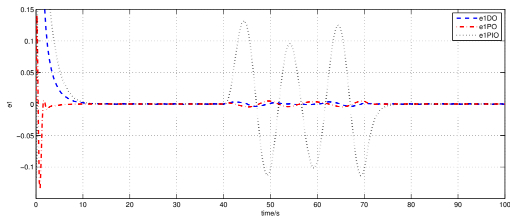

Fig. 3. estimation error e1(dashed line: DO; dot-dashed line: PO; dotted line: PIO)

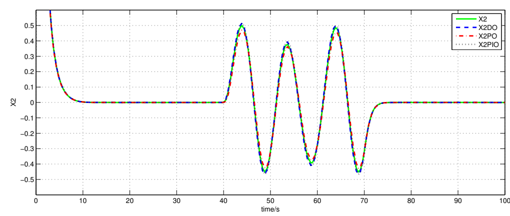

Fig. 4. estimation x2 (solid line: original state; dashed line: DO; dot-dashed line: PO; dotted line: PIO)

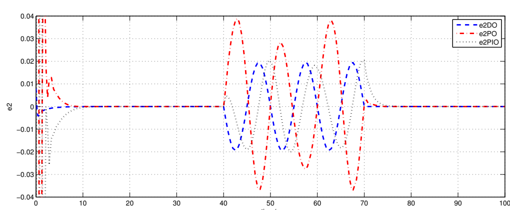

Fig. 5. estimation error e2 (dashed line: DO; dot-dashed line: PO; dotted line: PIO)

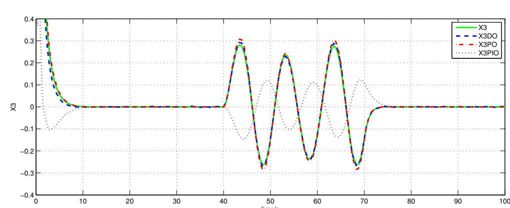

Fig. 6. estimation x3 (solid line: original state; dashed line: DO; dot-dashed line: PO; dotted line: PIO)

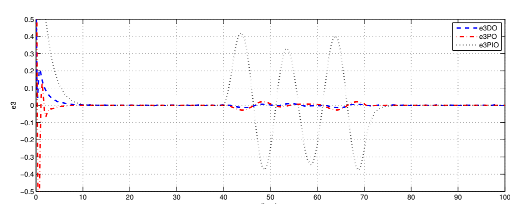

Fig. 7. estimation error e3 (dashed line: DO; dot-dashed line: PO; dotted line: PIO)

VI. Conclusion

In this paper, an H_∞ dynamic observer is proposed for LTI systems subject to disturbances. The proposed observer has a more generalized form, of which the popularly and widely used PO and PIO can be considered as particular cases. Furthermore, it has a simpler structure than standard DO. The design of the observer is derived from the solution of new LMIs. A numerical example is given to illustrate the observer design procedure and the better performances, compared with PO and PIO.

References

[1] J. O'Reilly, *Observers for linear systems*. Access Online via Elsevier, 1983, vol. 170.

[2] D. Luenberger, "An introduction to observers," *Automatic Control, IEEE Transactions on*, vol. 16, no. 6, pp. 596–602, 1971.

[3] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140 – 147, 2011.

[4] K.-S. Kim and K.-H. Rew, "Reduced order disturbance observer for discrete-time linear systems," *Automatica*, vol. 49, no. 4, pp. 968 – 975, 2013.

[5] Y. Wang, M. Sun, Z. Wang, Z. Liu, and Z. Chen, "A novel disturbance-observer based friction compensation scheme for ball and plate system," *ISA Transactions*, vol. 53, no. 2, pp. 671 – 678, 2014.

[6] J.-D. Chen, C.-D. Yang, C.-H. Lien, and J.-H. Horng, "New delay-dependent non-fragile H_∞ observer-based control for continuous time-delay systems," *Inf. Sci.*, vol. 178, no. 24, pp. 4699–4706, Dec. 2008.

[7] Y. Zhang and C. Liu, "Observer-based finite-time control of discrete-time markovian jump systems," *Applied Mathematical Modelling*, vol. 37, no. 6, pp. 3748 – 3760, 2013.

[8] M. H. Asemani and V. J. Majd, "A robust observer-based controller design for uncertain ts fuzzy systems with unknown premise variables via LMI," *Fuzzy Sets and Systems*, vol. 212, no. 0, pp. 21 – 40, 2013, theme: Control Engineering.

[9] B. Wojciechowski, "Analysis and synthesis of proportional-integral observers for single-input-single-output time-invariant continuous systems," Ph.D. dissertation, Gliwice, Poland, 1978.

[10] Z. Gao, T. Breikin, and H. Wang, "Discrete-time proportional and integral observer and observer-based controller for systems with both unknown input and output disturbances," *Optimal Control Applications and Methods*, vol. 29, no. 3, pp. 171–189, 2008.

[11] L. Bodizs, B. Srinivasan, and D. Bonvin, "On the Design of Integral Observers for Unbiased Output Estimation in the Presence of Uncertainty," *Journal of Process Control*, vol. 21, no. 3, pp. 379–390, 2011.

[12] C. Hua and X. Guan, "Synchronization of chaotic systems based on PI observer design," *Physics Letters A*, vol. 334, no. 56, pp. 382 – 389, 2005.

[13] J.-K. Park, D.-R. Shin, and T. M. Chung, "Dynamic observers for linear time-invariant systems," *Automatica*, vol. 38, no. 6, pp. 1083–1087, 2002.

[14] X.-J. Li and G.-H. Yang, "Dynamic observer-based robust control and fault detection for linear systems," *Control Theory Applications, IET*, vol. 6, no. 17, pp. 2657–2666, Nov 2012.

[15] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A unified algebraic approach to linear control design*. CRC Press, 1998.