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An LMI approach for the Integral Sliding Mode and H_∞ State Feedback Control Problem

Souad Bezzaoucha[1], **David Henry**[2]

[1] Interdisciplinary Centre for Security, Reliability and Trust, Campus Kirchberg, University of Luxembourg, L-1359, Luxembourg. *souad.bezzaoucha@uni.lu*
[2]University of Bordeaux, IMS Laboratory, F-33400 Talence, France.
david.henry@ims-bordeaux.fr

Abstract. This paper deals with the state feedback control problem for linear uncertain systems subject to both matched and unmatched perturbations. The proposed control law is based on an the Integral Sliding Mode Control (*ISMC*) approach to tackle matched perturbations as well as the H_∞ paradigm for robustness against unmatched perturbations. The proposed method also parallels the work presented in [1] which addressed the same problem and proposed a solution involving an Algebraic Riccati Equation (ARE)-based formulation. The contribution of this paper is concerned by the establishment of a Linear Matrix Inequality (LMI)-based solution which offers the possibility to consider other types of constraints such as $\mathcal{D}$-stability constraints (pole assignment-like constraints). The proposed methodology is applied to a pilot three-tank system and experiment results illustrate the feasibility. Note that only a few real experiments have been rarely considered using SMC in the past. This is due to the high energetic behaviour of the control signal.

It is important to outline that the paper does not aim at proposing a LMI formulation of an ARE. This is done since 1971 [2] and further discussed in [3] where the link between AREs and ARIs (algebraic Riccati inequality) is established for the H_∞ control problem. The main contribution of this paper is to establish the adequate LMI-based methodology (changes of matrix variables) so that the ARE that corresponds to the particular structure of the mixed ISMC/H_∞ structure proposed by [1] can be re-formulated within the LMI paradigm.

1. Motivations

Recent works on Sliding Mode Control (SMC) demonstrate that SMC approaches are able to withstand external disturbances and model uncertainties satisfying the matching condition, that is, perturbations that enter the state equation at the same point as the control input, see [4, 5] for instance. Perturbations not satisfying the matching condition are obviously called "unmatched perturbations". The matching condition is a sufficient condition for the existence of a SMC law able to reject matched perturbations (exact compensation or insensitivity) providing they are bounded by a known function $\in \mathcal{L}_\infty$. However, if the matching condition is not satisfied then the system behaviour depends on the influence of the disturbances [6, 5]. This motivates the work reported in this paper.

A sliding mode controller is composed of two modes: The first one called the *reaching mode* that drives the system onto the so-called sliding surface in a finite time. The second mode, known as the *sliding mode*, aims at maintaining the system on the sliding surface. Note that during the reaching phase, the system is sensitive, even to perturbations satisfying the matching condition.

Different solutions have been proposed to carry out the unmatched perturbations, see [7, 8, 9, 10, 11, 12, 13, 14] to name a few. The main idea is to define the control law as the sum of a continuous control (say u_0) designed by means of robust control techniques and a SMC signal (say u_1). u_0 is responsible of the attenuation of the unmatched disturbances on the system states and u_1, the discontinuous control, is used to reject the matched perturbations. The same philosophy is used in [15, 16] by adding an integral term in the definition of the sliding surface. This enables the system always to start at the sliding manifold. The technique is called Integral Sliding Mode Control (ISMC). The method proposed in [1] follows the same idea. The continuous signal u_0 is designed using H_∞ technique whereas u_1 is a ISMC-based law. Conditions are given for the existence of a stabilizing control law u_0 in terms of an algebraic Riccati equation (ARE), which leads the approach numerically suitable since it can be solved using matrix algebra theory. Note that, as outlined by the authors, the integral sliding mode controller, if improperly designed, while eliminating the matched perturbations, could lead to amplification of the unmatched ones. This further motivates the use of H_∞ techniques.

In this paper, a Semi Definite Programming (SDP) formulation involving Linear Matrix Inequality (LMI) constraints, is proposed. The major reason of using the LMI paradigm is that it is now well known that it offers the possibility to consider other types of constraints such as $\mathcal{D}$ stability constraints (pole assignment-like constraints) [17, 18, 19]. The proposed method also parallels the work presented in [1] which as already mentioned, proposed a ARE-based formulation.

It is important to outline that the paper does not aim at proposing a LMI formulation of an ARE. This is done since 1971 [2] and further discussed in [3] where the link between AREs and ARIs (algebraic Riccati inequality) is established for the H_∞ control problem. The main contribution of this paper is to establish the adequate LMI-based methodology (changes of matrix variables) so that the ARE that corresponds to the particular structure of the mixed ISMC/H_∞ structure proposed by [1] can be re-formulated as a SDP problem under LMI constraints. In fact, results from [3] cannot be used since some degrees of freedom in the control gain are used to ensure matched perturbation compensation by the SMC part. This will be highlighted in the paper, see remark 2. Establishing a LMI formulation of the ISMC/H_∞ control has not been done right now, see the literature in both the SMC and H_∞ communities.

The paper's structure is the following. In section 2, the Integral Sliding Mode Control (ISMC) combined with the H_∞ control is considered. The mathematical problem is stated formally in terms of LMI framework in section 3. In section 4, an application on a pilot three-tank system is given. The plant model is presented and the proposed control design is applied to the system. Experimental results are given to illustrate the developed approach efficiency. Finally, in section 5, conclusions and perspectives for future works are given.

2. Problem statement and material background

Similarly to the approach addressed in [1], the solution proposed in this paper to the state feedback control problem for linear uncertain systems subject to both matched and unmatched perturbations, consists in defining the control law as the sum of a continuous control signal u_0 and a discontinuous ISMC signal u_1, i.e.

$$u(t) = u_0(t) + u_1(t) \tag{1}$$

u_0 is in charge to manage the unmatched perturbations (this part is addressed through the H_∞ formalism) while u_1 is in charge to manage the matched perturbations (this part is addressed using sliding mode control). An integral term is included in the sliding manifold to guarantee that the system starts at the sliding manifold.

Thus, in the interest of brevity, throughout this section an earnest attempt will be made to avoid duplicating material presented in [1]. Towards this end, the focus of this section will lie wholly with the main results established in [1]. The reader is invited to refer to this paper for necessary backgrounds and proofs.

Consider the following state-feedback control problem

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ y(t) &= x(t) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the system's state vector, $u \in \mathbb{R}^m$ the control input vector, $w \in \mathbb{R}^q$ is an external perturbation due to model uncertainties or external disturbances (including here both matched and unmatched disturbances), $z \in \mathbb{R}^r$ is the objective vector (including tracking signals, regulation signals and actuator's output) and $y \in \mathbb{R}^n$ is the measurement signal. The following assumptions are made:

Assumption 1. $rank(B_1) = m$

Assumption 2. *$w(t)$ is assumed to be bounded by a known function $v(t) \in \mathcal{L}_\infty$, i.e. $||w(t)|| \leq v(t)$ for all t.*

Assumption 3. *The pair (A, B_2) is stabilisable, the pair (C_1, A) is detectable and $D_{12}^T[C_1 \ D_{12}] = [0 \ I]$.*

Assumption 1 is a necessary and sufficient condition to separate the matched from the unmatched part of w. Assumption 2 is common in the sliding mode theory and assumption 3 is common in the H_∞ theory. The interested reader can refer to [4, 5, 20].

From sliding mode control theory, the discontinuous control u_1 is selected as

$$u_1(x,t) = -\rho(x,t)\frac{(GB_2)^T s(x,t)}{||(GB_2)^T s(x,t)||} \tag{3}$$

where $\rho(x,t)$ is a matrix of adequate dimension so that its values are chosen high enough to enforce the sliding motion. The term $\dfrac{(GB_2)^T s(x,t)}{||(GB_2)^T s(x,t)||}$ plays the role of the "$sign$" function so that Eq. (3) is an alternative to the well known SM equation $u_1 = -\rho.sign(s)$. The sliding manifold s is defined by the set $\{x | s(x,t) = 0\}$ with:

$$s(x,t) = G\left[x(t) - x(t_0) - \int_{t_0}^{t}(Ax(t) + B_2 u_0(x,\tau))d\tau\right] \tag{4}$$

$G \in \mathbb{R}^{m \times n}$ is a projection matrix such that GB_2 is invertible. It can be noticed that at $t = t_0$, $s(x,t_0) = 0$ so the system always starts at the sliding manifold.

Following [1], the H_∞ norm of the transfer that goes from $w(t)$ to $z(t)$ satisfies $||T_{zw}||_\infty < \gamma$ if G is set to B_2^T and

$$u_0(t) = -B_2^T X x(t) \tag{5}$$

where X is a positive semi-definite matrix, solution of the ARE

$$XA + A^T X - X(B_2 B_2^T - \gamma^{-2}\overline{B}_w \overline{B}_w^T)X + C_1^T C_1 = 0 \tag{6}$$

$$\overline{B}_w = B_2^\perp (B_2^\perp)^+ B_1 \tag{7}$$

where the columns of $B_2^\perp$ span of null space of B_2^T. $(B_2^\perp)^+$ is the left inverse of $B_2^\perp$, that is $(B_2^\perp)^+ = ((B_2^\perp)^T B_2^\perp)^{-1}(B_2^\perp)^T$.

It follows that the sliding manifold is given by

$$s(x,t) = B_2^T \left[x(t) - x(t_0) - \int_{t_0}^{t} (A - B_2 B_2^T X) x(\tau) d\tau \right] \tag{8}$$

and the control law is defined according to

$$u(x,t) = -B_2^T X x(t) - \rho \frac{s}{||s||},\ \rho > ||B_2^+ B_1 w|| \tag{9}$$

The reader can refer to the paper [1] for additional explanations and proofs.

Remark 1. *It should be outlined that, even if the stability for the control laws* $u_1 = -\rho\frac{s}{||s||}$ *and* $u_0 = -B_2^T X x$ *has been proven separately, there does not exist any formal proof of stability for the overall control law (9). This is a known problem from the SMC community. Fortunately, different authors have demonstrated that the stability is practically preserved [12, 13, 14]. Thus, since the method we proposed in this paper follows the same principle, the same problem arises. This is a known problem which is under current research investigation.*

Remark 2. *Discussion on AREs and LMIs for ISMC:*
The particular structure of the control law u_0 *given by (5) follows the requirements for* u_1 *to mitigate the matched perturbations (exact compensation), see the theoretical developments in [1]. In a pure* H_∞ *problem, the control signal has the general form* $u(t) = -X_\infty x(t)$[1] *where* X_∞ *is the solution of an ARE. The consequence of the presence of* B_2^T *in the definition of* u_0 *is that one can not directly apply, e.g. the bounded and projection lemmas technique [3, 21], the changes of LMI variables approaches [17, 18] or [2, 22, 23] for the state-feedback case, to derive a LMI formulation. In other words, even if it seems trivial to derive a LMI formulation from Eq. (6), this is in fact not the case due to the particular structure of* u_0 *given by Eq (5). The next section provides a solution to this problem.*

3. Main result: a LMI formulation

Consider the disturbances vector w in the system model (2). Then, applying the theoretical developments proposed by [1], the following proposition yields

Proposition 1. *For any matrix satisfying* B_1 *satisfying assumption 1, the disturbances* w *can be decomposed as*

$$B_1 w(t) = B_w w(t) + \overline{B}_w w(t), \quad B_w = B_2 B_2^+ B_1, \quad \overline{B}_w = B_2^\perp (B_2^\perp)^+ B_1 w(t) \tag{10}$$

[1] The minus sign is kept there for similarity reason with eq. (5) but it can obviously be removed.

□

The first part of (10) represents the matched disturbances and will be eliminated by the discontinuous control u_1; the second one refers to the unmatched disturbances and will be attenuated using the continuous control u_0. The following proposition gives the solution to the problem:

Theorem 1. *Consider the system model given by Eq. (2) subject to both matched and unmatched perturbations. The state-feedback controller given by Eq. (1) minimizing the H_∞ norm of the transfer that goes from $w_u(t) = B_2^\perp(B_2^\perp)^+ B_1 w(t)$ (unmatched perturbations) to $z(t)$ such that $||T_{zw_u}||_\infty < \gamma$ and annihilating the effect of $w_m(t) = B_2 B_2^+ B_1 w(t)$ (matched perturbations) is defined according to*

$$u(x,t) = -B_2^T Q^{-1} x(t) - \rho \frac{s}{||s||},\ \rho > ||B_2^+ B_1 w|| \tag{11}$$

where $Q = Q^T > 0$ is a positive definite matrix solution of the following LMI constraint

$$\begin{pmatrix} AQ + QA^T - 2B_2B_2^T & -QC_1^T & B_2 & \overline{B}_w \\ & -I & 0 & 0 \\ & * & -I & 0 \\ & * & * & -\gamma^2 I \end{pmatrix} \leq 0 \tag{12}$$

□

Proof 1. *Let us first consider the following well know lemma that associates a so-called algebraic Riccati inequality (ARI) to an ARE:*

Lemma 1. *Suppose matrices $A, W = W^T > 0$ and Q are given. If the Algebric Riccati Equation (ARE)*

$$PA + A^T P + Q - PWP = 0 \tag{13}$$

has a positive definite symmetric solution P, then for any $0 < W_1 \leq W$ and $Q_1 \geq Q$, the equation

$$P_1 A + A^T P_1 + Q_1 - P_1 W_1 P_1 \leq 0 \tag{14}$$

has a positive definite symmetric solution $P_1 \geq P$. □

Using this lemma and the ARE (6), we know that there exists a matrix $X_1 = X_1^T$ which satisfies the following inequality:

$$X_1 A + A^T X_1 + C_1^T C_1 + \gamma^{-2} X_1 \overline{B}_w \overline{B}_w^T X_1 - X_1 B_2 B_2^T X_1 \leq 0 \tag{15}$$

Let us choose $K = -B_2^T X_1$, we get:

$$X_1 A + A^T X_1 + C_1^T C_1 + \gamma^{-2} X_1 \overline{B}_w \overline{B}_w^T X_1 + K^T B_2^T X_1 + X_1 B_2 K + K^T K \leq 0 \tag{16}$$

Let us note $Q = X_1^{-1}$ and $W = KQ$. From the congruence principle and applying the Schur's complement, the LMI (12) is then deduced where $X_1 = Q^{-1}$ and $K = WQ^{-1} = -B_2^T X_1$ which terminates the the proof. □

This theorem also gives a numerical tractable solution to the ISMC/H_∞ design problem, since noting that γ^2 enters linearly in (12), the H_∞ norm $||T_{zw_u}||$ can be minimized by minimizing γ.

3.1. ISMC, H_∞ and poles assignment

As it is known, a good time response specifications can be achieved by forcing the closed-loop poles into a specific regions. In this case, the H_∞ synthesis being formulated as a convex optimization problem involving LMIs will give wide range of flexibility in combining several constraints on the closed-loop system. This flexible nature of LMI schemes can be used to handle H_∞ controller through the LMI regions concept [24].

Direct application of the developments proposed in [24] to the definition of the control law $u_0(t) = -B_2^T Q^{-1} x(t)$ shows that the eigenvalues of the evolution matrix of the closed loop $A_{cl} = A - B_2 B_2^T Q^{-1}$ can be assigned into a prescribed $\mathcal{D} = \cap_{k=1}^{n_s} \mathcal{D}_k$ if there exist a common Lyapunov matrix $Y = Y^T > 0$ such that the set of n_s LMIs

$$\alpha_k \otimes Y + \beta_k \otimes A_{cl} Y + \beta_k^T \otimes Y A_{cl}^T < 0 \quad k = 1, 2, ..., n_s \tag{17}$$

is simultaneously satisfied.

In this expression, " $\otimes$ " denotes the Kronecker product of matrices, α_k and β_k are matrices of appropriate dimension defining each region $\mathcal{D}_k = \{\chi \in \mathbb{C} : f_{\mathcal{D}_k}(\chi) < 0\}$, where $f_{\mathcal{D}_k}(\chi) = \alpha_k + \chi\beta + \chi^*\beta^T$ is the characteristic function of the region $\mathcal{D}_k$. Here $\chi = x + yj$ is the complex variable and χ^* denotes the conjugate of χ.

Setting $Y = Q$, it is easy to see that solving the following optimization problem gives the solution of the ISMC problem with jointly H_∞ attenuation performance against unmatched perturbations and pole assignment.

$$\min \gamma^2 \quad s.t. \, Eq.(12) \text{ and } Eq.(17) \tag{18}$$

Remark 3. *Note that since theorem 1 provides only a sufficient condition for a solution of the ISMC/H_∞ problem, the solution of Eq. (12) may be more conservative than the solution of the ARE-based formulation. Thus, if it is not required, e.g. $\mathcal{D}$-stability, the solution described in section 2 may be preferred to the one proposed in section 3.*

4. Application to a pilot three-tank system

4.1. Three-tank system description and modelling

The proposed state-feedback controller is now applied to an experimental study. The experimental study is based on a pilot three-tank system, manufactured by "AMIRA" industry (fig 1). The plant consists of three cylinders T_1, T_2, T_3 with cross-section S. These are connected serially with one another cylindrical pipes with a cross-section S_n. The out-flowing liquid (usually distilled water) is collected in a reservoir, which supplies the pumps 1 and 2. Here, the circle is closed. The three water levels (in m) are denoted h_1, h_2 and h_3. They are measured via piezo-resistive pressure sensors. Q_1 and Q_2 (in m^3/s) are the flow rates of the pumps 1 and 2. The pumps are rotational-speed controlled such that a well-defined incoming mass flow corresponds to the reference input introduced by the pump controller.

To model the dynamics of this multi-input/multi-output (MIMO) system, the *Torricielli* rule is applied to each tank. Since modelling this system has already done by many authors, the detailed are omitted here. The interested reader can refer to, e.g. [25].

Performing a first-order approximation around an equilibrium point $h^* = (h_1^* \; h_2^* \; h_3^*)^T$ of the Torricielli equations leads to

$$\begin{cases} \dot{x}(t) &= A(\theta)x(t) + Kf(t) + B_2 u(t) \\ y(t) &= x(t) \end{cases} \tag{19}$$

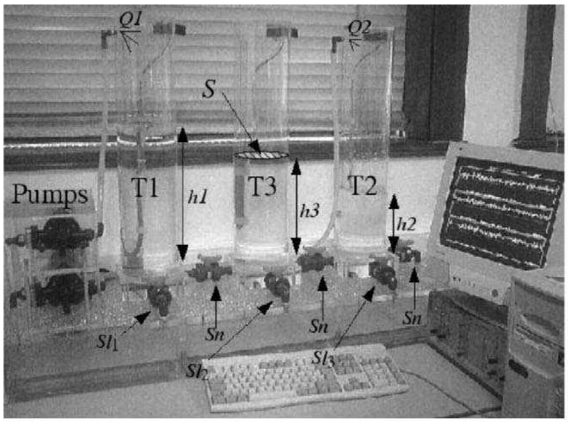

Figure 1. Physical structure of the three-tank system

where $u = (Q_1 \; Q_2)^T$ is the controlled inputs and $y = (h_1 \; h_2 \; h_3)^T$ represents the measured outputs. $f(t)$ models a leakage in the first tank. $\theta = (a_1 \; a_2 \; a_3)^T$ denotes the vector of outflow coefficients that are uncertain but assumed to be bounded. The system matrices are the following:

$$A(\theta) = \begin{pmatrix} -k_1a_1 & 0 & k_1a_1 \\ 0 & -k_3a_3 - k_2a_2 & k_3a_3 \\ k_1a_1 & k_3a_3 & -k_1a_1k_3a_3 \end{pmatrix}, K = \begin{pmatrix} \frac{S_n\sqrt{2g}a_{10}}{2S\sqrt{h_1}} \\ 0 \\ 0 \end{pmatrix}, B_2 = \frac{1}{S}\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (20)$$

The parameters $k_i, i = 1, 2, 3$ are defined as follows:

$$k_1 = \frac{S_n\sqrt{2g}}{2S\sqrt{h_1^* - h_3^*}}, \; k_2 = \frac{S_n\sqrt{2g}}{2S\sqrt{h_2^*}}, \; k_1 = \frac{S_n\sqrt{2g}}{2S\sqrt{h_3^* - h_2^*}} \quad (21)$$

where g the constant of the gravity. All numerical values are given in the appendix.

From the definition of K and B_2, it is easy to see that f satisfies the matching condition. Thus, following the methodology proposed in section 3, a state-feedback control law $u(t) = u_0(t) + u_1(t) = -B_2^TQ^{-1}x(t) - \rho\frac{s}{||s||}$ is proposed so that u_1 compensate exactly f, u_0 being in charge to attenuate, in the H_∞ sense, the effect of the uncertainties θ on the system state. u_1 being deduced from the ISMC theory according to section 2, the main drawback is concerned by the H_∞ part. So, f (the matched perturbation) is omitted in the following developments.

To proceed, the LFR (Linear Fractionnal Representation) formalism is used here. All parameters θ_i, $i = 1...3$ entering in $A(\theta)$ are "pulled out" so that the model appears as a LTI nominal model P subject to an artificial block diagonal operator $\Delta = blockdiag\left(\delta_1 I_{k_1}, ..., \delta_q I_{k_q}\right)$ specifying how δ_i enters P where $k_i > 1$ whenever the parameter δ_i is repeated, see [26, 27, 28, 29, 30] for more details. In this formalism, $|\delta_i| \le 1 \Leftrightarrow ||\Delta||_\infty \le 1$. This can be assumed without loss of generality since the model P can always be scaled. This boils down to the following model

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + E_1\eta(t) + B_2u(t) \\ \varepsilon(t) & = & E_2x(t) \\ y(t) & = & x(t) \end{array} \right. , \quad \begin{array}{l} \eta(t) = \Delta\varepsilon(t), \\ \Delta = diag(\delta_1, \delta_2, \delta_3) \\ A = A(\theta_0) \end{array} \quad (22)$$

$$E_1 = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & w_3 \\ -w_1 & 0 & -w_3 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -k_1 & 0 & k_1 \\ 0 & -k_2 & 0 \\ 0 & -k_3 & k_3 \end{pmatrix} \tag{23}$$

$w_i, i = 1, 2, 3$ are weights introduced to scale the uncertainties, that is $a_i = a_{i0} + w_i\delta_i, i = 1, 2, 3$. Then, by virtue of the small gain theorem, there exists a stabilizing H_∞ control law $u_0(t) = -\mathcal{K}x(t) = -B_2B_2^TQ^{-1}x(t)$ robust to Δ if $\mathcal{K}$ stabilizes the following model

$$\begin{cases} \dot{x}(t) &= Ax(t) + E_1\eta(t) + B_2u(t) \\ \varepsilon(t) &= E_2x(t) \\ y(t) &= x(t) \end{cases} \qquad \text{so that } ||T_{\eta\varepsilon}||_\infty < 1 \tag{24}$$

where $||T_{\eta\varepsilon}||_\infty$ denotes the transfer between η and ε. The dimension of the problem is $x \in \mathbb{R}^3$, $u \in \mathbb{R}^2$, $\eta \in \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}^3$ which clearly exhibits a MIMO problem.

4.2. Design of the control law

In terms of H_∞ objectives for u_0, it is only required attenuation of the unmatched perturbations on the system state, so the control penalty function is fixed to identity. Now, considering the above statement, it follows the following definition of the different matrices of the state-feedback problem (2)

$$B_w = K, \quad \overline{B}_w = E_1, \quad C_1 = \begin{pmatrix} E_2 \\ I_3 \end{pmatrix} \tag{25}$$

Theorem 1 is next used to derive the control law given by Eq. (11). In order to avoid fast dynamics of the control law (this prevents, e.g. measurement noise amplification), a pole clustering constraint is considered following the developments proposed in section 3.1. Especially, it is required pole clustering in the LMI region $\mathcal{D} = \{\lambda : \mathbb{R}\left(\lambda_i(A - B_2B_2^TQ^{-1})\right) > -\beta\}$ with $\beta = 70$. This leads to the following LMI:

$$2\beta Q + AQ + QA^T - 2B_2B_2^T > 0, \quad \beta = 70 \tag{26}$$

Noticed that the H_∞ LMI (12) enforces stability, it doesn't matter if this LMI contains the right-half complex plan. Finally, the SDP problem given by Eq. (18) is solved using the SDPT3 solver, leading to the following optimal solution.

$$\gamma \approx 0.847, \quad \lambda_i\left(A - B_2B_2^TQ^{-1}\right) \approx \{-0.0409,\ -67.3141,\ -67.3620\}, \quad ||B_2^TQ^{-1}||_2 \approx 1.27$$

Since $\gamma < 1$, robustness to the considered uncertainties (outflow coefficients) is guaranteed, thanks to the small gain theorem. Note that the numerical value of $||B_2^TQ^{-1}||_2$ indicates a noise amplification of $\approx 27\%$ on the control signal u_0. This may cause serious problems from a practical point of view. A solution to manage such a difficulty is proposed later using a Kalman filter, see remark 4.

With regards to the ISMC part, the main concern is the definition of the matrix ρ to enforce the sliding motion. Here, ρ has been chosen to $\rho = 5.10^{-5}I_2$. This numerical value has been fixed considering the saturation level of the pumps, i.e. it has been chosen to be close to the saturation level of the actuators $Q_{max} = 9.24.10^{-5}m^3/s$.

4.3. Simulation results

The proposed control law combining the *ISMC* and the H_∞ approach, is applied to the three-tank system presented above. In order to appreciate the effect of the $ISMC$ and H_∞ parts, simulation results from a nonlinear simulator is first presented, ignoring the measurement noises and the effect of the DAC-DCA. The numerical method used to solve the integrations and differential equations is based on the Euler method with a sampling period equal to $T_s = 10ms$. This configuration is retained since they correspond to the one used for the real experiments, see section 4.4.

The considered experience corresponds to the following simulation: The system starts at the equilibrium point h_1^*, h_2^* and h_3^* respectively equal to $0.4m$, $0.2m$ and $0.3m$, a change of reference occurs at $t = 150s$, the new references to attain being fixed to $10cm$ more, i.e. $0.5m$, $0.3m$ and $0.4m$ for the first, second and third outputs. A simple second order polynomial path planing is also implemented for each output to track. Then, a sinusoidal variation of the outflow coefficients is simulated for $t \geq 200s$. Finally, a leak occurs in the first tank for $t \geq 300s$.

Because we consider the problem of output tracking, the definition of the control signals u_0 and u_1 defined in section 3, needs to be updated. The major difficulty is concerned by the sliding mode part since it is well known that the new form of u_0 is $u_0(t) = \mathcal{K}e_x(t) = B_2 B_2^T Q^{-1} e_x(t)$ where $e_x(t) = x_c(t) - x(t)$ is the tracking error, x_c being the reference.
Based on Eq. (8), the following sliding function is chosen, the definition of u_1 being unchanged:

$$s(x,t) = B_2^T \left[e_x(t) - e_x(t_0) + \int_{t_0}^{t} (A - B_2 B_2^T X) e_x(\tau) d\tau \right] \tag{27}$$

Figures 2 give the obtained simulation results. For comparison, the solution proposed by [1], i.e. the ARE-based solution described in section 2, is too tested. The obtained results are given in the figures 3. The following remarks can be made from these figures:

- Even if it can be observed that the behaviour of u_0 and u_1 differs between the LMI and the ARE solutions, the tracking errors exhibit very similar behaviour. In other words, the two solutions exhibit very similar control performance, which is reassuring. Note however, that the H_∞ performance of the ARE-based solution is found to $\gamma \approx 0.098$ with $||B_2^T Q^{-1}|| \approx 1.0734$ which illustrates that the LMI-based solution may be more conservative than the ARE-based solution and thus that, if it is not required poles assignment, the ARE-based solution should be preferred, see remark 3.
- The sliding surface is reached at the beginning of the simulation and is never left, thanks to the definition of the sliding function $s(x,t)$ given by Eq. (27) where it can be noticed that at $t = t_0 = 0s$, $s(x, t_0) = 0$, so the system always starts at the sliding manifold.
- An immediate consequence is that at $t = 300s$, when a leak occurs in the first tank (matched perturbation), it is immediately mitigated by the SM part of the control since the sliding motion is not left. This can be easily observed on the tracking errors. This can be too noticed on the behaviour of $u_1(t)$ and the sliding function $s(t)$, especially at $t = 300s$ where a change can be observed.
- Finally, it can be observed that at $t = 200s$, when the uncertainties manifest themselves, the H_∞ part of the control, i.e. $u_0(t)$, reacts whereas the SMC part seems to not react. This highlights the benefit of the proposed control scheme.

4.4. Experimental results

Finally, the LMI solution is implement within the control software of the three-tanks system. The considered experience is close to those considered previously, except for the outflow coefficients since it is obvious that they are physically inaccessible. The obtained results are given on figures 4. Clearly, due to the presence of noise, the quantization effect of the DAC-DCA, transient saturations of the control signals and probably many other unknown perturbations

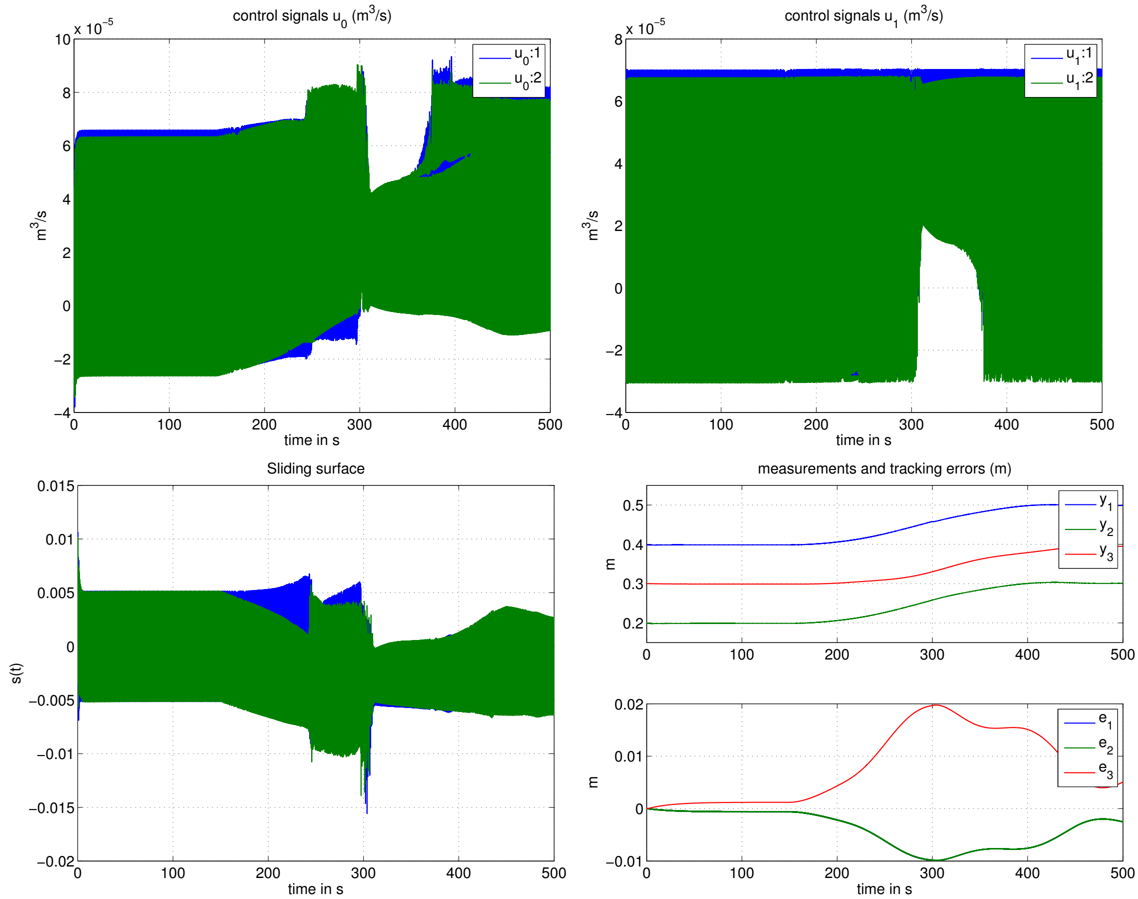

Figure 2. From top left to bottom right: control signals u_0, u_1, sliding function s and output signals $y = x$ and tracking error e_x. LMI solution

and disturbances, it is very hard to identify from the behaviour of $u0(t)$ and $u_1(t)$, how the H_∞ and SM parts of the control co-operate. However, it can be noticed the major benefits of both the techniques, i.e. the tracking errors are small, the leak is mitigated (but not instantaneously even if it seems that the sliding motion is not left), and the overall control system is stable despite the presence of the uncertainties. This tends to demonstrate that the proposed solution operates well. Note that the third output is correctly tracked as opposed to the simulations. A deeper investigation into the situation reveals that the outflow coefficients didn't really vary during the real experiment.

Remark 4. *It is well know that SM controllers are very sensitive to measurement noise. Thus, a Kalman filter is used to attenuate the measurement noise of the piezo-resistive pressure sensors, thus providing a "smooth" estimate* $\hat{h}_i, i = 1, 2, 3$ *of the three water levels* $h_i, i = 1, 2, 3$*. The model used correspond to Eq. (19) with* $\theta = \theta_0$ *and* $f = 0$ *and with covariance matrices fixed to* $Q = 0.5I_3$ *and* $R = I_3$*.*

5. Conclusions

In the proposed manuscript, a state feedback control based on *ISMC* and H_∞ approaches is proposed. The control gain (for H_∞ part) is calculated based on the LMI framework combined with poles placement constraints in order to improve the controller performances. An application on a three-tank pilot system is considered with satisfying results. Future work is concerned by

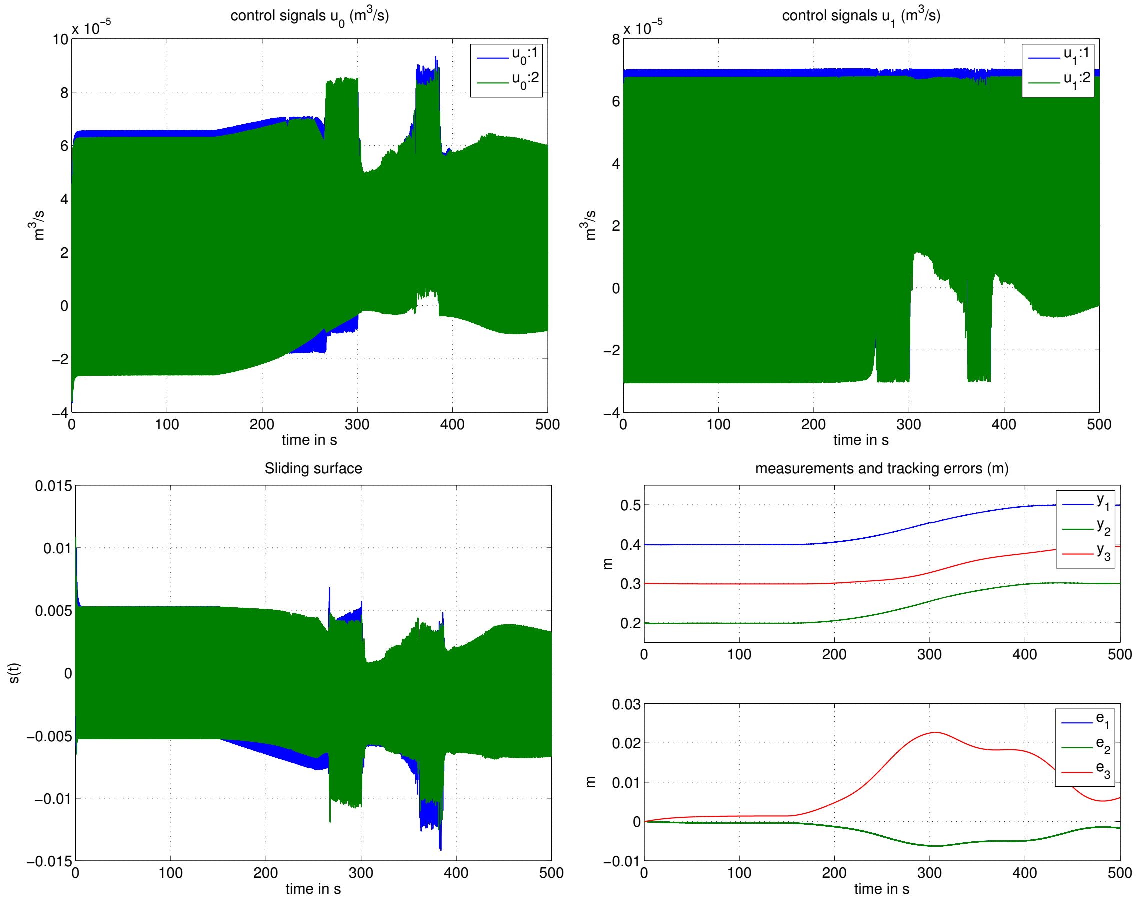

Figure 3. From top left to bottom right: control signals u_0, u_1, sliding function s and output signals $y = x$ and tracking error e_x. ARE solution

the establishment of a formal proof of stability of the overall control law, see remark 1.

Appendix A. Parameters of the three-tank system

$S = 0.0154m^2$, $Sn = 5.10^{-5}m^2$, $h_{max} = 60cm$, $Q_{max} = 9.24.10^{-5}m^3/s$, $0.58 \leq a_1 \leq 0.62$, $0.78 \leq a_2 \leq 0.82$, $0.58 \leq a_3 \leq 0.62$, $Sl = Sn$, $h_1^* = 40cm$, $h_2^* = 20cm$, $h_3^* = 30cm$.

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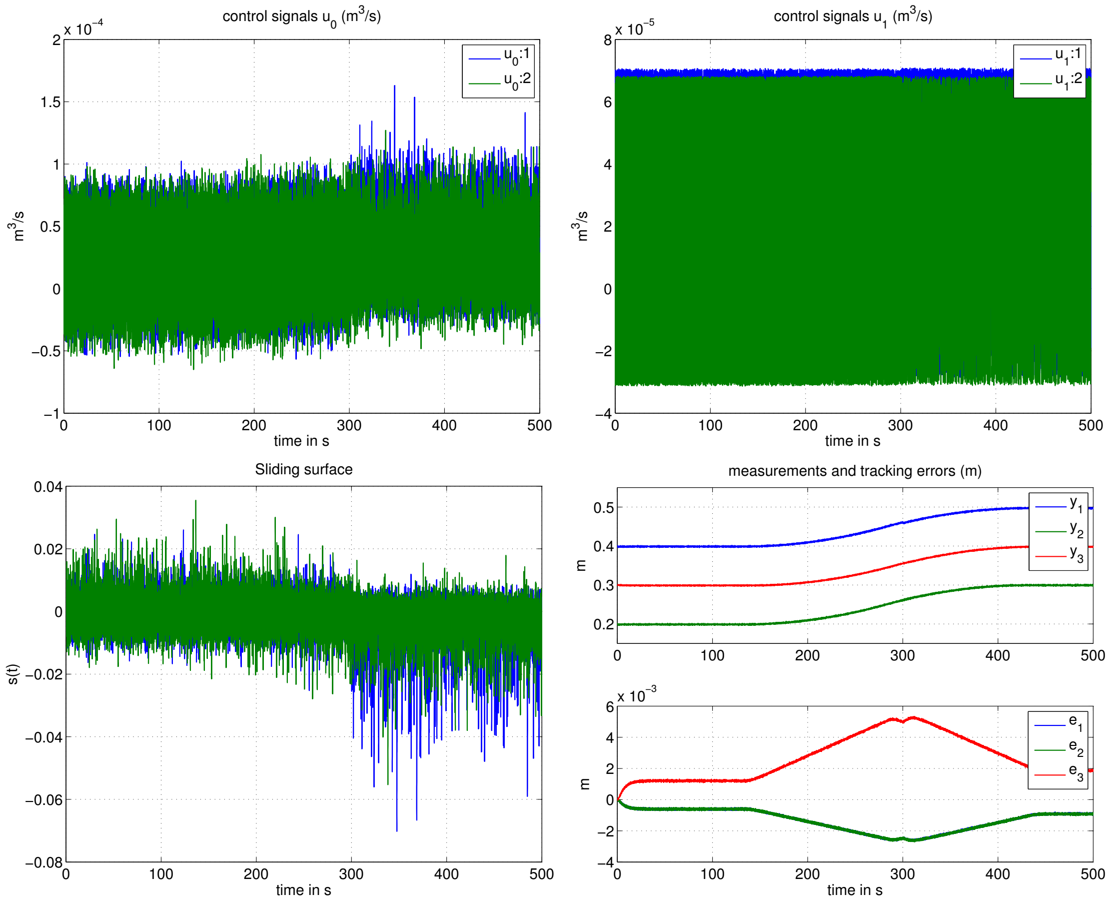

Figure 4. From top left to bottom right: control signals u_0, u_1, sliding function s and output signals $y = x$ and tracking error e_x. LMI solution

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