

ON LÉVY'S BROWNIAN MOTION INDEXED BY THE ELEMENTS OF COMPACT GROUPS

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ABSTRACT. We investigate positive definiteness of the Brownian kernel $K(x, y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$ on a compact group G and in particular for $G = SO(n)$.

1. INTRODUCTION

In 1959 P.Lévy [6] asked the question of the existence of a process X indexed by the points of a metric space (\mathcal{X}, d) and generalizing the Brownian motion, i.e. of a real Gaussian process which would be centered, vanishing at some point $x_0 \in \mathcal{X}$ and such that $\mathbb{E}(|X_x - X_y|^2) = d(x, y)$. By polarization, the covariance function of such a process would be

$$(1.1) \quad K(x, y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$$

so that this question is equivalent to the fact that the kernel K is positive definite. Positive definiteness of K for $\mathcal{X} = \mathbb{R}^m$ and d the Euclidean metric had been proved by Schoenberg [14] in 1938 and P.Lévy itself constructed the Brownian motion on $\mathcal{X} = \mathbb{S}^{m-1}$, the euclidean sphere of \mathbb{R}^m , d being the distance along the geodesics. Later Gangolli [12] gave an analytical proof of the positive definiteness of the kernel (1.1) for the same metric space (\mathbb{S}^{m-1}, d) , in a paper that dealt with this question for a large class of homogeneous spaces.

Finally Takenaka in [13] proved the positive definiteness of the kernel (1.1) for the Riemannian metric spaces of constant sectional curvature equal to $-1, 0$ or 1 , therefore adding the hyperbolic disk to the list. To be precise in the case of the hyperbolic space $\mathcal{H}_m = \{(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} : x_1^2 + \dots x_m^2 - x_0^2 = 1\}$, the distance under consideration is the unique, up to multiplicative constants, Riemannian distance that is invariant with respect to the action of $G = L_m$, the Lorentz group.

In this short note we investigate this question for the cases $\mathcal{X} = SO(n)$. The answer is that the kernel (1.1) is not positive definite on $SO(n)$ for $n > 2$. This is somehow surprising as, in particular, $SO(3)$ is locally isometric to $SU(2)$, where positive definiteness of the kernel K is immediate as shown below.

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We have been led to the question of the existence of the Brownian motion indexed by the elements of these groups - in particular of $SO(3)$ - in connection with the analysis and the modeling of the Cosmic Microwave Background which has become recently an active research field (see [5], [7], [8], [9] e.g.) and that has attracted the attention to the study of random fields ([1], [2], [11] e.g.). More precisely, in the modern cosmological models the CMB is seen as the realization of an invariant random field in a vector bundle over the sphere \mathbb{S}^2 and the analysis of its components (the polarization e.g.) requires the *spin* random fields theory. This leads naturally to the investigation of invariant random fields on $SO(3)$ enjoying particular properties and therefore to the question of the existence of a privileged random field i.e. Lévy's Brownian random field on $SO(3)$.

In §2 we recall some elementary facts about invariant distances and positive definite kernels. In §3 we treat the case $G = SU(2)$, recalling well known facts about the invariant distance and Haar measure of this group. Positive definiteness of K for $SU(2)$ is just a simple remark, but these facts are needed in §4 where we treat the case $SO(3)$ and deduce from the case $SO(n)$, $n \geq 3$.

2. SOME ELEMENTARY FACTS

In this section we recall some well known facts about Lie groups (see mainly [3] and also [4], [15]).

2.1. Invariant distance of a compact Lie group. From now on we denote by G a compact Lie group. It is well known that G admits at least a bi-invariant Riemannian metric (see [4] p.66 e.g.), that we shall denote $\{\langle \cdot, \cdot \rangle_g\}_{g \in G}$ where of course $\langle \cdot, \cdot \rangle_g$ is the inner product defined on the tangent space $T_g G$ to the manifold G at g and the family $\{\langle \cdot, \cdot \rangle_g\}_{g \in G}$ smoothly depends on g . By the bi-invariance property, for $g \in G$ the diffeomorphisms L_g and R_g (resp. the left multiplication and the right multiplication of the group) are isometries. Since the tangent space $T_g G$ at any point g can be translated to the tangent space $T_e G$ at the identity element e of the group, the metric $\{\langle \cdot, \cdot \rangle_g\}_{g \in G}$ is completely characterized by $\langle \cdot, \cdot \rangle_e$. Moreover, $T_e G$ being the Lie algebra \mathfrak{g} of G , the bi-invariant metric corresponds to an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} which is invariant under the adjoint representation Ad of G . Indeed there is a one-to-one correspondence between bi-invariant Riemannian metrics on G and Ad -invariant inner products on \mathfrak{g} . If in addition \mathfrak{g} is semisimple, then the negative Killing form of G is an Ad -invariant inner product on \mathfrak{g} itself.

If there exists a unique (up to a multiplicative factor) bi-invariant metric on G (for a sufficient condition see [4], Th. 2.43) and \mathfrak{g} is semisimple, then this metric is necessarily proportional to the negative Killing form of \mathfrak{g} . It is well known that this is the case for $SO(n)$, ($n \neq 4$) and $SU(n)$; furthermore, the (natural) Riemannian metric on $SO(n)$ induced by the embedding $SO(n) \hookrightarrow \mathbb{R}^{n^2}$ corresponds to the negative Killing form of $so(n)$.

Endowed with this bi-invariant Riemannian metric, G becomes a metric space, with a distance d which is bi-invariant. Therefore the function $g \in G \rightarrow d(g, e)$ is a class

function as

$$(2.1) \quad d(g, e) = d(hg, h) = d(hgh^{-1}, hh^{-1}) = d(hgh^{-1}, e), \quad g, h \in G.$$

It is well known that geodesics on G through the identity e are exactly the one parameter subgroups of G (see [10] p.113 e.g.), thus a geodesic from e is the curve on G

$$\gamma_X(t) : t \in [0, 1] \rightarrow \exp(tX)$$

for some $X \in \mathfrak{g}$. The length of this geodesic is

$$L(\gamma_X) = \|X\| = \sqrt{\langle X, X \rangle}.$$

Therefore

$$d(g, e) = \inf_{X \in \mathfrak{g} : \exp X = g} \|X\|.$$

2.2. Brownian kernels on a metric space. Let (\mathcal{X}, d) be a metric space.

Lemma 2.1. *The kernel K in (1.1) is positive definite on \mathcal{X} if and only if d is a restricted negative definite kernel, i.e., for every choice of elements $x_1, \dots, x_n \in \mathcal{X}$ and of complex numbers ξ_1, \dots, ξ_n with $\sum_{i=1}^n \xi_i = 0$*

$$(2.2) \quad \sum_{i,j=1}^n d(x_i, x_j) \xi_i \bar{\xi}_j \leq 0.$$

Proof. For every $x_1, \dots, x_n \in \mathcal{X}$ and complex numbers ξ_1, \dots, ξ_n

$$(2.3) \quad \sum_{i,j} K(x_i, x_j) \xi_i \bar{\xi}_j = \frac{1}{2} \left(\bar{a} \sum_i d(x_i, x_0) \xi_i + a \sum_j d(x_j, x_0) \bar{\xi}_j - \sum_{i,j} d(x_i, x_j) \xi_i \bar{\xi}_j \right)$$

where $a := \sum_i \xi_i$. If $a = 0$ then it is immediate that in (2.3) the l.h.s. is ≥ 0 if and only if the r.h.s. is ≤ 0 . Otherwise set $\xi_0 := -a$ so that $\sum_{i=0}^n \xi_i = 0$. The following equality

$$(2.4) \quad \sum_{i,j=0}^n K(x_i, x_j) \xi_i \bar{\xi}_j = \sum_{i,j=1}^n K(x_i, x_j) \xi_i \bar{\xi}_j$$

is then easy to check, keeping in mind that $K(x_i, x_0) = K(x_0, x_j) = 0$, which finishes the proof. \square

For a more general proof see [12] p. 127 in the proof of Lemma 2.5.

If \mathcal{X} is the homogeneous space of some topological group G , and d is a G -invariant distance, then (2.2) is satisfied if and only if for every choice of elements $g_1, \dots, g_n \in G$ and of complex numbers ξ_1, \dots, ξ_n with $\sum_{i=1}^n \xi_i = 0$

$$(2.5) \quad \sum_{i,j=1}^n d(g_i g_j^{-1} x_0, x_0) \xi_i \bar{\xi}_j \leq 0$$

where $x_0 \in \mathcal{X}$ is a fixed point. We shall say that the function $g \in G \rightarrow d(gx_0, x_0)$ is restricted negative definite on G if it satisfies (2.5).

In our case of interest $\mathcal{X} = G$ a compact (Lie) group and d is a bi-invariant distance as in §2.1. The Peter-Weyl development (see [3] e.g.) for the class function $d(\cdot, e)$ on G is

$$(2.6) \quad d(g, e) = \sum_{\ell \in \widehat{G}} \alpha_\ell \chi_\ell(g)$$

where \widehat{G} denotes the family of equivalence classes of irreducible representations of G and χ_ℓ the character of the ℓ -th irreducible representation of G .

Remark 2.2. A function ϕ with a development as in (2.6) is restricted negative definite if and only if $\alpha_\ell \leq 0$ but for the trivial representation.

Actually note first that, by standard approximation arguments, ϕ is restricted negative definite if and only if for every continuous function $f : G \rightarrow \mathbb{C}$ with 0-mean (i.e. orthogonal to the constants)

$$(2.7) \quad \int_G \int_G \phi(gh^{-1}) f(g) \overline{f(h)} dg dh \leq 0$$

dg denoting the Haar measure of G . Choosing $f = \chi_\ell$ in the l.h.s. of (2.7) and denoting d_ℓ the dimension of the corresponding representation, a straightforward computation gives

$$(2.8) \quad \int_G \int_G \phi(gh^{-1}) \chi_\ell(g) \overline{\chi_\ell(h)} dg dh = \frac{\alpha_\ell}{d_\ell}$$

so that if ϕ restricted negative definite, $\alpha_\ell \leq 0$ necessarily.

Conversely, if $\alpha_\ell \leq 0$ but for the trivial representation, then ϕ is restricted negative definite, as the characters χ_ℓ 's are positive definite and orthogonal to the constants.

3. $SU(2)$

The special unitary group $SU(2)$ consists of the complex unitary 2×2 -matrices g such that $\det(g) = 1$. Every $g \in SU(2)$ has the form

$$(3.1) \quad g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1.$$

If $a = a_1 + ia_2$ and $b = b_1 + ib_2$, then the map

$$(3.2) \quad \Phi(g) = (a_1, a_2, b_1, b_2)$$

is an homeomorphism (see [3], [15] e.g.) between $SU(2)$ and the unit sphere \mathbb{S}^3 of \mathbb{R}^4 . Moreover the right translation

$$R_g : h \rightarrow hg, \quad h, g \in SU(2)$$

of $SU(2)$ is a rotation (an element of $SO(4)$) of \mathbb{S}^3 (identified with $SU(2)$). The homeomorphism (3.2) preserves the invariant measure, i.e., if dg is the normalized

Haar measure on $SU(2)$, then $\Phi(dg)$ is the normalized Lebesgue measure on \mathbb{S}^3 . As the 3-dimensional polar coordinates on \mathbb{S}^3 are

$$(3.3) \quad \begin{aligned} a_1 &= \cos \theta, \\ a_2 &= \sin \theta \cos \varphi, \\ b_1 &= \sin \theta \sin \varphi \cos \psi, \\ b_2 &= \sin \theta \sin \varphi \sin \psi, \end{aligned}$$

$(\theta, \varphi, \psi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi]$, the normalized Haar integral of $SU(2)$ for an integrable function f is

$$(3.4) \quad \int_{SU(2)} f(g) dg = \frac{1}{2\pi^2} \int_0^\pi \sin \varphi d\varphi \int_0^\pi \sin^2 \theta d\theta \int_0^{2\pi} f(\theta, \varphi, \psi) d\psi$$

The bi-invariant Riemannian metric on $SU(2)$ is necessarily proportional to the negative Killing form of its Lie algebra $su(2)$ (the real vector space of 2×2 anti-hermitian complex matrices). We consider the bi-invariant metric corresponding to the Ad -invariant inner product on $su(2)$

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY), \quad X, Y \in su(2).$$

Therefore as an orthonormal basis of $su(2)$ we can consider the matrices

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The homeomorphism (3.2) is actually an isometry between $SU(2)$ endowed with this distance and \mathbb{S}^3 . Hence the restricted negative definiteness of the kernel d on $SU(2)$ is an immediate consequence of this property on \mathbb{S}^3 which is known to be true as mentioned in the introduction ([12], [6], [13]). In order to develop a comparison with $SO(3)$, we shall give a different proof of this fact in §5.

4. $SO(n)$

We first investigate the case $n = 3$. The group $SO(3)$ can also be realized as a quotient of $SU(2)$. Actually the adjoint representation Ad of $SU(2)$ is a surjective morphism from $SU(2)$ onto $SO(3)$ with kernel $\{\pm e\}$ (see [3] e.g.). Hence the well known result

$$(4.1) \quad SO(3) \cong SU(2)/\{\pm e\}.$$

Let us explicitly recall this morphism: if $a = a_1 + ia_2, b = b_1 + ib_2$ with $|a|^2 + |b|^2 = 1$ and

$$\tilde{g} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

then the orthogonal matrix $Ad(\tilde{g})$ is given by

$$(4.2) \quad g = \begin{pmatrix} a_1^2 - a_2^2 - (b_1^2 - b_2^2) & -2a_1a_2 - 2b_1b_2 & -2(a_1b_1 - a_2b_2) \\ 2a_1a_2 - 2b_1b_2 & (a_1^2 - a_2^2) + (b_1^2 - b_2^2) & -2(a_1b_2 + a_2b_1) \\ 2(a_1b_1 + a_2b_2) & -2(-a_1b_2 + a_2b_1) & |a|^2 - |b|^2 \end{pmatrix}$$

The isomorphism in (4.1) might suggest that the positive definiteness of the Brownian kernel on $SU(2)$ implies a similar result for $SO(3)$. This is not true and actually it turns out that the distance $(g, h) \rightarrow d(g, h)$ on $SO(3)$ induced by its bi-invariant Riemannian metric *is not a restricted negative definite kernel* (see Lemma 2.1).

As for $SU(2)$, the bi-invariant Riemannian metric on $SO(3)$ is proportional to the negative Killing form of its Lie algebra $so(3)$ (the real 3×3 antisymmetric real matrices). We shall consider the Ad -invariant inner product on $so(3)$ defined as

$$\langle A, B \rangle = -\frac{1}{2} \text{tr}(AB) , \quad A, B \in so(3) .$$

An orthonormal basis for $so(3)$ is therefore given by the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly to the case of $SU(2)$, it is easy to compute the distance from $g \in SO(3)$ to the identity. Actually g is conjugated to the matrix of the form

$$\Delta(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(tA_1)$$

where $t \in [0, \pi]$ is the rotation angle of g . Therefore if d still denotes the distance induced by the bi-invariant metric,

$$d(g, e) = d(\Delta(t), e) = t$$

i.e. the distance from g to e is the rotation angle of g .

Let us denote $\{\chi_\ell\}_{\ell \geq 0}$ the set of characters for $SO(3)$. It is easy to compute the Peter-Weyl development in (2.6) for $d(\cdot, e)$ as the characters χ_ℓ are also simple functions of the rotation angle. More precisely, if t is the rotation angle of g (see [8] e.g.),

$$\chi_\ell(g) = \frac{\sin \frac{(2\ell+1)t}{2}}{\sin \frac{t}{2}} = 1 + 2 \sum_{m=1}^{\ell} \cos(mt) .$$

We shall prove that the coefficient

$$\alpha_\ell = \int_{SO(3)} d(g, e) \chi_\ell(g) dg$$

is positive for some $\ell \geq 1$. As both $d(\cdot, e)$ and χ_ℓ are functions of the rotation angle t , we have

$$\alpha_\ell = \int_0^\pi t \left(1 + 2 \sum_{j=1}^{\ell} \cos(jt) \right) p_T(t) dt$$

where p_T is the density of $t = t(g)$, considered as a r.v. on the probability space $(SO(3), dg)$. The next statements are devoted to the computation of the density p_T .

This is certainly well known but we were unable to find a reference in the literature. We first compute the density of the trace of g .

Proposition 4.1. *The distribution of the trace of a matrix in $SO(3)$ with respect to the normalized Haar measure is given by the density*

$$(4.3) \quad f(y) = \frac{1}{2\pi} (3-y)^{1/2} (y+1)^{-1/2} 1_{[-1,3]}(y) .$$

Proof. The trace of the matrix (4.2) is equal to

$$\mathrm{tr}(g) = 3a_1^2 - a_2^2 - b_1^2 - b_2^2 .$$

Under the normalized Haar measure of $SU(2)$ the vector (a_1, a_2, b_1, b_2) is uniformly distributed on the sphere \mathbb{S}^3 . Recall the normalized Haar integral (3.4) so that, taking the corresponding marginal, θ has density

$$(4.4) \quad f_1(\theta) = \frac{2}{\pi} \sin^2(\theta) d\theta .$$

Now

$$3a_1^2 - a_2^2 - b_1^2 - b_2^2 = 4 \cos^2 \theta - 1 .$$

Let us first compute the density of $Y = \cos^2 X$, where X is distributed according to the density (4.4). This is elementary as

$$F_Y(t) = \mathbb{P}(\cos^2 X \leq t) = \mathbb{P}(\arccos(\sqrt{t}) \leq X \leq \arccos(-\sqrt{t})) = \frac{2}{\pi} \int_{\arccos(\sqrt{t})}^{\arccos(-\sqrt{t})} \sin^2(\theta) d\theta .$$

Taking the derivative it is easily found that the density of Y is, for $0 < t < 1$,

$$F'_Y(t) = \frac{2}{\pi} (1-t)^{1/2} t^{-1/2} .$$

By an elementary change of variable the distribution of the trace $4Y - 1$ is therefore given by (4.3). □

Corollary 4.2. *The distribution of the rotation angle of a matrix in $SO(3)$ is*

$$p_T(t) = \frac{1}{\pi} (1 - \cos t) 1_{[0,\pi]}(t) .$$

Proof. It suffices to remark that if t is the rotation angle of g , then its trace is equal to $2 \cos t + 1$. p_T is therefore the distribution of $W = \arccos(\frac{Y-1}{2})$, Y being distributed as (4.3). The elementary details are left to the reader. □

Now it is easy to compute the Fourier development of the function $d(\cdot, e)$.

Proposition 4.3. *The kernel d on $SO(3)$ is not restricted negative definite.*

Proof. It is enough to show that in the Fourier development

$$d(g, e) = \sum_{\ell \geq 0} \alpha_\ell \chi_\ell(g)$$

$\alpha_\ell > 0$ for some $\ell \geq 1$ (see Remark 2.2). We have

$$\begin{aligned} \alpha_\ell &= \int_{SO(3)} d(g, e) \chi_\ell(g) dg = \frac{1}{\pi} \int_0^\pi t \left(1 + 2 \sum_{m=1}^\ell \cos(mt) \right) (1 - \cos t) dt = \\ &= \frac{1}{\pi} \underbrace{\int_0^\pi t(1 - \cos t) dt}_{:=I_1} + \frac{2}{\pi} \sum_{m=1}^\ell \underbrace{\int_0^\pi t \cos(mt) dt}_{:=I_2} - \frac{2}{\pi} \sum_{m=1}^\ell \underbrace{\int_0^\pi t \cos(mt) \cos t dt}_{:=I_3} . \end{aligned}$$

Now integration by parts gives

$$I_1 = \frac{\pi^2}{2} + 2, \quad I_2 = \frac{(-1)^m - 1}{m^2} ,$$

whereas, if $m \neq 1$, we have

$$I_3 = \int_0^\pi t \cos(mt) \cos t dt = \frac{m^2 + 1}{(m^2 - 1)^2} ((-1)^m + 1)$$

and for $m = 1$,

$$I_3 = \int_0^\pi t \cos^2 t dt = \frac{\pi^2}{4} .$$

Putting things together we find

$$\alpha_\ell = \frac{2}{\pi} \left(1 + \sum_{m=1}^\ell \frac{(-1)^m - 1}{m^2} + \sum_{m=2}^\ell \frac{m^2 + 1}{(m^2 - 1)^2} ((-1)^m + 1) \right) .$$

If $\ell = 2$, for instance, we find $\alpha_2 = \frac{2}{9\pi} > 0$, but it is easy to see that $\alpha_\ell > 0$ for every ℓ even. □

Consider now the case $n > 3$. $SO(n)$ contains a closed subgroup H that is isomorphic to $SO(3)$ and the restriction to H of any bi-invariant distance d on $SO(n)$ is a bi-invariant distance \tilde{d} on $SO(3)$. By Proposition 4.3, \tilde{d} is not restricted negative definite, therefore there exist $g_1, g_2, \dots, g_m \in H$, $\xi_1, \xi_2, \dots, \xi_m \in \mathbb{R}$ with $\sum_{i=1}^m \xi_i = 0$ such that

$$(4.5) \quad \sum_{i,j} d(g_i, g_j) \xi_i \xi_j = \sum_{i,j} \tilde{d}(g_i, g_j) \xi_i \xi_j > 0 .$$

We have therefore

Corollary 4.4. *Any bi-invariant distance d on $SO(n)$, $n \geq 3$ is not a restricted negative definite kernel.*

Remark that the bi-invariant Riemannian metric on $SO(4)$ is not unique, meaning that it is not necessarily proportional to the negative Killing form of $so(4)$. In this case Corollary 4.4 states that every such bi-invariant distance cannot be restricted negative definite.

5. FINAL REMARKS

We were intrigued by the different behavior of the invariant distance of $SU(2)$ and $SO(3)$ despite these groups are locally isometric and decided to compute also for $SU(2)$ the development

$$(5.1) \quad d(g, e) = \sum_{\ell} \alpha_{\ell} \chi_{\ell}(g) .$$

This is not difficult as, denoting by t the distance of g from e , the characters of $SU(2)$ are

$$\chi_{\ell}(g) = \frac{\sin((\ell+1)t)}{\sin t}, \quad t \neq k\pi$$

and $\chi_{\ell}(e) = \ell + 1$ if $t = 0$, $\chi_{\ell}(-) = (-1)^{\ell}(\ell + 1)$ if $t = \pi$. Then it is elementary to compute, for $\ell > 0$,

$$\alpha_{\ell} = \frac{1}{\pi} \int_0^{\pi} t \sin((\ell+1)t) \sin t \, dt = \begin{cases} -\frac{8}{\pi} \frac{m+1}{m^2(m+2)^2} & \ell \text{ odd} \\ 0 & \ell \text{ even} \end{cases}$$

thus confirming the restricted negative definiteness of d (see Remark 2.2). Remark also that the coefficients corresponding to the even numbered representations, that are also representations of $SO(3)$, here vanish.

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