VOLUME AND NON-EXISTENCE OF COMPACT CLIFFORD–KLEIN FORMS

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Abstract. This article studies the volume of compact quotients of reductive homogeneous spaces. Let G/H be a reductive homogeneous space and Γ a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. We prove that the volume of $\Gamma\backslash G/H$ is the integral, over a certain homology class of Γ, of a G-invariant form on G/K (where K is a maximal compact subgroup of G).

As a corollary, we obtain that, in all known examples of compact reductive Clifford–Klein forms that admit deformations, the volume is constant under these deformations.

We also derive a new obstruction to the existence of compact Clifford–Klein forms for certain homogeneous spaces. In particular, we obtain that $\mathrm{SO}(p, q+1)/SO(p,q)$ does not admit compact quotients when p is odd, and that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ does not admit compact quotients when m is even.

Introduction

The problem of understanding compact quotients of homogeneous spaces has a long history that can be traced back to the "Erlangen program" of Felix Klein [13]. In the second half of the last century, several existence and rigidity theorems where proved (by Borel [5], Mostow [22] and Margulis [20] among others), leading to a rather good understanding of quotients of *Riemannian* homogeneous spaces. Comparatively, little is known about the non-Riemannian case, and in particular about quotients of pseudo-Riemannian homogeneous spaces.

In this paper we focus on reductive homogeneous spaces, i.e. quotients of a semi-simple Lie group G by a closed reductive subgroup H. The G-homogeneous space $X = G/H$ carries a natural G-invariant pseudo-Riemannian metric (coming from the Killing metric of G) and therefore a G-invariant volume form vol_X. Interesting examples include the homogeneous spaces

$$\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p,q)$$

whose pseudo-Riemannian metric has signature (p,q) and constant negative sectional curvature.

A quotient of X by a discrete subgroup Γ of G acting properly discontinuously and cocompactly is called a *compact Clifford–Klein form* of X. The study of compact Clifford–Klein forms of reductive homogeneous spaces was initiated by Kulkarni and Kobayashi in the 80's. Here, we are interested in the following questions: do Clifford–Klein forms exist? If so, are they rigid? If not, is their volume rigid?

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Volume of Compact Clifford–Klein forms. The question of the volume rigidity was raised in [2] in the specific case of *anti-de Sitter* 3*-manifolds*, which are locally modelled on $\mathrm{SO}_0(2,1)$ with its Killing metric. In that case, compact Clifford–Klein forms are known to exist and to have a rich deformation space (see [23], [12] or [25]). In a recent paper, the author studies more generally the volume of compact quotients of $\mathrm{SO}_0(n,1)$ by a discrete subgroup of $\mathrm{SO}_0(n,1)\times \mathrm{SO}_0(n,1)$ [26]. By the work of Kassel [11], such quotients have the form

$$j\times\rho(\Gamma)\backslash \mathrm{SO}_0(n,1)\ ,$$

where Γ is a cocompact lattice in $\mathrm{SO}_0(n,1)$, j the inclusion and ρ another representation of Γ into $\mathrm{SO}_0(n,1)$. Moreover, Guéritaud and Kassel proved in [9] that these quotients have the structure of a $\mathrm{SO}(n)$ bundle over $\Gamma\backslash\mathbb{H}^n$ (see Theorem 6.1). In [26], we proved the following formula:

$$\mathbf{Vol}\left(j\times\rho(\Gamma)\backslash \mathrm{SO}_0(n,1)\right) = \mathbf{Vol}(\mathrm{SO}(n))\int_{j(\Gamma)\backslash\mathbb{H}^n} vol_{\mathbb{H}^n} + (-1)^n f^* vol_{\mathbb{H}^n}\ ,$$

where f is any smooth (j,ρ)-equivariant map. It follows that the volume of these compact Clifford–Klein forms is rigid.

The primary purpose of this paper is to extend this result to compact Clifford–Klein forms of reductive homogeneous spaces. The main issue is that we don't have a structure theorem similar to the one of Guéritaud–Kassel in general. However, denoting L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L, we see that $\Gamma\backslash G/H$ is homotopy equivalent to $\Gamma\backslash G/L$, which is a K/L bundle over $\Gamma\backslash G/K$. Let q be the dimension of K/L and $p+q$ the dimension of G/H. Using spectral sequences, one can deduce that Γ has homological dimension p and that $\mathrm{H}_p(\Gamma,\mathbb{Z})$ is generated by an element $[\Gamma]$ (Proposition 2.1). Since G/K is contractible, $\mathrm{H}_p(\Gamma,\mathbb{Z})$ is naturally isomorphic to $\mathrm{H}_p(\Gamma\backslash G/K,\mathbb{Z})$ and $[\Gamma]$ can thus be seen as a singular p-cycle in $\Gamma\backslash G/K$. We will prove the following:

Theorem 1.
Let G/H be a reductive homogeneous space, with G and H connected.Let L be a maximal compact subgroup of H and K a maximal compact subgroup of G containing L. Set $p=\dim G/H-\dim K/L$. Then there exists a G-invariant p-form ω_H on G/K such that, for any torsion-free discrete subgroup $\Gamma\subset G$ acting properly discontinuously and cocompactly on G/H, we have

$$\mathbf{Vol}\left(\Gamma\backslash G/H\right) = \left|\int_{[\Gamma]}\omega_H\right|\ .$$

Example 0.1. If $G=\mathrm{SO}_0(n,1)\times\mathrm{SO}_0(n,1)$ and $H=\mathrm{SO}_0(n,1)$ embedded diagonally, then the symmetric space G/K is $\mathbb{H}^n\times\mathbb{H}^n$. The main Theorem in [26] asserts that, in this case,

$$\omega_H = \mathbf{Vol}(\mathrm{SO}(n))\left(vol_1 + (-1)^n vol_2\right)\ ,$$

where vol_1 and vol_2 denote respectively the volume forms on the first and second copy of $\mathbb{H}^n$.

Rigidity of the volume. The construction of the form ω_H is rather explicit, though hard to compute in general. For certain Lie groups G, a theorem of Cartan and Borel guaranties that every G-invariant form on G/K is essentially a characteristic class (Theorem 4.3). This implies that the volume is locally rigid (i.e. invariant by small deformations of Γ).

Corollary 2.
Assume that G and K have the same complex rank. Then the volume of $\Gamma\backslash G/H$ is locally rigid. In particular, the volume is locally rigid in the following cases:

(*i*) $G = \mathrm{SO}(p, q+1)$, $H = \mathrm{SO}(p, q)$, *p even,*
(*ii*) $G = \mathrm{SO}(2n, 2)$, $H = \mathrm{U}(n, 1)$,
(*iii*) $G = \mathrm{SU}(n, 1) \times \mathrm{SU}(n, 1)$, $H = \mathrm{SU}(n, 1)$ *(embedded diagonally).*

Together with the case $G = \mathrm{SO}(n, 1) \times \mathrm{SO}(n, 1)$ and $H = \mathrm{SO}(n, 1)$ embedded diagonally (which was covered in [26]), this proves the volume rigidity of all known examples of compact reductive Clifford–Klein forms admitting "interesting" deformations (see section 4 for details).

A new obstruction to the existence of compact quotients. Contrary to the Riemannian setting, compact pseudo-Riemannian Clifford–Klein forms do not always exist, and it is a long standing problem to characterize which reductive homogeneous spaces admit compact quotients. Let us recall two famous conjectures in this field.

Kobayashi's Space-form Conjecture. *The homogeneous space $\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p, q)$ ($p, q > 0$) admits a compact Clifford–Klein form if and only if one of the following holds:*

- *p is even and $q = 1$,*
- *p is a multiple of 4 and $q = 3$,*
- *$p = 8$ and $q = 7$.*

Conjecture (See for instance [12], Section 0.1.5)**.** *The homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ ($1 < m < n$) never admits a compact Clifford–Klein form.*

It is known that $\mathbb{H}^{p,q}$ does not admit a compact Clifford–Klein form when $p \leq q$ [28] and when p and q are odd [17]. As for the second conjecture, it is known for $m \leq n-3$ ([18], [19]) and for $n = m+1$ odd [3] (see Section 5 for details).

In section 5, we prove that, in some cases, the form ω_H vanishes. This provides a rather powerful obstruction to the existence of compact Clifford–Klein forms. In particular, we obtain new results toward the conjectures above:

Theorem 3.
- *The homogeneous space $\mathbb{H}^{p,q}$ ($p, q \geq 1$) does not admit a compact Clifford–Klein form when p is odd.*
- *The homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ ($1 < m < n$) does not admit a compact Clifford–Klein form when m is even. In particular, $\mathrm{SL}(2n+2, \mathbb{R})/\mathrm{SL}(2n, \mathbb{R})$ does not admit a compact Clifford–Klein form.*

Theorem 5.1 gives a longer list of homogeneous spaces where our obstruction applies.

Organization of the paper. In Section 1 we give some precisions about the setting of this work. In Section 2, we explain why compact Clifford–Klein forms can always be seen as fibrations over an Eilenberg–McLane space "at the homology level". In Section 3 we construct the form ω_H as the contraction of a $p+q$-form on G/L along the fibers gK/L and we prove Theorem 1. In Section 4, we state a theorem of Cartan and Borel and explain why and when it allows us to prove the volume rigidity. In Section 5 we prove that the form ω_H vanishes for certain homogeneous spaces G/H, and deduce the non-existence of compact Clifford–Klein forms for these homogeneous spaces. Finally in Section 6, we prove that the vanishing of the form ω_H is also an obstruction to the existence of certain local foliations of G/H by compact homogeneous subspaces, and we formulate a conjecture about the geometry of compact reductive Clifford–Klein forms.

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1. Reductive Clifford–Klein forms and their volume

In all this paper, we fix a connected semi-simple Lie group G and a connected reductive subgroup H. The quotient $X = G/H$ is called a *reductive homogeneous space*. We also fix L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L. Finally we denote respectively by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{l}$ the Lie algebras of G, H, K, L.

The homogeneous space X is always *pseudo-Riemannian*, meaning that the action of G on X preserves a pseudo-Riemannian metric. Indeed, the restriction of the Killing form κ_G to $\mathfrak{h}$ in non-degenerate and κ_G thus splits as an orthogonal sum of two bilinear forms κ_H and κ_X, respectively on $\mathfrak{h}$ and $\mathfrak{h}^\perp$. Since $\mathfrak{h}^\perp$ naturally identifies with the tangent space to X at the point $x_0 = H$ and since κ_X is invariant by the adjoint action of H, it extends to a G-invariant pseudo-Riemannian metric on X that we still denote κ_X. Moreover, since H is connected, the space X is orientable and the metric κ_X induces a G-invariant volume form vol_X on X.

A *compact Clifford–Klein form* of X is a quotient of X by a discrete subgroup Γ of G acting properly discontinuously and cocompactly. The volume form vol_X then descends to a volume form on $\Gamma\backslash X$ (that we still denote by vol_X) and we can define the *volume* of $\Gamma\backslash X$ by

$$\mathbf{Vol}\,(\Gamma\backslash X) = \left| \int_{\Gamma\backslash X} vol_X \right| .$$

We say that the Clifford–Klein form $\Gamma\backslash X$ is *(locally) rigid* if, for every representation $\rho : \Gamma \to G$ in some neigbourhood of the inclusion, if $\rho(\Gamma)$ acts properly discontinuously on X, then ρ is conjugate to the inclusion. (Note that this is a priori weaker than the local rigidity of Γ.)

Finally, we say that the volume of the Clifford–Klein form $\Gamma\backslash X$ is *(locally) rigid* if, for every representation $\rho : \Gamma \to G$ in some neigbourhood of the inclusion, if $\rho(\Gamma)$ acts properly discontinuously on X, then

$$\mathbf{Vol}\,(\rho(\Gamma)\backslash X) = \mathbf{Vol}\,(\Gamma\backslash X)\ .$$

It is easy to see that, if a compact Clifford–Klein form is rigid, then its volume is rigid.

2. Clifford–Klein forms are fibrations at the homology level

Let us fix Γ a torsion-free discrete subgroup of G acting properly discontinuously and cocompactly on G/H, and denote by M the Clifford–Klein form

$$M = \Gamma\backslash G/H\ .$$

We introduce two auxiliary Clifford–Klein forms:

$$E = \Gamma\backslash G/L$$

and

$$B = \Gamma\backslash G/K\ .$$

We remark the following facts:

- (i) E fibers over M with fibers isomorphic to H/L. Since H/L is contractible, it follows that this fibration is a homotopy equivalence.
- (ii) E is also a fibration over B with fibers isomorphic to K/L.
- (iii) Since G/K is contractible, B is a *classifying* space for Γ.

From the first point, we deduce in particular that the homology of M is the same as the homology of E. The third point implies that the homology of B is the homology of Γ. Finally, (ii) implies that the homologies of B, E and K/L are linked (in a complicated way) by *Serre's spectral sequence.* We will use the following consequence:

Proposition 2.1 (See [17] and [14])**.** *The group* Γ *has homological dimension* p *and*

$$\mathrm{H}_p(\Gamma,\mathbb{Z}) \simeq \mathrm{H}_{p+q}(M,\mathbb{Z}) = \mathbb{Z}\ .$$

Proof. Let p', q' and r' denote respectively the homological dimensions of B, K/L and E. By Serre's theorem, the spectral sequence given by

$$\mathbf{E}^2_{k,l} = \mathrm{H}_k\left(B, \mathrm{H}_l(K/L,\mathbb{Z})\right)$$

converges to $\mathrm{H}_{k+l}(E,\mathbb{Z})$. A classical consequence is that

$$r' = p' + q'$$

and that

$$\mathrm{H}_{p'+q'}(E,\mathbb{Z}) \simeq \mathrm{H}_{p'}\left(B, \mathrm{H}_{q'}(K/L,\mathbb{Z})\right)\ . \tag{1}$$

Since K/L is a closed oriented manifold of dimension q, we have $q' = q$ and $\mathrm{H}_q(K/L,\mathbb{Z}) \simeq \mathbb{Z}$. Since E is homotopy equivalent to M which is a closed oriented manifold of dimension $p+q$, we also have $r' = p + q$. Therefore $p' = p$.

Moreover, since L is connected, the action of Γ on G/L preserves an orientation of the fibers of the fibration

$$G/L \to G/K$$

and Γ thus acts trivially on $\mathrm{H}_q(K/L,\mathbb{Z})$. From (1), we obtain

$$\mathbb{Z} \simeq \mathrm{H}_{p+q}(E,\mathbb{Z}) \simeq \mathrm{H}_p(B,\mathbb{Z}) \ .$$

The proposition follows since E is homotopy equivalent to M and B is a classifying space for Γ. □

To go further, we need to explictly describe the isomorphism $\mathrm{H}_{p+q}(E,\mathbb{Z}) \simeq \mathrm{H}_p(B,\mathbb{Z})$. Let $[\Gamma]$ denote a generator of $\mathrm{H}_p(B,\mathbb{Z}) \simeq \mathrm{H}_p(\Gamma,\mathbb{Z})$, and π the fibration of E over B. The general idea is that, if one thinks of $[\Gamma]$ as a closed submanifold of B of dimension p, then the isomorphism $\mathrm{H}_p(B,\mathbb{Z}) \to \mathrm{H}_{p+q}(E,\mathbb{Z})$ maps $[\Gamma]$ to $\pi^{-1}([\Gamma])$, which is a submanifold of E of dimension $p+q$.

However, we don't know that $[\Gamma]$ is realized by a submanifold. One way to overcome this difficulty would be to work with simplicial complexes. However, since we will use differential geometry later, it is more convenient to use Thom's realization theorem:

Theorem 2.2 (Thom, [27])**.** *There exists a closed oriented p-manifold B', a smooth map $\varphi : B' \to B$ and an integer k such that*

$$k[\Gamma] = \varphi_*[B'] \ .$$

Let $\pi' : E' \to B'$ be the pull-back of the fibration $\pi : E \to B$ by φ and $\hat{\varphi} : E' \to E$ the lift of φ. The total space of the fibration E' is a closed orientable $(p+q)$-manifold.

Proposition 2.3. *Let $[E]$ denote a generator of $\mathrm{H}_{p+q}(E)$. Then, up to switching the orientation of E', we have*

$$k[E] = \hat{\varphi}_*[E'] \ .$$

Proof. The Serre spectral sequence shows that the fibrations π and π' respectively induce isomorphisms

$$\pi^* : \mathrm{H}_p(B) \to \mathrm{H}_{p+q}(E)$$

and

$$\pi'^* : \mathrm{H}_p(B') \to \mathrm{H}_{p+q}(E') \ .$$

By naturality of the Serre spectral sequence, we have the following commuting diagram:

$$\begin{array}{ccc} \mathrm{H}_p(B') & \xrightarrow{\varphi^*} & \mathrm{H}_p(B) \\ \downarrow{\scriptstyle \pi'^*} & & \downarrow{\scriptstyle \pi^*} \\ \mathrm{H}_{p+q}(E') & \xrightarrow{\hat{\varphi}^*} & \mathrm{H}_{p+q}(E) \ . \end{array}$$

Since $\varphi_*[B'] = k[\Gamma]$, we thus have

$$\hat{\varphi}_*[E'] = k[E] \ .$$

□

To summarize, we proved that the rational homology of E in dimension $p+q$ is represented by a cycle that "fibers" over a p-cycle of B.

3. Fiberwise integration of the volume form

Let E', B', φ, $\hat{\varphi}$ and π, π' be as in the previous section. Denote by ψ the projection from E to M. Recall that the volume form vol_X on $X = G/H$ induces a volume form on M that we still denote vol_X.

Proposition 3.1. *We have*

$$\mathbf{Vol}(M) = \frac{1}{k}\left|\int_{E'} \hat{\varphi}^*\psi^* vol_X\right| .$$

Proof. Since ψ is a homotopy equivalence, we have

$$\mathbf{Vol}(M) = \left|\int_M vol_X\right| = \left|\int_{[E]} \psi^* vol_X\right| .$$

Since $k[E] = \hat{\varphi}_*[E']$, we have

$$\left|\int_{[E]} \psi^* vol\right| = \frac{1}{k}\left|\int_{E'} \hat{\varphi}^*\psi^* vol\right| .$$

□

Now, since E' fibers over B', we can "average" the form $\hat{\varphi}^*\psi^* vol$ along the fibers to obtain a p-form on B' whose integral will give the volume of M. Let us give more precisions.

Let x be a point in G/K and let F denote the fiber $\pi^{-1}(x)$. Choose some volume form vol_F on F and let ξ denote the section of $\Lambda^q TF$ such that $vol_F(\xi) = 1$. At every point y of F, the p-form obtained by contracting $\psi^* vol$ with ξ has T_yF in its kernel and therefore induces a p-form ω_y on T_xG/K.

Definition 3.2. The form ω_H on G/K is defined at the point x by

$$(\omega_H)_x = \int_F \omega_y \,\mathrm{d}vol_F(y) .$$

On easily checks that this definition does not depend on the choice of vol_F. Since the maps ψ and π are equivariant with respect to the actions of G, the volume forms $\psi^* vol_X$ and ω_H are G-invariant. By a slight abuse of notation, we still denote by ω_H the induced p-form on $B = \Gamma\backslash G/H$.

Proposition 3.3. *We have*

$$\int_{E'} \hat{\varphi}^*\psi^* vol_X = \int_{B'} \varphi^*\omega_H . \tag{2}$$

Proof. This is presumably a classical result of differential geometry. Let U be an open set in B' over which the bundle $\pi' : E' \to B'$ is trivial. Let us identify ${\pi'}^{-1}(U)$ with $K/L \times U$. We can locally write the form $\hat{\varphi}^*\psi^* vol$ as $f(y,x) vol_F \wedge vol_U$ for some function f on $K/L \times U$ and some volume forms vol_F and vol_U on K/L and U respectively. Let ξ be the section of $\Lambda^q TK/L$ such that $vol_F(\xi) = 1$. The contraction of $\hat{\varphi}^*\psi^* vol$ with ξ is thus $f(y,x) vol_U$. By construction, we thus have

$$(\varphi^*\omega_H)_x = \left(\int_F f(x,y)\mathrm{d}vol_F(y)\right) vol_U ,$$

and therefore

$$\begin{aligned}\int_{\pi'^{-1}(U)} \hat{\varphi}^*\psi^* vol_X &= \int_{F\times U} f(y,x)\mathrm{d}vol_F(y)\mathrm{d}vol_U(x)\\ &= \int_U \varphi^*\omega_H\ .\end{aligned}$$

□

We can now conclude the proof of Theorem 1. Indeed, we have

$$\begin{aligned}\mathbf{Vol}(M) &= \frac{1}{k}\left|\int_{E'} \hat{\varphi}^*\psi^* vol\right|\\ &= \frac{1}{k}\left|\int_{B'} \varphi^*\omega_H\right|\\ &= \left|\int_{[\Gamma]} \omega_H\right|\ .\end{aligned}$$

4. Characteristic classes an rigidity of the volume

In this section, we deduce from Theorem 1 that for certain Lie groups G the volume of compact Clifford–Klein forms $\Gamma\backslash G/H$ is always locally rigid.

Recall that the volume of $\Gamma\backslash G/H$ is trivially rigid if Γ itself is rigid in G. In particular, the rigidity theorems of Mostow [22] and Margulis [20] imply that the volume of a compact Riemannian Clifford–Klein form is always rigid when G has no factor isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. When G is $\mathrm{PSL}(2,\mathbb{R})$, Riemannian Clifford–Klein forms are either hyperbolic surfaces or their unit tangent bundle, and the volume rigidity follows from the Gauss–Bonnet formula.

There is no analogous rigidity theorem for non-Riemannian Clifford–Klein forms, and a few such non-rigid forms are known:

- Let L be $\mathrm{SO}_0(n,1)$ or $\mathrm{SU}(n,1)$. Let G be $L\times L$ and H the diagonal embedding of L into $L\times L$, so that G/H identifies with the space L with the action of $L\times L$ by left and right multiplication. If Γ is a uniform lattice in L and $u:\Gamma\to L$ a representation sufficiently close to the trivial representation (a more precise criterion is given in [8]), then the group
$$\Gamma_u = \{(\gamma, u(\gamma)), \gamma\in\Gamma\}$$
acts properly discontinuously on $L = G/H$ [16]. When $\mathrm{H}_1(\Gamma,\mathbb{Z})$ does not vanish, such non-trivial representations u exist and provide deformations of the group $\Gamma_0 = \{(\gamma,\mathbf{1}),\gamma\in\Gamma\}$.

- Let G be $\mathrm{SO}_0(2n,2)$ and H be $\mathrm{U}(n,1)$. Let Γ be a cocompact lattice of $\mathrm{SO}_0(2n,1)\subset\mathrm{SO}_0(2n,2)$. Then Γ acts properly discontinuously and cocompactly on G/H [17]. The group Γ can sometimes be deformed into a Zariski dense subgroup of $\mathrm{SO}_0(2n,2)$ [10]. By results of Barbot [1] and Guéritaud–Guichard–Kassel–Wienhard [8], for every representation $\rho:\Gamma\to\mathrm{SO}_0(2n,2)$ in the connected component of the inclusion, $\rho(\Gamma)$ acts properly discontinuously on G/H.

These give, to the best of our knowledge, the only examples of non Riemannian reductive Clifford–Klein forms $\Gamma\backslash G/H$ such that Γ is Zariski dense in G.

Remark 4.1. The dual case where H is $\mathrm{SO}_0(2n,1)$ and Γ a lattice in $\mathrm{U}(n,1)$ is essentially rigid. A theorem of Ragunathan implies that Γ can only be deformed inside $\mathrm{U}(n,1)$ by "adding" a representation of Γ into the center of $\mathrm{U}(n,1)$ (see [12, Section 6.1.2]). This implies the volume rigidity.

Here we prove the volume rigidity in these examples. The proof will follow from Theorem 1, together with the fact that, for certain Lie groups G, every G-invariant form on G/K is essentially a characteristic class.

Let G be a semi-simple Lie group and K a maximal compact subgroup. We see the group G as the total space of a principal K-bundle over G/K. The distribution orthogonal to the fibers (with respect to the Killing metric) provides a G-invariant principal connexion ∇ on this bundle. The curvature F_∇ of ∇ is a G-invariant 2-form on G/K with values in $\mathfrak{k}$. We say that a G-invariant p-form ω on G/K is an *equivariant Chern–Weil class* if there exists a K-invariant polynomial P on $\mathfrak{k}$ such that

$$\omega = P(F_\nabla)\ .$$

Remark 4.2. This terminology is justified by the fact that the algebra of equivariant Chern–Weil classes of G/K is isomorphic to the algebra of equivariant Chern–Weil classes of the compact dual of G/K, which is isomorphic to the algebra of its Chern–Weil classes.

The study of the cohomology of symmetric spaces, initiated by Cartan in [6], was completed by Borel in his thesis [4]. They obtain in particular le following result:

Theorem 4.3 (Cartan, Borel)**.** *Let G be a Semi-simple Lie group and K a maximal compact subgroup. Assume that the complexifications of the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ have the same rank. Then the exterior algebra of G-invariant forms on G/K is exactly the algebra of equivariant Chern–Weil classes.*

We say that a semi-simple Lie group G satisfies the Borel–Cartan property if the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ have the same rank.

Corollary 4.
Let G be a semi-simple Lie group, H a closed reductive subgroup and Γ a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. If G satisfies the Borel–Cartan property, then the volume of $\Gamma\backslash G/H$ is rigid.

The list of simple Lie groups G satisfying the Borel–Cartan property is the following:

- $\mathrm{SU}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ and more generally every Lie group whose symmetric space is Hermitian,
- $\mathrm{SO}_0(p,q)$ for p or q even,
- $\mathrm{Sp}(p,q)$,
- the exceptional Lie groups $\mathrm{E}_{6(2)}$, $\mathrm{E}_{7(7)}$, $\mathrm{E}_{7(-5)}$, $\mathrm{E}_{8(8)}$, $\mathrm{E}_{8(-24)}$, $\mathrm{F}_{4(4)}$, $\mathrm{F}_{4(-20)}$ and $\mathrm{G}_{2(2)}$.

Besides, the product of two Lie groups satisfying the Borel–Cartan property still satifies the Borel–Cartan property. In particular, the groups $\mathrm{SO}_0(2n, 2)$, $\mathrm{SO}_0(2n, 1) \times \mathrm{SO}_0(2n, 1)$ and $\mathrm{SU}(n, 1) \times \mathrm{SU}(n, 1)$ satisfy the Borel–Cartan property. Therefore, the volume of compact Clifford–Klein forms of $\mathrm{SO}_0(2n, 2)/\mathrm{U}(n, 1)$, $\mathrm{SO}_0(2n, 1) \times \mathrm{SO}_0(2n, 1)/\mathrm{SO}_0(2n, 1)$ and $\mathrm{SU}(n, 1) \times \mathrm{SU}(n, 1)/\mathrm{SU}(n, 1)$ is rigid. The volume ridity of compact Clifford–Klein forms of $\mathrm{SO}_0(2n + 1, 1) \times \mathrm{SO}_0(2n + 1, 1)/\mathrm{SO}_0(2n + 1, 1)$ is proved in [26]. To our knowledge, this covers all known examples of compact non-rigid reductive pseudo-Riemannian Clifford–Klein forms that admit Zariski dense deformations.

Proof of Corollary 4. It is a rather classical fact that, if a G-invariant p-form ω on a symmetric space G/K is what we called an equivariant Chern–Weil class, then for any continuous family ρ_t of representations of a group Γ into G, the cohomology class $\rho_t^*\omega \in \mathrm{H}^p(\Gamma, \mathbb{R})$ is constant. Let us give some more details in our setting.

By Selberg's Lemma, we can replace Γ by a torsion-free finite index subgroup. Let $\rho_0 : \Gamma \to G$ denote the inclusion and let ρ_t be a continuous deformation of ρ_0 such that $\rho_t(\Gamma)$ still acts properly discontinuously and cocompactly on G/H. We denote respectively by B_t and E_t the spaces $\rho_t(\Gamma)\backslash G/K$ and $\rho_t(\Gamma)\backslash G$. Recall that E_t is a principal K-bundle over B_t with a natural connection ∇.

Let B' be a closed orientable manifold of dimension p and $\varphi_0 : B' \to B_0$ such that $\varphi_{0*}[B'] = k[\Gamma]$. One can deform φ_0 into a family of smooth maps $\varphi_t : B' \to B_t$ depending continuously on t. By continuity, we have

$$\varphi_{t*}[B'] = k\rho_{t*}[\Gamma]$$

for all t. (Here, we denote by $\rho_{t*}[\Gamma]$ the class $[\Gamma]$ seen as a homology class of $\rho_t(\Gamma)\backslash G/K$.)

We obtain the volume rigidity by using Chern–Weil theory. Let BK be a classifying space for K and $EK \to BK$ the associated universal principal K-bundle. There exists a continuous map $f_t : B' \to BK$ (unique up to homotopy) such that the principal K-bundle $\varphi_t^* E_t$ is isomorphic to $f_t^* EK$.

By the theorem of Cartan and Borel, we know that

$$\omega_H = P(F_\nabla)$$

for some K-invariant polynomial P on $\mathfrak{k}$. We thus have

$$\varphi_t^*\omega_H = P(F_{\varphi_t^*\nabla}) ,$$

where $\varphi_t^*\nabla$ denotes the pulled back connection on $\varphi_t^* E_t$.

The Chern–Weil isomorphism implies the existence of a cohomology class $c_P \in \mathrm{H}^p(BK, \mathbb{R})$ such that the form $P(F_{\varphi_t^*\nabla})$ represents the cohomology class $f_t^* c_P$ in $\mathrm{H}^p(B', \mathbb{R})$. We thus obtain

$$\begin{aligned}\mathbf{Vol}\,(\rho_t(\Gamma)\backslash G/H) &= \frac{1}{k}\int_{B'} \varphi_t^*\omega_H \\ &= \frac{1}{k}\int_{[B']} f_t^* c_P \ .\end{aligned}$$

Now, since ρ_t depends continuously on t, one can choose f_t so that it depends continuously on t. Therefore, the f_t are all homotopic and the class $f_t^* c_P$ is constant. We conclude that the volume of $\rho_t(\Gamma)\backslash G/H$ is constant. □

5. An obstruction to the existence of compact Clifford–Klein forms

In this section, we prove that, in some homogeneous spaces G/H, the form ω_H vanishes. By Theorem 1, every compact Clifford–Klein form $\Gamma\backslash G/H$ should have volume 0. Since this is clearly absurd, we conclude that compact Clifford–Klein forms of these homogeneous spaces do not exist.

Theorem 5.1. *For the following pairs (G, H), the volume form ω_H vanishes and G/H does not admit any compact Clifford–Klein form.*

(1) $G = \mathrm{SO}_0(p, q + r)$, $H = \mathrm{SO}_0(p, q)$, $p, q, r > 0$, *p odd;*
(2) $G = \mathrm{SL}(n, \mathbb{R})$, $H = \mathrm{SL}(m, \mathbb{R})$, $1 < m < n$, *m even;*
(3) $G = \mathrm{SL}(p + q, \mathbb{R})$, $H = \mathrm{SO}_0(p, q)$, $p, q > 0$, *$p + q$ odd;*
(4) $G = \mathrm{SO}(n, \mathbb{C})$, $H = \mathrm{SO}(m, \mathbb{C})$, $1 < m < n$, *m even;*
(5) $G = \mathrm{SO}(p + q, \mathbb{C})$, $H = \mathrm{SO}(p, q)$, $p, q > 0$, *$p + q$ odd.*

In the past decades, many different works have been devoted to finding various obstructions to the existence of compact Clifford–Klein forms. Let us detail where Theorem 5.1 fits in this litterature.

- Case (1) extends a result of Kulkarni [17] and its recent improvement by Morita [21], where both p and q are assumed to be odd. When specified to $r = 1$, we obtain in particular that $\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q + 1)/\mathrm{SO}_0(p, q)$ does not admit a compact quotient when p is odd. This is an important step toward Kobayashi's space form conjecture.

- The case of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ has been extensively studied as a "test case". It is conjectured that $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ never admits a compact quotient (for $2 \leq m < n$). Kobayashi proved that such quotients do not exist for $n < \lceil 3/2m \rceil$ [15] and Labourie, Mozes and Zimmer extended the result to $m \leq n - 3$ with completely different methods ([29], [18], [19]). On the other side, Benoist proved that $\mathrm{SL}(2n+1, \mathbb{R})/\mathrm{SL}(2n, \mathbb{R})$ does not admit a compact quotient [3]. Case (2) recovers Benoist's result[1] and also implies that $\mathrm{SL}(2n + 2, \mathbb{R})/\mathrm{SL}(2n, \mathbb{R})$ does not admit a compact quotient, which was previously known only for $n = 1$ [24].

- To the best of our knowledge, case (3) is new. It complements a recent result of Morita according to which $\mathrm{SL}(p + q)/\mathrm{SO}_0(p, q)$ does not admit a compact quotient when p and q are odd [21]. When $p = q + 1$, Benoist proved that every discrete group acting properly discontinuously on $\mathrm{SL}(p + q)/\mathrm{SO}_0(p, q)$ is virtually Abelian (in particular, its action is not cocompact) [3]. He also constructed proper

[1] Benoist's result is actually stronger: every discrete group acting properly discontinuously on $\mathrm{SL}(2n + 1, \mathbb{R})/\mathrm{SL}(2n, \mathbb{R})$ is virtually abelian.

actions of a free group of rank 2 as soon as $p \neq q$ or $q+1$.

- Case (4) seems new in general. However, for $m \leq n-5$, it follows from Constantine's generalization of the work of Labourie–Mozes–Zimmer [7]. Conjecturally, very few *complex* reductive homogeneous spaces admit compact Clifford–Klein forms. In particular, Kobayashi conjectured that the homogeneous space $\mathrm{SO}(n+1,\mathbb{C})/\mathrm{SO}(n,\mathbb{C})$ (which can be thought of as a complexification of the real sphere) only admits compact quotients for $n = 1$, 3 and 7. Unfortunately, case (4) only gives the non-existence of compact quotients of $\mathrm{SO}(2n+1,\mathbb{C})/\mathrm{SO}(2n,\mathbb{C})$, which is already a consequence of the *Calabi–Markus phenomenon* (see [14]).

- Finally, case (5) is new to the best of our knowledge.

There are more examples of homogeneous spaces where the form ω_H vanishes. For instance the form ω_H vanishes when $G = \mathrm{SL}(n,\mathbb{R})$, $H = \mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(n-m,\mathbb{R})$, $0 < m < n$, n odd, or when $G = \mathrm{SO}(n,\mathbb{C})$, $H = \mathrm{SO}(m,\mathbb{C}) \times \mathrm{SO}(n-m,\mathbb{C})$, $0 < m < n$, n odd (in these cases the non-existence of compact Clifford–Klein forms was proven by Benoist [3]).

In order to prove Theorem 5.1, we need to give a more explicit description of the form ω_H. Recall that the tangent space of G/H at the point $x_0 = H$ can be identified with the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}$. The form ω_H is uniquely determined by its restriction to $T_{x_0}G/H$.

If $\mathfrak{v}$ is a subspace of $\mathfrak{g}$ of dimension d in restriction to which the Killing form κ_G is non degenerate, we denote by $\omega_{\mathfrak{v}}$ the d-form on $\mathfrak{g}$ given by composing the orthogonal projection on $\mathfrak{v}$ with the volume form on $\mathfrak{v}$ induced by the restriction of the Killing form.

Finally, we provide K/L with the left invariant volume form $\omega_{K/L}$ induced by the restriction of the metric on G/H.

Lemma 5.2. *The form ω_H at the point x_0 is given by*

$$(\omega_H)_{x_0} = \int_{K/L} \mathrm{Ad}_u^* \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, \mathrm{d}\omega_{K/L}(u) \ .$$

Proof. In the construction of ω_H (Definition 3.2), we choose $\omega_{K/L}$ as our volume form on $F_{x_0} = K/L$. Let ξ be the q-vector on $\omega_{K/L}$ such that $\omega_{K/L}(\xi) = 1$.

At $y_0 = L$, the pull-back of vol_X by the projection $\psi : G/L \to G/H$ identifies with the form $\omega_{\mathfrak{h}^\perp}$ on $\mathfrak{g}$. Since the q-vector ξ at y_0 is given by $e_1 \wedge \ldots \wedge e_q$, where $(e_1, \ldots, e_q)$ is an orthonormal frame of $\mathfrak{k} \cap \mathfrak{h}^\perp$, we have

$$(i_\xi \omega_{\mathfrak{h}^\perp})_{y_0} = \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \ .$$

By left invariance, we also have

$$(i_\xi \psi^* vol_X)_{u \cdot y_0} = u_* \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \ .$$

Now, identifying $T_{u \cdot y_0}$ with $u_* \mathfrak{l}^\perp$, the differential of $\pi : G/L \to G/K$ is given at $u \cdot y_0$ by

$$\mathrm{d}\pi_{u \cdot y_0}(u_* v) = p_{\mathfrak{k}^\perp} \mathrm{Ad}_u(v) \ ,$$

where $p_{\mathfrak{k}^\perp}$ denotes the orthogonal projection on $\mathfrak{k}^\perp$.

Therefore, the form $(i_\xi \psi^* vol_X)$ at $u \cdot y_0$, whose kernel contains $u_* \mathfrak{k}$, induces by projection the form $\mathrm{Ad}_{u*}\omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp}$ at x_0. By construction of the form ω_H, we thus obtain

$$(\omega_H)_{x_0} = \int_{K/L} \mathrm{Ad}_{u*}\omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, \mathrm{d}\omega_{K/L}(u) \ .$$

□

Proof of Theorem 5.1. Though the proof requires a case-by-case study, the strategy is always the same: we exhibit an element $\Omega \in K$ whose action on $\mathfrak{g}$ stabilizes $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ and whose induced action on $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ has determinant -1. It follows that

$$\begin{aligned} \omega_H &= \int_{K/L} \mathrm{Ad}_{U*}\omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, \mathrm{d}vol_{K/L}(U) \\ &= \int_{K/L} \mathrm{Ad}_{U\Omega} * \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, \mathrm{d}vol_{K/L}(U) \\ &= \int_{K/L} -\mathrm{Ad}_{U*}\omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, \mathrm{d}vol_{K/L}(U) \\ &= -\omega_H \ , \end{aligned}$$

hence $\omega_H = 0$.

For each case in Theorem 5.1, we now describe $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ as a space of matrices and we give a choice of an element Ω. This element Ω just multiplies certain coefficients of the matrices in $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ by -1 and we leave to the reader the verification that the induced action on $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ has determinant -1.

We denote by $D_n(a_1, \dots, a_k)$ the diagonal matrix of size n whose i-th diagonal coefficient is -1 if $i \in \{a_1, \dots, a_k\}$ and 1 otherwise.

- $G = \mathrm{SO}_0(p, q+r)$, $H = \mathrm{SO}_0(p, q)$, $p, q, r > 0$, p odd:

  In this case, $K = \mathrm{SO}(p) \times \mathrm{SO}(q+r)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

  $$\left(\begin{array}{c|c|c} 0 & 0 & A^T \\ \hline 0 & \multicolumn{2}{c}{} \\ \cline{1-1} A & \multicolumn{2}{c}{0} \end{array}\right) ,$$

  with $A \in \mathcal{M}_{r,p}(\mathbb{R})$. We can choose

  $$\Omega = D_{p+q+r}(p+q, p+q+1) \ .$$

- $G = \mathrm{SL}(n, \mathbb{R})$, $H = \mathrm{SL}(m, \mathbb{R})$, m even:

In this case, $K = \mathrm{SO}(n)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c} \lambda \mathrm{I}_m & A \\ \hline A^T & B \end{array}\right) ,$$

with $A \in \mathcal{M}_{m,n-m}(\mathbb{R})$, $B \in \mathrm{Sym}_{n-m}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ satisfying $\mathrm{Tr}(B) + m\lambda = 0$. We choose

$$\Omega = D_n(m, m+1) .$$

- $G = \mathrm{SL}(p+q, \mathbb{R})$, $H = \mathrm{SO}_0(p,q)$, $p, q > 0$, $p+q$ odd:

In this case, $K = \mathrm{SO}(p+q)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) ,$$

with $A \in \mathrm{Sym}_p(\mathbb{R})$ and $B \in \mathrm{Sym}_q(\mathbb{R})$. We can choose

$$\Omega = D_n(p, p+1) .$$

- $G = \mathrm{SO}(n, \mathbb{C})$, $H = \mathrm{SO}(m, \mathbb{C})$, m even:

In this case, $K = \mathrm{SO}(n)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c} 0 & iA \\ \hline -iA^T & iB \end{array}\right) ,$$

with $A \in \mathcal{M}_{m,n-m}(\mathbb{R})$ and $B \in \mathrm{Antisym}_{m-n}(\mathbb{R})$. Again, we choose

$$\Omega = D_n(m, m+1) .$$

- $G = \mathrm{SO}(p+q, \mathbb{C})$, $H = \mathrm{SO}_0(p,q)$, $p, q > 0$, $p+q$ odd,:

To embed $\mathrm{SO}_0(p,q)$ into $\mathrm{SO}(p+q, \mathbb{C})$, we see $\mathrm{SO}(p+q, \mathbb{C})$ as the group of complex matrices P satisfying

$$P^T \mathrm{I}_{p,q} P = \mathrm{I}_{p,q} .$$

Then $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c} iA & 0 \\ \hline 0 & iB \end{array}\right) ,$$

with $A \in \mathrm{Antisym}_p(\mathbb{R})$ and $B \in \mathrm{Antisym}_q(\mathbb{R})$. Again, we choose

$$\Omega = D_{p+q}(p, p+1) \ .$$

□

This proof unfortunately requires a case by case study and does not provide a general criterion for the vanishing of the form ω_H. Note, however, that the form ω_H vanishes if, for some reason, G/K cannot admit any non-trivial G-invariant form of the right degree. In particular, if G satisfies the Borel–Cartan criterion, then every G-invariant form on G/K has even degree, and we obtain:

Proposition 5.3. *If the complexifications of* $\mathfrak{g}$ *and* $\mathfrak{k}$ *have the same rank and* $\dim G/H - \dim K/L$ *is odd, then* G/H *does not admit a compact Clifford–Klein form.*

We haven't found any new example where this criterion applies. It gives for instance the non-existence of compact quotients of $\mathbb{H}^{p,q}$ when both p and q are odd, recovering a result of Kulkarni [17].

6. Local foliations of G/H and global foliations of $\Gamma\backslash G/H$

The results of this paper where motivated by the fact that compact Clifford–Klein forms should "look like" K/L-bundles over a classifying space for Γ. This was suggested by the following theorem:

Theorem 6.1 (Guéritaud–Kassel, [9])**.** *Let* Γ *be a discrete torsion-free subgroup of* $\mathrm{SO}_0(n,1) \times \mathrm{SO}_0(n,1)$ *acting properly discontinuously and cocompactly on* $\mathrm{SO}_0(n,1)$ *(by left and right multiplication). Then* Γ *is isomorphic to the fundamental group of a closed hyperbolic* n*-manifold* B*, and* $\Gamma\backslash\mathrm{SO}_0(n,1)$ *admits a fibration over* B *with fibers of the form*

$$g\mathrm{SO}(n)h^{-1}, \quad g, h \in \mathrm{SO}_0(n,1) \ .$$

More generally, we conjecture the following:

Conjecture. *Let* G/H *be a reductive homogeneous space (with* G *and* H *connected),* L *a maximal compact subgroup of* H *and* K *a maximal compact subgroup of* G *containing* L*. Let* Γ *be a torsion free discrete subgroup of* G *acting properly discontinuously and cocompactly on* G/H*. Then there exists a closed manifold* B *of dimension* p *such that*

- *the fundamental group of* B *is isomorphic to* Γ*,*
- *the universal cover of* B *is contractible,*

- *$\Gamma\backslash G/H$ admits a fibration over B with fibers of the form gK/L for some $g \in G$.*

To support this conjecture, we note that the vanishing of the form ω_H (which implies the non-existence of compact Clifford–Klein forms) is actually an obstruction to the existence of a *local* fibration by copies of K/L.

Proposition 6.2. *Let G/H be a reductive homogeneous space (with G and H connected), L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L. If the form ω_H on G/K vanishes (and in particular for all the pairs (G,H) in Theorem 5.1), then no non-empty open domain of G/H admits a foliation with leaves of the form gK/L.*

The non-existence of such local foliations in certain homogeneous spaces may be quite surprising. For instance, if $G = \mathrm{SO}_0(2n-1,2)$ and $H = \mathrm{SO}_0(2n-1,1)$, then G/H is the *anti-de Sitter space* AdS_{2n} (for which the non-existence of compact Clifford–Klein forms was proven by Kulkarni [17]). In that case, K/L is a timelike geodesic and we obtain the following corollary:

Corollary 6.3. *No open domain of the even dimensional anti-de Sitter space can be foliated by complete timelike geodesics.*

This leads to the following more general question, that may be of independent interest:

Question 6.4. *Let G/H be a reductive homogeneous space, G' a closed subgroup of G and $H' = G' \cap H$. When does G/H admit an open domain with a foliation by leaves of the form gG'/H'?*

Proof of Proposition 6.2. Assume that there exists a non-empty domain U in $X = G/H$ with a foliation by leaves $(F_v)_{v\in V}$ of the form $g_v K/L$. Since the stabilizer in G of $K/L \subset G/H$ is exactly K, the space of leaves V can be seen as a submanifold of dimension p in G/K. Set $U' = \pi^{-1}(V)$, where π is the projection from G/L to G/K. Then the projection ψ from G/L to G/H induces a diffeomorphism from U' to U. We thus have

$$\int_U vol_X = \int_{U'} \psi^* vol_X \ .$$

On the other hand, by construction of ω_H, we have

$$\int_{U'} \psi^* vol_X = \int_V \omega_H \ .$$

Since U is non-empty, its volume is non-zero, hence the form ω_H cannot vanish. □

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