APPROXIMATE NORMALITY OF HIGH-ENERGY HYPERSPHERICAL EIGENFUNCTIONS

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Abstract. The Berry heuristic has been a long standing *ansatz* about the high energy (i.e. large eigenvalues) behaviour of eigenfunctions (see [8]). Roughly speaking, it states that under some generic boundary conditions, these eigenfunctions exhibit Gaussian behaviour when the eigenvalues grow to infinity. Our aim in this paper is to make this statement quantitative and to establish some rigorous bounds on the distance to Gaussianity, focussing on the hyperspherical case (i.e., for eigenfunctions of the Laplace-Beltrami operator on the normalized d-dimensional sphere - also known as spherical harmonics). Some applications to non-Gaussian models are also discussed.

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1. Introduction and Notation

The Berry heuristic has been a long standing *ansatz* about the high energy (i.e. large eigenvalues) behaviour of eigenfunctions (see [8]). Roughly speaking, it states that under some generic boundary conditions, these eigenfunctions exhibit Gaussian behaviour when the eigenvalues grow to infinity (see below for more discussion and details). Our aim in this paper is to make this statement quantitative and to establish some rigorous bounds on the distance to Gaussianity, focussing on the hyperspherical case (i.e., for eigenfunctions of the Laplace-Beltrami operator on the normalized d-dimensional unit sphere - also known as spherical harmonics).

The first of our formulations involves a wide class of geometric functionals satisfying mild regularity conditions, including for example the excursion area, the density of critical points above some threshold and the normalized Euler-Poincaré characteristic of excursion sets (see below for details). By the Berry heuristic, one should expect that for high energy, such functionals, when evaluated at a typical eigenfunction, should be close to the expectation of the same functionals evaluated in the corresponding Gaussian case. To make this statement quantitative, we show that the Lebesgue measure of eigenfunctions for which the difference of the two aforementioned quantities lies above some vanishing threshold tends to zero in the high-energy limit with some explicit rate. In the same spirit, we prove that an analogous statement remains true if the geometric functionals are replaced by supremum norms, again with quantitative bounds. In particular, we show that the square of the supremum norms is typically of logarithmic order;

this latter finding complements some earlier investigation on the L^4-norm of spherical eigenfunctions in dimension two by Sogge and Zelditch (see [29]).

As a second characterization, we focus on eigenfunctions which are evaluated at a random point on the sphere and establish tight bounds on the probability distances between these random variables and standard Gaussians. In this setting, for dimension $d \geq 3$, we are furthermore able to establish a form of almost sure convergence: consider sequences of eigenfunctions containing a subsequence whose Kolmogorov distance to a Gaussian stays above some vanishing threshold. Then the measure of such sequences is zero (see below for precise statements). In this second characterization, asymptotic Gaussianity is also exhibited if the eigenvalue is kept fixed while the dimension of the underlying hypersphere grows to infinity; this is related to the approach by Meckes (see [20]).

Finally, as an application, we investigate some non-Gaussian models. Assuming some regularity on the sequence of probability measures (existence of Lebesgue densities with suitable growth constraint), we show that the asymptotic behaviour of functionals of excursion sets can be expressed in terms of the Gaussian limits, evaluated at random excursion levels and depending on the L^2-norm of sample paths. In particular, when this random norm converges to a deterministic limit, Gaussian behaviour follows; this way, we partially confirm an earlier conjecture from [15] on the relationship between convergence of sample norms and asymptotic Gaussianity.

Let us now fix the mathematical framework. For a given dimension $d \geq 2$, consider the orthonormal family

$$\{\, Y_{\ell m} \mid \ell \in \mathbb{N}_0, m = 1, \dots, n_{\ell d}\}$$

of real hyperspherical harmonics on the normalized hypersphere S^d, i.e.

$$\Delta_{S^d} Y_{\ell m} = -\ell(\ell + d - 1) Y_{\ell m}, \qquad \ell \in \mathbb{N}_0, \qquad m = 1, \dots, n_{\ell d},$$

where

$$n_{\ell d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1}$$

is the dimension of the eigenspace corresponding to the eigenvalue $-\ell(\ell + d - 1)$ of the Laplace-Beltrami operator Δ_{S^d} (see for example [3, 4, 14, 16]). Elementary computations show that

$$\lim_{\ell \to \infty} \frac{n_{\ell d}}{\ell^{d-1}} = \frac{2}{(d-1)!} \qquad \text{and} \qquad \lim_{d \to \infty} \frac{n_{\ell d}}{d^\ell} = \frac{1}{(\ell - 1)!}. \tag{1}$$

As anticipated above, we consider hyperspherical eigenfunctions $h_{\alpha,\ell}$ defined by

$$h_{\alpha,\ell}(x) = \sum_{m=1}^{n_{\ell d}} \alpha_{\ell m} Y_{\ell m}(x), \tag{2}$$

where $\alpha_\ell = (\alpha_{\ell 1}, \dots, \alpha_{\ell n_{\ell d}}) \in S^{n_{\ell d}}$ is a vector of coefficients. Taking $\alpha_\ell \in S^{n_{\ell d}}$ corresponds to the normalization

$$\|h_{\alpha,\ell}\|_{L^2}^2 = \int_{S^d} h_{\alpha,\ell}^2(x) d\mu(x) = \sum_{m=1}^{n_{\ell d}} \alpha_{\ell m}^2 = 1,$$

where μ represents the normalized Lebesgue measure on S^d. The link to Gaussian eigenfunctions comes through the following natural and standard randomization of the coefficient vector: Considering $(S^{n_{\ell d}}, \mathcal{F}_\ell, \mu_\ell)$ as a probability space, where $\mathcal{F}_\ell$ is the Borel σ-algebra and μ_ℓ the normalized Lebesgue measure, we construct the probability space $(\Omega_\ell, \mathcal{F}^*, \mu_\ell^*)$, where $\Omega_\ell = \mathbb{R}_+ \times S^{n_{\ell d}}$, $\mathcal{F}_\ell^* = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\ell$ and $\mu_\ell^* = \nu_\ell \otimes \mu_\ell$. Here, ν_ℓ is the measure induced by a random radius R_ℓ defined as

$$R_\ell = \sqrt{\frac{X_{\ell,d}}{n_{\ell d}}},$$

where $X_{\ell,d} \sim \chi^2(n_{\ell d})$ is a random varioable distributed as a chi-square with $n_{\ell d}$ degrees of freedom. It is straightforward to check that under the measure μ_ℓ^*, the random vector

$$u_\ell = (u_{\ell 1}, \dots, u_{\ell n_{\ell d}}) \colon \Omega_\ell \to \mathbb{R}^{n_{\ell d}},$$

defined by $u_\ell(r, \alpha) = r \cdot \alpha$, have a multivariate Gaussian distribution with covariance matrix $n_{\ell d}^{-1} I_{n_{\ell d}}$, I_n denoting as usual the identity matrix of order n. We can hence introduce the zero-mean, unit variance Gaussian eigenfunctions

$$T_\ell(x) = T_\ell(x; r, \alpha) := \sum_{m=1}^{n_{\ell d}} u_{\ell m}(r, \alpha) Y_{\ell m}(x). \tag{3}$$

Note that because we normalize the spherical measure to unity, we also have $\mathbb{E} \|T_\ell\|^2_{L^2(S^d)} = \mathbb{E}[R_\ell^2] = 1$, where, here and in the following, $\mathbb{E}$ denotes mathematical expectation.

The rest of this paper is organized as follows: the main results are stated in Section 2 and applied in Section 3 to some non-Gaussian models. Most proofs are in Section 4, while we have collected some technical lemmas in an appendix.

2. Main results

As stated in the introduction, we establish some quantitative result on alternative versions of the Berry heuristics. Our first approach is to focus on geometric functionals; we shall then consider the behaviour of supremum norms, and finally eigenfunctions evaluated on random points.

2.1. Excursion functionals.

Let us first start from the standard definition of excursion sets and monotonic functionals (see for example [1]).

Definition 1 (Excursion set). *Let $f \colon S^d \to \mathbb{R}$ be a real valued function; the excursion set of f above $u \in \mathbb{R}$ is defined by*

$$A_u(f; S^d) := \left\{ x \in S^d : f(x) \geq u \right\} .$$

Definition 2 (Monotonic excursion functional). *A functional $g \colon \mathcal{B}(S^d) \to \mathbb{R}$ is called* monotonic *if either $A \subset B \Rightarrow g(A) \leq g(B)$ for all $A, B \in \mathcal{B}(S^d)$ or $A \subset B \Rightarrow g(A) \geq g(B)$ for all $A, B \in \mathcal{B}(S^d)$.*

Remark 3. *For excursion sets, we shall typically adopt the simpler notation*

$$g(A_u(f;S^d)) =: g(f,u)\ .$$

Note that, for every $c>0$, it holds that

$$g(f,u) = g(cf,cu), \tag{4}$$

and of course g is monotonic with respect to u, indeed either $u' \le u \Rightarrow g(f,u') \ge g(f,u)$ or $u' \le u \Rightarrow g(f,u') \le g(f,u)$ for fixed $f \in L^2(S^d)$.

Examples of excursion functionals which have been studied in Gaussian circumstances are excursion volumes (see [19, 17]) and the cardinality of critical points (see [22, 9, 10]); as we shall show, other functionals such as the normalized Euler-Poincaré characteristics satisfy more general conditions, but can be easily be brought into the scope of our results below by simple manipulations. For our results, we need to impose the following regularity property. As discussed below, this condition is met by all the examples we mentioned.

Definition 4 (regular family of monotonic excursion functionals)**.** *Let T_ℓ be a Gaussian random field of the form* (3)*. A family $\{g_\ell \colon \ell \in \mathbb{N}\}$ of monotonic excursion functionals is called* regular*, if the following conditions are verified.*

(1) For all $u \in \mathbb{R}$ it holds that

$$\mathbb{E}\left[g_\ell(T_\ell,u)\right] = \Psi_\ell(u) \to \Psi(u), \qquad (\ell \to \infty), \tag{5}$$

where Ψ_ℓ is differentiable and, uniformly over ℓ,

$$\left|\Psi'_\ell(u)\right| \le \frac{c}{1+|u|} \tag{6}$$

for some $c>0$.

(2)

$$\sup_u \operatorname{Var}\left(g_\ell(T_\ell,u)\right) = o(\ell). \tag{7}$$

Remark 5. *Note that the sequences of functionals g_ℓ can be constant over ℓ; we use this formulation for cases where we focus on quantities whose expected values is bounded, as for instance for the excursion volumes. On the other hand, to deal with diverging expected values, as for instance the number of critical points above a threshold u or the Euler-Poincaré characteristic (see below), it is convenient to introduce a normalization depending on ℓ.*

Example 6. *Simple examples satisfying the previous conditions are obtained by suppressing the dependence on u and considering functionals of the form $\int_{S^d} M(T_\ell)dx$, where M is measurable such that $\mathbb{E}\left[M(Z)^2\right] < \infty$, Z being standard Gaussian. Indeed, under these circumstances it was proved in* [17] *that the variance is of order $1/n_{\ell d}$. Likewise, we could consider functionals of the form $\int_{A_u(f;S^d)} M(T_\ell)dx$ where the function M is even and nonnegative: under these circumstances, it is indeed possible to show that the variance is itself a monotonic (increasing) functional of the excursion set.*

Let us introduce some more notation: as before, let μ_ℓ be the normalized Lebesgue measure on $S^{n_{\ell d}}$; moreover, let $\{g_\ell \colon \ell \in \mathbb{N}\}$ be a sequence of regular excursion functionals. Set

$$\sigma_\ell^2(u) = \mathrm{Var}\left(g_\ell(T_\ell, u)\right), \qquad \sigma_\ell^2 = \sup_u \sigma_\ell^2(u)$$

and

$$G_\ell(\varepsilon, u) = \left\{\alpha_\ell \in S^{n_{\ell d}} \colon \left|g_\ell(h_{\alpha,\ell}, u) - \mathbb{E}\left[g_\ell(T_\ell, u)\right]\right| > \varepsilon\right\},$$

where the eigenfunctions $h_{\alpha,\lambda}$ and T_ℓ are defined by (2) and (3), respectively. A quantitative version of the Berry heuristic for regular families of excursion functionals can now be stated as follows.

Theorem 7. *For all sequences $(\varepsilon_\ell)_{\ell \geq 0}$ with values in $(0,1)$ it holds that*

$$\sup_u \mu_\ell(G_\ell(\varepsilon_\ell, u)) \leq \frac{2(1+c)}{\varepsilon_\ell^2}\left(\frac{1}{n_{\ell d}} + \sigma_\ell^2\right), \tag{8}$$

where $c > 0$ is such that the regularity condition (6) holds. In particular, for vanishing sequences $\{\varepsilon_\ell\}_{\ell \geq 1}$ such that $1/n_{\ell d} + \sigma_\ell^2 = o(\varepsilon_\ell^2)$, we have that

$$\sup_u \mu_\ell(G_\ell(\varepsilon_\ell, u)) \to 0, \qquad (\ell \to \infty).$$

Loosely speaking, this Theorem gives a bound on the Lebesgue measure of eigenfunctions such that the corresponding geometric functionals diverge from Gaussian behaviour by more than a (vanishing) sequence $\{\varepsilon_\ell\}$.

Remark 8. *A careful examination of the proof of Theorem 7 shows that locally, the improved bound*

$$\mu_\ell(G_\ell(\varepsilon_\ell, u)) \leq \frac{2(1+c)}{\varepsilon_\ell^2}\left(\frac{1}{n_{\ell d}} + \max\left\{\sigma_\ell^2(u_-), \sigma_\ell^2(u_+)\right\}\right)$$

holds, where $u_- = \sqrt{1 - \frac{\varepsilon}{1+c}}u$ and $u_+ = \sqrt{1 + \frac{\varepsilon}{1+c}}u$.

Example 9. *The following examples are easily seen to be covered by Theorem 7; in particular, for excursion volumes we can take the functionals $\{g_\ell\}$ to be constant with respect to ℓ, while the remaining functionals require a normalization depending upon ℓ. Technical details to show that the assumptions of the Theorem hold are proved in the Appendix (see Lemmas 28, 29 and 30).*

(1) Excursion volume (see [17, 19, 24]) *The excursion volume $S(h_{\alpha,\ell}; u)$ is defined as*

$$S(h_{\alpha,\ell}; u) = \int_{S^d} 1_{[u,\infty)} h_{\alpha,\ell}(x)) dx, \tag{9}$$

where 1_A denotes as usual the indicator function of the set A. In this case, the function $\Psi(u) = 1 - \Phi(u)$ is simply the Gaussian tail probability, which is easily seen to satisfy the required regularity conditions. Moreover, it is shown in Lemma 28 below (see also [19, 17]*) that, for all $d \geq 2$*

$$\mathrm{Var}\left(S(T_\ell; u) - [1 - \Phi(u)]\right) = O\left(\frac{1}{n_{\ell d}}\right),$$

uniformly over u; *hence* (7) *is also fulfilled. Thus, Theorem 7 implies that*

$$\mu_\ell \left\{\alpha_\ell \colon \left|S(h_{\alpha,\ell};u) - 1 + \Phi(u)\right| > \varepsilon_\ell\right\} = O\left(\frac{1}{n_{\ell d}\varepsilon_\ell^2}\right)$$

and in particular, for all fixed $\varepsilon > 0$

$$\mu_\ell \left\{\alpha_\ell \colon \left|S(h_{\alpha,\ell};u) - 1 + \Phi(u)\right| > \varepsilon\right\} \to 0 \ , \ \text{as } \ell \to \infty.$$

(2) Normalized critical points, extrema and saddles (see [9]). *The critical points* N^c, *minima* N^{min}, *maxima* N^{max} *and saddles* N^s *are defined by*

$$\begin{aligned} N^c(h_{\alpha,\ell};u) &= \#\left\{x \in S^d \colon \nabla h_{\alpha,\ell}(x) = 0 \text{ and } h_{\alpha,\ell}(x) \geq u\right\}, \\ N^{min}(h_{\alpha,\ell};u) &= \#\left(N^c(h_{\alpha,\ell};u) \cap \left\{x \in S^d \colon \operatorname{Ind}\left\{\nabla^2 h_{\alpha,\ell}(x)\right\} = 0\right\}\right), \\ N^s(h_{\alpha,\ell};u) &= N^c(h_{\alpha,\ell};u) \setminus \left(N^{min}(h_{\alpha,\ell};u) \cup N^{max}(h_{\alpha,\ell};u)\right) \end{aligned}$$

where ∇ *and* ∇^2 *denote as usual the gradient and Hessian on the sphere, and, for a matrix* A, $\operatorname{Ind} A$ *denotes the number of negative eigenvalues of* A. *Fixing* $d = 2$ *it is shown in* [9] *that*

$$\lim_{\ell \to \infty} \frac{\mathbb{E}\left[N^b(T_\ell;u)\right]}{\ell^2} = \Psi^b(u)$$

where $b \in \{c, e, s\}$ *for critical points, extrema and saddles, respectively, and* $\Psi^b(u) = \int_u^\infty \psi^b(z)dz$, *where*

$$\begin{aligned} \psi^c(u) &= \frac{3}{\sqrt{2\pi}}(2e^{-t^2} + t^2 - 1)e^{-t^2}, \\ \psi^e(u) &= \frac{3}{\sqrt{2\pi}}(e^{-t^2} + t^2 - 1)e^{-t^2}, \\ \psi^s(u) &= \frac{3}{\sqrt{2\pi}}e^{-3t^2/2}. \end{aligned}$$

The functions $\Psi^b(u)$ *are easily seen to satisfy the conditions listed in Definition 4; in particular, it is shown in* [9] *that*

$$\operatorname{Var}\left(\frac{N^b(T_\ell;u)}{\ell^2} - \Psi^b(u)\right) = O\left(\frac{1}{\ell}\right)$$

uniformly over u *for* $b \in \{c, e, s\}$ *(in fact, an analytic expression for the leading constant is given, as a function of* u*). The fact that identity* (4) *is satisfied is shown in the appendix in Lemma 30. Hence, by Theorem 7, we have the asymptotics*

$$\mu_\ell\left\{\alpha_\ell \colon \left|\frac{N^b(h_{\alpha,\ell};u)}{\ell^2} - \Psi^b(u)\right| > \varepsilon_\ell\right\} = O\left(\frac{1}{\ell\varepsilon_\ell^2}\right),$$

and again this measure converges to zero for any fixed $\varepsilon > 0$.

Example 10 (Euler-Poincaré characteristic)**.** *For a number of alternative definitions of the Euler-Poincaré characteristic* χ *and its main properties, we refer to the monographs* [1, 2]. *For our purposes, it suffices to focus on the two-dimensional case* $d = 2$ *and excursion sets* $A_u(h_{\alpha,\ell})$ *(see Definition 1),* *where $\chi(A_u(h_{\alpha,\ell}))$ can be expressed by means of Morse's Theorem as the number of extrema minus the number of saddles, i.e.*

$$\chi(A_u(h_{\alpha,\ell};S^d)) = N^e(h_{\alpha,\ell};u) - N^s(h_{\alpha,\ell};u).$$

Equivalently, $\chi(A_u(h_{\alpha,\ell};S^d))$ can be viewed as the number of connected regions in A_u, minus the number of "holes". Its expected value for Gaussian random fields can be obtained by the celebrated Gaussian Kinematic Formula (see for example [1, 2]*). For $d=2$, we have that*

$$\mathbb{E}\left[\chi(A_u(T_\ell,S^2))\right] = 2(1-\Phi(u)) + \sqrt{\frac{2}{\pi}}\frac{\ell(\ell+1)}{2}\frac{u\phi(u)}{2} \; . \tag{10}$$

The variance has been given in [10] *to be*

$$\mathrm{Var}\left(\chi(A_u(T_\ell,S^2))\right) = \left((u^3+2u)^2\phi^2(u)\right)^2\frac{\ell^3}{8\pi} + O(\ell^2).$$

The function on the right-hand side of (10) *does not fulfill the monotonicity conditions of Definition 2. However, exploiting our previous results we obtain*

$$\begin{aligned}
&\mu_\ell\left\{\alpha_\ell : \left|\frac{\chi(A_u(h_{\alpha,\ell};S^2))}{\ell^2} - \frac{1}{\sqrt{2\pi}}ue^{-u^2/2}\right| \geq \varepsilon_\ell\right\}\\
&= \mu_\ell\left\{\alpha_\ell : \left|\frac{\mathcal{N}^e(h_{\alpha,\ell};u) - \mathcal{N}^s(h_{\alpha,\ell}\ell;u)}{\ell^2} - \frac{1}{\sqrt{2\pi}}ue^{-u^2/2}\right| \geq \varepsilon_\ell\right\}\\
&\leq \mu_\ell\left\{\alpha_\ell : \left|\frac{\mathcal{N}^e(h_{\alpha,\ell};u)}{\ell^2} - \Psi^e(u)\right| + \left|\frac{\mathcal{N}^s(h_{\alpha,\ell};u)}{\ell^2} - \Psi^s(u)\right| \geq \varepsilon_\ell\right\}\\
&\leq \mu_\ell\left\{\alpha_\ell : \left|\frac{\mathcal{N}^e(h_{\alpha,\ell};u)}{\ell^2} - \Psi^e(u)\right| \geq \frac{\varepsilon_\ell}{2}\right\}\\
&\qquad + \mu_\ell\left\{\alpha_\ell : \left|\frac{\mathcal{N}^s(h_{\alpha,\ell};u)}{\ell^2} - \Psi^s(u)\right| \geq \frac{\varepsilon_\ell}{2}\right\}\\
&= O\left(\frac{1}{\ell\varepsilon_\ell^2}\right),
\end{aligned}$$

where we have used the identity

$$\frac{1}{\sqrt{2\pi}}ue^{-u^2/2} = \frac{1}{\sqrt{2\pi}}\int_u^\infty (t^2-1)e^{-t^2/2}dt = \Psi^e(u) - \Psi^s(u),$$

and the first step can be found for example in [1, eq.11.6.12, p.289].

2.2. **The behaviour of L^∞-norms.** In this subsection, using the same approach as above, we show that the set of hyperspherical eigenfunctions whose squared L^∞-norm is of logarithmic order has asymptotically measure one. To do so, we first investigate two results on upper and lower bounds on the L^∞-norms in the Gaussian case, which are of some independent interest.

Proposition 11. *For all $d \geq 2$, the following is true.*

(1) There exists a constant $M>0$ such that

$$\mathbb{E}\left[\|T_\ell\|_\infty\right] \leq M\sqrt{\log \ell} \tag{11}$$

for all $\ell \in \mathbb{N}$.

(2) *For M satisfying* (11) *and $\beta > 0$, it holds that*

$$\mu_\ell^* \left\{ \|T_\ell\|_\infty \geq \left(M + \sqrt{2\beta}\right)\sqrt{\log \ell} \right\} \leq \frac{1}{\ell^\beta}.$$

(3) *For $0 < K < \sqrt{d/(12d+2)}$ and $2K^2/d < \alpha < 1/(6d+1)$ it holds that*

$$\mu_\ell^* \left\{ \|T_\ell\|_\infty \leq K\sqrt{\log \ell} \right\} = O\left(\frac{1}{\ell^{(1-(6d+1)\alpha)/2} \log \ell} \right).$$

Now, exploiting the same Gaussian approximation ideas as in the previous subsection, we can establish the following quantitative results on the behaviour of the supremum-norms for typical eigenfunctions.

Theorem 12. *For $d \geq 2$, the following is true.*

(1) *Let M be such that* (11) *is verified. Then, for all constants β, β' such that $0 < \beta' < \beta$ it holds that*

$$\mu_\ell \left\{ \alpha_\ell \colon \|h_{\alpha,\ell}\|_\infty \geq \left(M + \sqrt{2\beta}\right)\sqrt{\log \ell} \right\} = O\left(\frac{1}{\ell^{\beta'}} \right).$$

(2) *For $0 < K < \sqrt{d/(12d+2)}$ and $2K^2/d < \alpha < 1/(6d+1)$ it holds that*

$$\mu_\ell \left\{ \alpha_\ell \colon \|h_{\alpha,\ell}\|_\infty \leq K\sqrt{\log \ell} \right\} = O\left(\frac{1}{\ell^{(1-(6d+1)\alpha)/2} \log \ell} \right).$$

A standard Borel-Cantelli argument immediately yields the following result on the product space $\bigotimes_{\ell \geq 1} \Omega_\ell$.

Corollary 13. *In the setting and with the notation of Theorem 12, the set of eigenfunctions with fluctuations which are infinitely often larger than $\left(M + \sqrt{2\beta}\right)\sqrt{\log \ell}$ is zero, i.e.*

$$\mu_\infty \left\{ (\alpha_\ell)_{\ell \geq 1} \colon \|h_{\alpha,\ell}\|_\infty \geq (M + \sqrt{2\beta})\sqrt{\log \ell} \text{ infinitely often} \right\} = 0, \tag{12}$$

where μ_∞ denotes the product measure $\bigotimes_{\ell \geq 1} \mu_\ell$. Likewise, we have that

$$\mu_\infty \left\{ (\alpha_\ell)_{\ell \geq 1} \colon \|h_{\alpha,\ell}\|_\infty \leq K\sqrt{\log \ell} \text{ infinitely often} \right\} = 0. \tag{13}$$

Remark 14. *The behaviour of L^p norms for spherical eigenfunctions has been investigated by many authors. For eigenfunctions e_λ of the Laplace-Beltrami operator on a Riemannian manifold M such that $\Delta e_\lambda = -\lambda^2 e_\lambda$ and $\|e_\lambda\|_{L^2(M)} = 1$, Sogge* [28] *obtained the asymptotics*

$$\|e_\lambda\|_{L^p(M)} = O(\lambda^{\sigma(p)}), \qquad 2 < p \leq \infty,$$

where

$$\sigma(p) = \begin{cases} 2(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2} & \text{if } p \geq 6, \\ \frac{1}{2}(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 < p \leq 6. \end{cases}$$

On S^2, this implies in particular that $\|Y_{\ell m}\|_{L^4(S^2)} = O(\ell^{1/8})$ and $\|Y_{\ell m}\|_{L^\infty(S^2)} = O(\ell^{1/2})$; these estimates are known to be sharp, with the two limits achieved by $Y_{\ell\ell}$ and $Y_{\ell 0}$, respectively. However, for $d = 2$ it is also known that the typical eigenfunction has much smaller norms; in particular, it has been *shown more recently by Sogge and Zelditch in* [29] *that the average* L^4 *norm is logarithmic, more precisely*

$$\frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}\int_{S^2}|Y_{\ell m}(x)|^4\,dx = O(\log\ell)\ ,\ as\ \ell\to\infty\ .$$

Remark 15. *The behaviour of* L^∞ *norms for spherical harmonics plays a crucial role in the analysis of the recovery properties for sparse spherical signals by compressed sensing techniques, see for instance* [26],[25]*. The results we derived here may lead to improved bounds on reconstruction errors - we leave this as an issue for future research.*

2.3. **Random evaluation of eigenfunctions.** The previous subsections were concerned with the path behaviour of eigenfunctions, showing that the set where the latter differ from Gaussian behaviour is asymptotically negligible. In this section, we address a slightly different question, namely we investigate the behaviour of typical eigenfunctions when evaluated at a random point on the sphere; this is the same setting as considered in [20]. Let $X \sim U(S^d)$ be uniformly distributed on the d-dimensional sphere. In our notation, it is proved in [20] that for any fixed $\alpha \in S^{n_{\ell,d}}$, the random variable $h_{\alpha,\ell}(X) = \sum_m \alpha_{\ell m} Y_{\ell m}(X)$ is such that

$$d_{TV}(h_{\alpha,\ell}(X), Z) \le \frac{2}{\ell(\ell+d-1)}\int_{S^d}\left|\|\nabla h_{\alpha,\ell}(x)\|^2_{\mathbb{R}^d} - \left(\int_{S^d}\|\nabla h_{\alpha,\ell}(y)\|_{\mathbb{R}^d}\,dy\right)^2\right|d\,x$$

where d_{TV} denotes as usual the total variation distance between random variables, i.e.

$$d_{TV}(h_{\alpha,\ell}(X), Z) = \sup_{A\in\mathcal{B}(\mathbb{R})}\left|\int_{S^d} 1_A\left(h_{\alpha,\ell}(x))dx\right)dx - \int_A \phi(u)du\right| .$$

In the sequel, we focus instead on the (weaker) Kolmogorov distance, which is defined by

$$d_{Kol}(h_{\alpha,\ell}(X), Z) = \sup_u\left|\int_{S^d} 1_{[u,\infty)}\left(h_{\alpha,\ell}(x)\right)dx - \Phi(u)\right| ,$$

(see [11, 23] for more discussion on probability metrics). Our next result establishes the rate of convergence to asymptotic Gaussianity in the Kolmogorov distance for the typical eigenfunctions evaluated at a random point.

Theorem 16. *There exists a universal constant* K *such that for fixed* d *and for all positive sequences* $\varepsilon_{\ell,d}$ *it holds that*

$$\mu_\ell\left(\{\alpha_\ell\colon d_{Kol}(h_{\alpha,\ell,d}(X), Z) > \varepsilon_{\ell,d}\}\right) \le \frac{K}{n_{\ell d}\,\varepsilon^3_{\ell,d}} = O\left(\frac{1}{\ell^{d-1}\varepsilon^3_{\ell,d}}\right) \tag{14}$$

as $\ell\to\infty$*. Likewise, there exists a universal constant* K' *such that for fixed* ℓ *and for all positive sequences* $\varepsilon_{\ell,d}$ *it holds that*

$$\mu_\ell\left(\{\alpha_{\ell,d}\colon d_{Kol}(h_{\alpha,\ell,d}(X), Z) > \varepsilon_{\ell,d}\}\right) \le \frac{K}{n_{\ell d}\,\varepsilon^3_{\ell,d}} = O\left(\frac{1}{d^{\ell}\varepsilon^3_{\ell,d}}\right) \tag{15}$$

as $d\to\infty$*.*

Clearly, the previous proposition implies that for sequences $\varepsilon_{\ell,d} = \frac{s_{\ell,d}}{n_{\ell d}^{1/3}}$ where $s_{\ell,d} \to \infty$ it holds that

$$\mu_\ell\left(\{\alpha_{\ell,d}\colon d_{\mathrm{Kol}}(h_{\alpha,\ell,d}(X), Z) > \varepsilon_{\ell,d}\}\right) \to 0$$

as $\ell \to \infty$. It is important to stress that the second statement in the previous Theorem implies that asymptotic Gaussianity also holds for fixed ℓ and growing dimension $d \to \infty$. This is a setting closer to the assumptions of [20]; in particular, in Theorem 10 of that reference, it is proved that (again in our notation)

$$\mathbb{E}\left[d_{TV}(h_{\alpha,\ell,d}(X), Z)\right] \leq \frac{K}{\sqrt{d}}, \text{ some } K > 0, \tag{16}$$

where the expected value is taken with respect to the measure μ_ℓ, i.e. the uniform distribution of the coefficients $\alpha_{\ell,d}$. Working with Kolmogorov distance rather than Total Variation, we can strengthen the bound as follows.

Corollary 17. *In the setting of Theorem 16 and for fixed ℓ it holds that*

$$\mathbb{E}\left[d_{Kol}(h_{\alpha,\ell,d}(X), Z)\right] = O\left(\frac{1}{d^{\ell/3}}\right)$$

as $d \to \infty$. Also, for fixed d, it holds that

$$\mathbb{E}\left[d_{Kol}(h_{\alpha,\ell,d}(X), Z)\right] = O\left(\frac{1}{\ell^{(d-1)/3}}\right)$$

as $\ell \to \infty$.

For d large enough, the sequence $\frac{1}{n_{\ell d}\varepsilon_{\ell,d}^{1/3}}$ can be chosen to be summable over ℓ and the results we established can be formulated in terms of almost sure convergence by standard Borel-Cantelli arguments. The precise result is as follows.

Proposition 18. *Let $(\Omega, \mathcal{F}, \mu)$ be the product probability space of the spaces $(\Omega_\ell, \mathcal{F}_\ell, \mu_\ell)$. For all sequences $\{n_{\ell d}\}$ and $\{\varepsilon_\ell > 0\}$ such that*

$$\sum_{\ell=0}^{\infty} \frac{1}{n_{\ell d}\varepsilon_\ell^3} < \infty$$

it holds that

$$\mu\left(\{\, (\alpha_\ell)_{\ell=1}^\infty \colon d_{Kol}(h_{\alpha,\ell}(X), Z) > \varepsilon_\ell \text{ for infinitely many } \ell \,\}\right) = 0$$

Example 19. *For $d = 3$, $n_{\ell d}$ is of order ℓ^2 and the previous almost sure convergence result holds for all sequences ε_ℓ such that $1/\varepsilon_\ell = O(\ell^{1/3-\delta})$ for some $\delta > 0$. More generally, for $d \geq 3$, convergence will hold provided that ε_ℓ is of the form $\ell^{(2-d)/3+\delta}$ for some $\delta > 0$. Again, this shows that the Gaussian approximation becomes more and more accurate in higher dimension.*

3. Applications: Some non-Gaussian Models

The literature on geometric properties of random eigenfunctions has so far focused only on the case where the latter are normally distributed; in this section, we shall show how one can exploit the previous results to investigate the asymptotic behaviour of geometric functionals under some non-Gaussian circumstances. To do so, let us consider the eigenfunctions

$$\widetilde{T}_\ell(x) = \sum_{m=1}^{n_{\ell d}} \widetilde{u}_{\ell m} Y_{\ell m}(x), \tag{17}$$

defined analogously to (1), but now allowing non-Gaussian distributions for the sequence of coefficient vectors $\widetilde{u}_\ell = (\widetilde{u}_{\ell 1}, \dots, \widetilde{u}_{\ell n_{ld}})$, $\widetilde{u}_\ell \colon \Omega_\ell \to \mathbb{R}^d$, distributed according to the (non-Gaussian) sequence of measures $(\widetilde{\mu}_\ell)$. We take $\widetilde{\mu}_\ell$ to be absolutely continuous with respect to the Lebesgue measure μ_ℓ on the unit sphere; in other words, we assume that the vector $\widetilde{u}_\ell$ admits a probability density, for all ℓ.

Our first result in this section is a straightforward extension of Theorem 7.

Proposition 20. *In the setting of Theorem 7, denote by* $p_\ell := \frac{\mathrm{d}\mu_\ell}{\mathrm{d}\widetilde{\mu}_\ell}$ *the sequence of Radon-Nikodym derivatives and let* $(g_\ell)_{\ell \geq 0}$ *be a regular family of excursion functionals. Then for all positive sequences* $(\varepsilon_\ell)_{\ell \geq 0}$ *it holds that*

$$\Pr\left(\sup_u \left| g_\ell(\widetilde{T}_\ell; u) - \Psi(u) \right| > \varepsilon_\ell \right) = O\left(\|p_\ell\|_\infty \left(\frac{1}{\sigma_\ell^2 \varepsilon_\ell^3} + \frac{1}{n_{\ell;d} \varepsilon_\ell^2} \right) \right) ,$$

as $\ell \to \infty$.

Remark 21. *Of course, Proposition 20 only yields interesting results if* $\|p_\ell\|_\infty$ *grows slower than* $\sigma_\ell^2 + n_{\ell d}$.

Proof. Because each g_ℓ is by assumption monotonic in the second variable, the proof that

$$\mu_\ell\left(\left\{ \alpha_\ell : \sup_u |g_\ell(h_{\alpha_\ell, d}; u) - \Psi(u)| > \varepsilon_\ell \right\} \right) = O\left(\frac{1}{\sigma_\ell^2 \varepsilon_\ell^3} + \frac{1}{n_{\ell;d} \varepsilon_\ell^2} \right)$$

can be given exactly as in the proof of Theorem 16, i.e. by showing first that

$$\mu_\ell^*\left(\left\{ (r_\ell, \alpha_\ell) \colon \sup_u |g_\ell(T_\ell; u) - \Psi(u)| > \varepsilon_\ell \right\} \right) = O\left(\frac{1}{\sigma_\ell^2 \varepsilon_\ell^3} \right)$$

and then exploiting the fact that

$$\mu_\ell^*\left(\{ |R_\ell - 1| > \varepsilon_\ell \} \right) = O\left(\frac{1}{n_{\ell;d} \varepsilon_\ell^2} \right).$$

To conclude the argument, it then suffices to define as before

$$\widetilde{G}_\ell(\varepsilon) = \left\{ \widetilde{u}_\ell \in S^{n_{\ell d}} \colon \sup_u \left| g_\ell(\widetilde{T}_\ell; u) - \Psi(u) \right| > \varepsilon_\ell \right\}$$

and to note that

$$\Pr\left(\sup_u \left| g_\ell(\widetilde{T}_\ell; u) - \Psi(u) \right| > \varepsilon_\ell \right)$$

$$\begin{aligned}
\widetilde{\mu}_\ell(\widetilde{G}_\ell(\varepsilon)) &= \int_{\widetilde{G}_\ell(\varepsilon)} \mathrm{d}\,\widetilde{\mu}_\ell = \int_{\left\{\alpha_\ell : \sup_u \left|g_\ell(h_{\alpha_\ell,d};u)-\Psi(u)\right|>\varepsilon_\ell\right\}} p_\ell \,\mathrm{d}\mu_\ell \\
&\leq \|p_\ell\|_\infty \,\mu_\ell\left(\left\{\alpha_\ell : \sup_u |g_\ell(h_{\alpha_\ell,d};u)-\Psi(u)| > \varepsilon_\ell\right\}\right) \\
&= O\left(\frac{1}{\sigma_\ell^2\varepsilon_\ell^3} + \frac{1}{n_{\ell;d}\varepsilon_\ell^2}\right)
\end{aligned}$$

as claimed. □

Remark 22. *It should be noticed that under isotropy, the random coefficients of any non-Gaussian model must satisfy*

$$D_\ell(g)\widetilde{u}_{\ell.} \overset{d}{=} \widetilde{u}_{\ell.} \text{ for all } g \in SO(d+1),$$

where D_ℓ denotes the ℓth irreducible representation of $SO(d+1)$ (for $d=2$ the set $\{D_\ell(g)\colon g \in SO(3)\}$ is the well known family of $(2\ell+1)\times(2\ell+1)$ unitary Wigner matrices). In the Gaussian case, this identity in distribution is actually obvious, because the distribution of the vector $\widetilde{u}_\ell$ is uniform on a sphere of random radius. Our result in this section heuristically suggests that, under regularity conditions, the distribution of the vector of random coefficients should be close to uniform as well in the high energy limit. We leave it for further research to relate this behaviour with the mixing properties of the matrices $D_\ell(g)$.

A simple example of non-Gaussian eigenfunctions is provided by

$$\widetilde{T}_\ell(x) = \xi_\ell T_\ell(x), \qquad \ell \in \mathbb{N},$$

where the sequence (T_ℓ) is as before Gaussian and isotropic with zero mean and unit variance, while (ξ_ℓ) is an arbitrary sequence of random variables. Of course, in this case one trivially obtains a Gaussian limiting behaviour normalizing the sequence $(\widetilde{T}_\ell)$ by its (random) L^2-norm $|\xi_\ell|\,\|T_\ell\|_{L^2}$. It is natural to ask whether such a result could hold under more general circumstances and the following simple Corollary provides a partial positive answer.

Corollary 23. *Let $(\widetilde{T}_\ell)$ be a sequence of isotropic non-Gaussian eigenfunctions of the form* (17) *such that the density sequence of the vectors $\left(\widetilde{u}_\ell/\sqrt{\widetilde{C}_\ell}\right)$ of normalized spherical harmonic coefficients is uniformly bounded as $\ell \to \infty$, where*

$$\widetilde{C}_\ell = \sum_{m=1}^{n_{\ell d}} \widetilde{u}_{\ell m}^2 \tag{18}$$

denotes the sample angular power spectrum. Then, for a regular family (g_ℓ) of monotonic excursion functionals it holds that

$$\Pr\left(\sup_{u\in\mathbb{R}}\left|g_\ell(\widetilde{T}_\ell;u) - \Psi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right| > \varepsilon_\ell\right) = O\left(\frac{1}{\sigma_\ell^2\varepsilon_\ell^3} + \frac{1}{n_{\ell;d}\varepsilon_\ell^2}\right) \tag{19}$$

as $\ell \to \infty$.

Proof. We have

$$\sup_{u\in\mathbb{R}}\left|g_\ell(\widetilde{T}_\ell;u)-\Psi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right| = \sup_{u\in\mathbb{R}}\left|g_\ell\left(\frac{\widetilde{T}_\ell}{\sqrt{\widetilde{C}_\ell}};\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)-\Psi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right|$$
$$\leq \sup_{v\in\mathbb{R}}\left|g_\ell\left(\frac{\widetilde{T}_\ell}{\sqrt{\widetilde{C}_\ell}};v\right)-\Psi(v)\right|.$$

Thus, the result follows from Proposition 20. □

In words, Corollary 23 is stating that under regularity assumptions, and conditionally on the value of the norm of its random coefficients, the geometric functionals evaluated at non-Gaussian eigenfunctions converge to a rescaled version of the Gaussian limit - or, equivalently, to the Gaussian limit evaluated at a random point depending on the norm of the coefficients. Let us now reconsider the previous examples.

Example 24 (Excursion volume)**.** (See [17, 19, 24]) *Using the notation introduced in Example 9, we get that*

$$\sup_{u\in\mathbb{R}}\left|S(\widetilde{T}_\ell,u)-\left(1-\Phi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right)\right| \to_p 0,$$

where $\to_p$ denotes convergence in probability. Assuming for instance that $\widetilde{C}_\ell$ converges in distribution to some limiting random variable C_∞, we have that $S(\widetilde{T}_\ell;u)$ converges in distribution to $1-\Phi\left(\frac{u}{\sqrt{C_\infty}}\right)$. In particular, the Gaussian limiting behaviour is obtained once more if and only if $C_\infty=1$ with probability one.

Example 25 (Critical points)**.** (See [9]) *Recall the Definition of N^c and Ψ^c from Example 9.2. Under the assumptions of Corollary 23, exactly the same argument as before yields*

$$\sup_{u\in\mathbb{R}}\left|\frac{N^c(\widetilde{T}_\ell;u)}{\ell^2}-\Psi^c\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right| \to_p 0$$

as $\ell\to\infty$. As before, assuming $\widetilde{C}_\ell$ converges in distribution to some limiting random variable C_∞, we have

$$\frac{N^c(\widetilde{T}_\ell;u)}{\ell^2}\to_d \Psi^c\left(\frac{u}{\sqrt{C_\infty}}\right).$$

Examples 24-25 are basically stating that the limiting behaviour in these non-Gaussian circumstances corresponds to a mixture of the Gaussian limiting expressions with a (random) scaling factor.

As a final remark, we note that the previous results suggest that convergence of the random norm of the spherical harmonic coefficients to a constant may be closely related to the asymptotic Gaussianity of hyperspherical eigenfunctions. This statement is made rigorous in the following corollary; we recall the standard notation $X_n=O_p(d_n)$ to denote that the sequence $\left|\frac{X_n}{d_n}\right|$ is bounded in probability, i.e., for all $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that $\mathrm{P}\left(\left|\frac{X_n}{d_n}\right| > K\right) < \varepsilon$ for all $n > n_0$.

Corollary 26. *Let $\widetilde{T}_\ell(x)$ be given as in* (17)*, and assume moreover that*

$$\sum_{m=1}^{n_{\ell d}} \widetilde{u}_{\ell m}^2 - 1 = O_p(\gamma_\ell) \ , \ \gamma_\ell \to 0 \ \text{as} \ \ell \to \infty.$$

Then, under the assumptions of Corollary 23, we have that

$$d_{Kol}(\widetilde{T}_\ell(x), Z) = O\left(\frac{1}{\sigma_\ell^2 \varepsilon_\ell^3} + \frac{1}{n_{\ell;d}\varepsilon_\ell^2} + \gamma_\ell\right)$$

for all $x \in S^d$, where Z is standard Gaussian.

Proof. In view of Corollary 23 and the isotropy of T, we know that

$$\begin{aligned}
\Pr\left(\frac{\widetilde{T}_\ell(x)}{\sqrt{\widetilde{C}_l}} \le u\right) - \Phi(u) &= \mathbb{E}\left[1_{(-\infty,u]}\left(\frac{\widetilde{T}_\ell(x)}{\sqrt{\widetilde{C}_l}}\right)\right] - \Phi(u) \\
&= \mathbb{E}\left[\int_{S^d} 1_{(-\infty,u]}\left(\frac{\widetilde{T}_\ell(x)}{\sqrt{\widetilde{C}_l}}\right) \mathrm{d}x - \Phi(u)\right] \\
&= O\left(\frac{1}{\sigma_\ell^2 \varepsilon_\ell^3} + \frac{1}{n_{\ell;d}\varepsilon_\ell^2}\right),
\end{aligned}$$

uniformly in u. Therefore, as by assumption $\sqrt{\widetilde{C}_\ell} \to_p 1$, it holds that

$$\sup_u \left|\Pr\left(\widetilde{T}_\ell(x) \le u\right) - \Phi(u)\right| = \sup_u \left|\Pr\left(\frac{\widetilde{T}_\ell(x)}{\sqrt{\widetilde{C}_\ell}} \le \frac{u}{\sqrt{\widetilde{C}_\ell}}\right) - \Phi(u)\right|$$

$$\le \sup_u \left|\Pr\left(\frac{\widetilde{T}_\ell(x)}{\sqrt{\widetilde{C}_\ell}} \le \frac{u}{\sqrt{\widetilde{C}_\ell}}\right) - \Phi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right)\right| + \sup_u \left|\Phi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right) - \Phi(u)\right| \to 0,$$

as $\ell \to \infty$. To conclude the proof, it suffices to notice that, by a simple application of the Mean Value Theorem,

$$\begin{aligned}
\sup_u \left|\Phi\left(\frac{u}{\sqrt{\widetilde{C}_\ell}}\right) - \Phi(u)\right| &\le \sup_u \left|\Phi(u) - \Phi\left(\sqrt{\widetilde{C}_\ell} u\right)\right| \\
&\le \sup_u |\phi(u)|\,|u| \left|\sqrt{\widetilde{C}_\ell} - 1\right| \\
&\le \frac{1}{\sqrt{2\pi \mathrm{e}}\left(\sqrt{\widetilde{C}_\ell} + 1\right)} \left|\widetilde{C}_\ell - 1\right| \\
&= O_p(\gamma_\ell).
\end{aligned}$$

□

Of course, if $\widetilde{T}_\ell(x)$ is Gaussian, the reverse implication is obvious by the standard law of large numbers (note that under isotropy the $\widetilde{u}_{\ell m}$ are necessarily uncorrelated , while they are independent if and only if the field is Gaussian; see for instance [6, 7], or [16], Chapter 5). The previous corollary hence partially confirms a conjecture on the relationship between high frequency ergodicity and high frequency Gaussianity that was raised a few years ago by [15].

4. Proofs

We begin with the proof of Theorem 7 from Section 2.1, which is rather straightforward; the details are as follows.

Proof of Theorem 7. Recall the construction of the Gaussian measure $\mu_\ell^* = \nu_\ell \otimes \mu_\ell$ from the introduction, obtained by adjoining a random radius distributed as $\sqrt{\frac{X_{\ell,d}}{n_{\ell d}}}$, where $X_{\ell,d} \sim \chi^2_{n_{\ell d}}$, to the normalized Lebesgue measure μ_ℓ on $S^{n_{\ell d}-1}$. Define $\Psi_\ell(u) = \mathbb{E}\left[g_\ell(T_\ell, u)\right]$. It holds that

$$\begin{aligned}\mu_\ell(G_\ell(\varepsilon, u)) &= \mu_\ell\left\{\alpha_\ell\colon\ |g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u)| \geq \varepsilon\right\}\\ &= \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon\ |g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u)| \geq \varepsilon\right\},\end{aligned}$$

where, here and in the following, we tacitly assume that $r_\ell \in \mathbb{R}_+$ and $\alpha_\ell \in S^{n_{\ell d}-1}$. By the law of total probability, the above can be bounded by

$$\begin{aligned}&\nu_\ell\left\{r_\ell\colon\ \left|r_\ell^2 - 1\right| \geq \lambda\right\}\\ &\qquad\qquad + \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon\ |g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u)| \geq \varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\\ &\quad = \nu_\ell\left\{r_\ell\colon\ \left|r_\ell^2 - 1\right| \geq \lambda\right\}\\ &\qquad\qquad + \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u) \geq \varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\\ &\qquad\qquad + \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u) \leq -\varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\end{aligned} \tag{20}$$

For the first measure on the right hand side, Chebyshev's inequality yields

$$\nu_\ell\left\{r_\ell\colon\ \left|r_\ell^2 - 1\right| \geq \lambda\right\} = \Pr\left(\left|R_\ell^2 - \mathbb{E}\left[R_\ell^2\right]\right| \geq \lambda\right) \leq \frac{2}{n_{\ell d}\lambda^2}.$$

We show how to bound the other two measures for the case where $u \geq 0$ (the negative case follows analogously). As $\left|r_\ell^2 - 1\right| < \lambda$ is equivalent to $\sqrt{1-\lambda} < r_\ell < \sqrt{1+\lambda}$ (recall that $\lambda \in (0,1)$), the monotonicity and invariance under scaling of g_ℓ yields

$$\begin{aligned}&\mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(h_{\alpha,\ell}, u) - \Psi_\ell(u) \geq \varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\\ &\quad = \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(r_\ell\, h_{\alpha,\ell}, r_\ell\, u) - \Psi_\ell(u) \geq \varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\\ &\quad \leq \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(r_\ell\, h_{\alpha,\ell}, \sqrt{1-\lambda}\, u) - \Psi_\ell(u) \geq \varepsilon,\ \left|r_\ell^2 - 1\right| < \lambda\right\}\\ &\quad \leq \mu_\ell^*\left\{(r_\ell, \alpha_\ell)\colon g_\ell(r_\ell\, h_{\alpha,\ell}, \sqrt{1-\lambda}\, u) - \Psi_\ell(u) \geq \varepsilon\right\}\end{aligned} \tag{21}$$

Now note that, using the regularity property 6 and the Mean Value Theorem

$$\left|\Psi_\ell(u) - \Psi_\ell(\sqrt{1-\lambda} u)\right| \leq c\left(1 - \sqrt{1-\lambda}\right) \leq c\lambda,$$

where $c > 0$ does not depend on u. Therefore, we can continue to bound (21) by

$$\begin{aligned}
&\le \mu_\ell^* \left\{ (r_\ell, \alpha_\ell) \colon g_\ell(r_\ell\, h_{\alpha,\ell}, \sqrt{1-\lambda}\, u) - \Psi_\ell(\sqrt{1-\lambda}\, u) \ge \varepsilon - c\lambda \right\} \\
&= \Pr\left(g_\ell(T_\ell, \sqrt{1-\lambda}\, u) - \Psi_\ell(\sqrt{1-\lambda}\, u) \ge \varepsilon - c\lambda \right) \\
&\le \frac{\sigma_\ell^2}{(\varepsilon - c\lambda)^2},
\end{aligned}$$

where we have applied Chebyshev's inequality to arrive at the last bound. Similarly, we deduce that

$$\begin{aligned}
&\mu_\ell^* \left\{ (r_\ell, \alpha_\ell) \colon \Psi_\ell(u) - g_\ell(h_{\alpha,\ell}, u) \le -\varepsilon,\ \left| r_\ell^2 - 1 \right| < \lambda \right\} \\
&\quad \le \Pr\left(g_\ell(T_\ell, \sqrt{1+\lambda}\, u) - \Psi_\ell(\sqrt{1+\lambda}\, u) \ge \varepsilon - c\lambda \right) \\
&\quad \le \frac{\sigma_\ell^2}{(\varepsilon - c\lambda)^2}.
\end{aligned}$$

Plugged back into (20), we obtain

$$\mu_\ell(G_\ell(\varepsilon, u)) \le 2 \left(\frac{1}{n_{\ell d}\lambda^2} + \frac{\sigma_\ell^2}{(\varepsilon - c\lambda)^2} \right).$$

Choosing $\lambda = \varepsilon/(c+1)$ and taking the supremum completes the proof. □

Our argument to follow establishes an upper bound on the L^∞-norms of Gaussian eigenfunctions by means of metric entropy ideas and the Borel-TIS inequality. The computations require a careful analysis of the high-frequency behaviour of Gegenbauer polynomials. For the lower bound, we construct a nearly equi-spaced grid of points, where values of the random field can be viewed as asymptotically independent, and then use standard results on the supremum of i.i.d. Gaussian variables.

Proof of Proposition 11. Let us define the canonical metric (see i.e. [1, p.12]) on S^d by

$$d_\ell(x, y) = \sqrt{\mathbb{E}\left[T_\ell(x) - T_\ell(y)\right]^2},$$

which, by isotropy of T_ℓ, can also be written as

$$d_\ell(x, y) = \sqrt{2 - 2G_{\ell;d}(\cos\vartheta)},$$

where $G_{\ell;d}$ is the ℓth Gegenbauer polynomial (see i.e. [30]) and $\vartheta = \arccos\langle x, y\rangle_{\mathbb{R}^d}$ is the usual geodesic distance on S^d.

By Hilb's asymptotic formula for Jacobi polynomials (see for example [30, Thm. 8.21.12]), we have uniformly for $\ell \ge 1$, $\vartheta \in [0, \frac{\pi}{2}]$ that

$$\begin{aligned}
(22)\quad & G_{\ell;d}(\cos\vartheta) \\
&= \frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{l}} (\sin\vartheta)^{-\frac{d}{2}+1} \left(a_{\ell,d} \left(\frac{\vartheta}{\sin\vartheta} \right)^{\frac{1}{2}} J_{\frac{d}{2}-1}(L\vartheta) + \delta(\vartheta) \right),
\end{aligned}$$

where $L = \ell + \frac{d-1}{2}$, J_α denotes the Bessel function of the first kind of order α,

$$a_{\ell,d} = \frac{\Gamma(\ell + \frac{d}{2})}{(\ell + \frac{d-1}{2})^{\frac{d}{2}-1}\ell!} \ \sim \ 1$$

as $\ell \to \infty$ and the remainder is given by

$$\delta(\vartheta) = \begin{cases} O\left(\sqrt{\vartheta}\, \ell^{-\frac{3}{2}}\right) & (K\ell)^{-1} < \vartheta < \frac{\pi}{2} \ , \\ O\left(\vartheta^{\left(\frac{d}{2}-1\right)+2}\, \ell^{\frac{d}{2}-1}\right) & 0 < \vartheta < (K\ell)^{-1} \ , \end{cases}$$

for some $K > 0$.

Therefore, using the asymptotic relation

$$\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}} = \frac{2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!}{l^{\frac{d}{2}-1}} + o(1),$$

we have for $\vartheta < (K\ell)^{-1}$ that

$$G_{\ell;d}(\cos\vartheta) = \frac{2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!}{\ell^{\frac{d}{2}-1}} (\sin\vartheta)^{-\frac{d}{2}+1} \left(a_{\ell,d} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{1}{2}} J_{\frac{d}{2}-1}(L\vartheta) + \delta(\vartheta) \right)$$

with

$$\delta(\vartheta) = O\left(\vartheta^2\right) \ .$$

Recall that J_α is defined as

$$J_\alpha(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$

which implies that

$$J_{\frac{d}{2}-1}(x) = \left(\frac{x}{2}\right)^{\frac{d}{2}-1} \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma\left(m+\frac{d}{2}\right)} \left(\frac{x}{2}\right)^{2m} . \tag{23}$$

Hence,

$$\begin{aligned} J_{\frac{d}{2}-1}(x) &= \frac{x^{\frac{d}{2}-1}}{2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!} - \frac{x^{\frac{d}{2}+1}}{2^{\frac{d}{2}+1}\left(\frac{d}{2}\right)!} \\ &\qquad + \left(\frac{x}{2}\right)^{\frac{d}{2}-1} \sum_{m=2}^{+\infty} \frac{(-1)^m}{m!\Gamma\left(m+\frac{d}{2}\right)} \left(\frac{x}{2}\right)^{2m} \\ &= \frac{x^{\frac{d}{2}-1}}{2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!} \left(1 - \frac{x^2}{2d}\right) \\ &\qquad + \left(\frac{x}{2}\right)^{\frac{d}{2}-1} \sum_{m=2}^{+\infty} \frac{(-1)^m}{m!\Gamma\left(m+\frac{d}{2}\right)} \left(\frac{x}{2}\right)^{2m} . \end{aligned}$$

Note that

$$\lim_{K\to+\infty}\sup_{x\le (K)^{-1}}\left|\frac{1-\frac{2^{\frac{d}{2}-1}(\frac{d}{2}-1)!}{x^{\frac{d}{2}-1}}J_{\frac{d}{2}-1}(x)}{x^2}-\frac{1}{2d}\right|=0.$$

This implies that for $\varepsilon>0$ there exists $K_\varepsilon>0$ such that

$$\left(\frac{1}{2d}-\varepsilon\right)\ell^2\vartheta^2\le 1-\frac{2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!}{(L\vartheta)^{\frac{d}{2}-1}}J_{\frac{d}{2}-1}(L\vartheta)\le\left(\frac{1}{2d}+\varepsilon\right)\ell^2\vartheta^2$$

for $\vartheta<K_\varepsilon\ell^{-1}$. Plugged back into (22), we get for $\vartheta<K_\epsilon\ell^{-1}$ that

$$\left(\frac{1}{2d}-\varepsilon\right)\ell^2\vartheta^2+O(\vartheta^2)\le 1-G_{\ell;d}(\cos\vartheta)\le\left(\frac{1}{2d}+\varepsilon\right)\ell^2\vartheta^2+O(\vartheta^2)\ .$$

Therefore, there exist two constants $c_1,c_2>0$ such that for any $x,y\in S^d$ it holds that

$$c_1d^2(x,y)\le\frac{d_\ell^2(x,y)}{\ell^2}\le c_2d^2(x,y).$$

From here we use exactly the same spherical cap argument as in the proof of Proposition 2 in [18]. To briefly recall, we consider some sequence of balls of radius ε in the canonical metric, which indeed are hyperspherical caps whose radius is asymptotically equal to $\frac{\varepsilon}{\ell}$, $\varepsilon>0$. Their Euclidean volume is asymptotically equal to $\frac{\varepsilon^d}{\ell^d}$, therefore the number $N_\ell(\varepsilon)$ of such caps needed to cover the hypersphere is asymptotically equal to $\frac{\ell^d}{\varepsilon^d}$. By [1, Thm. 1.3.3], there exists a constant K^*, only depending on d, such that

$$\mathbb{E}\left[\sup_{S^d}T_\ell\right]\le K^*\Big(\int_0^{C/\ell}\sqrt{\log N_\ell(\varepsilon)}\,\mathrm{d}\varepsilon+\int_{C/\ell}^{\delta}\sqrt{\log N_\ell(\varepsilon)}\,\mathrm{d}\varepsilon\Big)\ .$$

Note that analogous steps yield the upper bound $c\sqrt{2d\log\ell}$ for both of the previous summands, where $c>0$ is some constant.

To prove part (ii), we apply the Borel-TIS inequality (see for example Thm. 2.1.1, p.50 in [1]), which yields for $t>\mathrm{E}\left[\|T_\ell\|_\infty\right]$ that

$$\Pr\left(\|T_\ell\|_\infty>t\right)\le \mathrm{e}^{-\left(t-\mathbb{E}[\|T_\ell\|_\infty]\right)^2/2}. \tag{24}$$

The result now follows from part (i). The proof of part (iii) uses a similar argument as given in the case of needlet random fields by [18]. In particular, we have that

$$\Pr\left(\sup_{x\in S^d}|T_\ell(x)|\ge K\sqrt{\log\ell}\right)\ge\Pr\left(\sup_{\xi_k\in\Xi_\ell}|T_\ell(\xi_k)|\ge K\sqrt{\log\ell}\right) \tag{25}$$

where, for some $\alpha\in(0,1)$, Ξ_ℓ is a grid of points $\{\xi_k\}$ such that

$$\min_{k\ne k'}d(\xi_k,\xi_{k'})\ge\ell^{-\alpha}$$

and such that $\#\Xi_\ell=N_\ell$ is of order $\ell^{d\alpha}$. The existence of such grids is well-known and has for instance been exploited in the construction of cubature points for spherical wavelets (see [21, 5]).

Now define events A and B by

$$A = \left\{ \left\| \{Z_{\ell 1}, ...Z_{\ell, N_\ell}\} \right\|_\infty \geq 2K\sqrt{\log \ell} \right\}$$
$$B = \left\{ \left\| \{Z_{\ell 1}, ...Z_{\ell, N_\ell}\} - \left\{T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right\} \right\|_\infty \leq K\sqrt{\log \ell} \right\}$$

Then $A \cap B$ implies the event

$$\left\{ \sup_{\xi_k \in \Xi_\ell} |T_\ell(\xi_k)| \geq K\sqrt{\log \ell} \right\}.$$

Therefore,

$$\begin{aligned} \Pr\left(\sup_{\xi_k \in \Xi_\ell} |T_\ell(\xi_k)| \geq K\sqrt{\log \ell} \right) &\geq \Pr(A \cap B) \\ &\geq \Pr(A) + \Pr(B) - 1 \\ &= 1 - \Pr(A^c) - \Pr(B^c), \end{aligned} \tag{26}$$

which, together with (25), implies that

$$\Pr\left(\|T_\ell\| \leq K\sqrt{\log \ell} \right) \leq \Pr(A^c) + \Pr(B^c). \tag{27}$$

Using Mill's inequality

$$\frac{2z}{1+z^2}\phi(z) \leq \Pr\{Z > z\} \leq \frac{2}{z}\phi(z),$$

we can evaluate the probability $\Pr(A^c)$ as follows.

$$\begin{aligned} \Pr(A^c) &= \Pr\left(\left\| \{Z_{\ell 1}, ...Z_{\ell, N_\ell}\} \right\|_\infty < 2K\sqrt{\log \ell} \right) \\ &= \prod_{k=1}^{N_\ell} \Phi(2K\sqrt{\log \ell}) \simeq \left\{ 1 - \frac{1}{2K\sqrt{\log \ell}}\phi(2K\sqrt{\log \ell}) \right\}^{N_\ell} \\ &= \left\{ 1 - \frac{1}{2K\sqrt{\log \ell}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(2K\sqrt{\log \ell})^2) \right\}^{N_\ell} \\ &= \left\{ 1 - \frac{1}{\sqrt{8\pi}K^2 \ell^{2K^2}\sqrt{\log \ell}} \right\}^{\ell^{d\alpha}}. \end{aligned}$$

Using the asymptotics $\log(1+z) \sim z$ for small z, we get that

$$\Pr(A^c) \sim \exp\left(-\frac{\ell^{d\alpha - 2K^2}}{\sqrt{8\pi}K^2\sqrt{\log \ell}} \right). \tag{28}$$

To estimate the probability $\mathrm{P}(B^c)$, write Σ_ℓ for the covariance matrix of the vector $\left\{T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right\}$, and I_{N_ℓ} for the identity matrix of order N_ℓ; it should be recalled that, due to our choice of the grid Ξ_ℓ, the elements of the vector Z_ℓ are asymptotically independent. Note also that

$$\begin{aligned}
\Sigma_\ell^{-1/2} - I_{N_\ell} &= Q\Lambda^{-1/2}Q^* - I_{N_\ell} \\
&= Q\left(\Lambda^{-1/2} - I_{N_\ell}\right)Q^* \\
&= Q\left(\left(\Lambda^{-1/2} - I_{N_\ell}\right)\left(\Lambda^{-1/2} + I_{N_\ell}\right)\left(\Lambda^{-1/2} + I_{N_\ell}\right)^{-1}\right)Q^* \\
&= Q\left(\left(\Lambda^{-1} - I_{N_\ell}\right)\left(\Lambda^{-1/2} + I_{N_\ell}\right)^{-1}\right)Q^* \\
&= Q\left(\Lambda^{-1}\left(I_{N_\ell} - \Lambda\right)\left(\Lambda^{-1/2} + I_{N_\ell}\right)^{-1}\right)Q^*,
\end{aligned}$$

and thus

$$\begin{aligned}
\left\|\Sigma_\ell^{-1/2} - I_{N_\ell}\right\|_2^2 &\leq \left\|Q\left(\Lambda^{-1}\left(I_{N_\ell} - \Lambda\right)\left(\Lambda^{-1/2} + I_{N_\ell}\right)^{-1}\right)Q^*\right\|_2^2 \\
&\leq \|Q\|_2^2\,\|Q^*\|_2^2\,\|\Lambda^{-1}\|_2^2\left\|\left(\Lambda^{-1/2} + I_{N_\ell}\right)^{-1}\right\|_2^2\|I_{N_\ell} - \Lambda\|_2^2 \\
&= O(\|I_{N_\ell} - \Lambda\|_2^2). = O\left(\frac{N_\ell^2}{\sqrt{\ell d(\xi_k, \xi_{k'})}}\right) \\
&= O\left(\frac{N_\ell^2}{\ell^{(1-\alpha)/2}}\right) = O(\ell^{2d\alpha+\alpha/2-1/2}).
\end{aligned}$$

Consequently, for the i.i.d. standard Gaussian array

$$(Z_{\ell 1}, ...Z_{\ell,N_\ell}) = \Sigma_\ell^{-1/2}\left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)$$

it holds that

$$\left\|(Z_{\ell 1}, ...Z_{\ell,N_\ell}) - \left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_\infty^2 \leq \left\|(Z_{\ell 1}, ...Z_{\ell,N_\ell}) - \left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_2^2,$$

and for the expectation of the right hand side we have

$$\begin{aligned}
&\mathbb{E}\left\|(Z_{\ell 1}, ...Z_{\ell,N_\ell}) - \left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_2^2 \\
\leq\ &\left\|\Sigma_\ell^{-1/2} - I_{N_\ell}\right\|_2^2\mathbb{E}\left\|\left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_2^2 \\
\leq\ &\left\|\Sigma_\ell^{-1/2} - I_{N_\ell}\right\|_2^2 N_\ell \\
\leq\ &O(N_\ell \times \ell^{2d\alpha+\alpha/2-1/2}) \\
=\ &O(\ell^{3d\alpha+\alpha/2-1}) = O(\ell^{((6d+1)\alpha-1)/2})\ .
\end{aligned}$$

By Chebyshev's inequality, we thus get

$$\begin{aligned}
\Pr(B^c) &= \Pr\left(\left\|(Z_{\ell 1}, ...Z_{\ell,N_\ell}) - \left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_2 \geq K\sqrt{\log\ell}\right) \\
&\leq \frac{1}{K\log\ell}E\left\|(Z_{\ell 1}, ...Z_{\ell,N_\ell}) - \left(T_\ell(\xi_1), ...T_\ell(\xi_{N_\ell})\right)\right\|_2^2 \\
&= O\left(\frac{\ell^{((6d+1)\alpha-1)/2}}{\log\ell}\right). \qquad (29)
\end{aligned}$$

Plugging (28) and (29) back into (27) yields for any $\alpha \in (0,1)$ that

$$\Pr\left(\|T_\ell\| \le K\sqrt{\log \ell}\right) = O\left(\exp\left(-\frac{\ell^{d\alpha - 2K^2}}{\sqrt{8\pi}K^2\sqrt{\log \ell}}\right) + \frac{\ell^{((6d+1)\alpha-1)/2}}{\log \ell}\right).$$

Now note that the right hand side tends to zero if and only if we have $0 < K < \sqrt{d/(12d+2)}$ and $2K^2/d < \alpha < 1/(6d+1)$. Furthermore, in this case the exponential term on the right hand side is dominated and can be neglected. □

The idea in the proof below is again to associate a Gaussian measure to Lebesgue by introducing a random radius. In this case, however, we shall exploit a Large Deviation Principle on the radius itself, and because of this we will obtain a sharper bound on the rate of convergence to zero for the involved measures.

Proof of Theorem 12. We have for all $K, a > 0$ that

$$\begin{aligned}
&\mu_\ell\left\{\alpha_\ell\colon \|h_{\alpha,\ell}\|_\infty \ge K\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|h_{\alpha,\ell}\|_\infty \ge K\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \ge r_\ell K\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \ge r_\ell K\sqrt{\log \ell},\, r_\ell \ge a^{-1}\right\} \\
&\qquad\quad+ \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \ge r_\ell K\sqrt{\log \ell},\, r_\ell < a^{-1}\right\} \\
&\quad\le \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \ge \frac{K}{a}\sqrt{\log \ell}\right\} + \nu_\ell\left\{r_\ell < a^{-1}\right\}.
\end{aligned}$$

Now set $K = M + \sqrt{2\beta}$ and $a = \left(M + \sqrt{2\beta}\right) / \left(M + \sqrt{2\beta'}\right) > 1$. Then, by Lemma 27, the second measure in the above bound vanishes exponentially in ℓ. The result now follows from part (ii) of Theorem 11. To prove part (ii), we proceed analogously and write

$$\begin{aligned}
&\mu_\ell\left\{\alpha_\ell\colon \|h_{\alpha,\ell}\|_\infty \le K'\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|h_{\alpha,\ell}\|_\infty \le K'\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \le r_\ell K'\sqrt{\log \ell}\right\} \\
&\quad= \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \le r_\ell K'\sqrt{\log \ell},\, r_\ell \le a^{-1}\right\} \\
&\qquad\quad+ \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \le r_\ell K'\sqrt{\log \ell},\, r_\ell > a^{-1}\right\} \\
&\quad\le \mu_\ell^*\left\{(r_\ell,\alpha_\ell)\colon \|r_\ell\, h_{\alpha,\ell}\|_\infty \le \frac{K'}{a}\sqrt{\log \ell}\right\} + \nu_\ell\left\{r_\ell > a^{-1}\right\}.
\end{aligned}$$

For any $a > 1$, the second measure vanishes exponentially in ℓ by Lemma 27 and the result now follows from part (iii) of Theorem 11 (clearly, given $K' \in (0, \sqrt{d/(12d+2)})$, we can find K in the same interval and $a > 1$ such that $K' = K/a$).

□

Proof of Corollary 13. In the setting of Theorem 12, choose β' such that $1 < \beta' < \beta$. Then, the Borel-Cantelli Lemma gives

$$\begin{aligned}
&\mu_\infty \left\{ (\alpha_\ell)_{\ell \geq 1} \colon \left\| h_{\alpha,\ell} \right\|_\infty \geq \left(M + \sqrt{2\beta} \right) \sqrt{\log \ell} \text{ infinitely often} \right\} \\
&\quad \leq \lim_{L \to \infty} \sum_{\ell = L}^{\infty} \mu_\ell \left\{ \alpha_\ell \colon \left\| h_{\alpha,\ell} \right\|_\infty \geq (M + \sqrt{2\beta}) \sqrt{\log \ell} \right\} \\
&\quad \leq \text{const} \times \lim_{L \to \infty} \sum_{\ell = L}^{\infty} \frac{1}{\ell^{\beta'}} = 0 \ .
\end{aligned}$$

Identity (13) can be show in the same way. □

Finally, we provide the proof for the convergence in Kolmogorov distance. This requires some uniform bound, which is obtained by means of chaining arguments, similar for instance to the one given by [12]. Details require some care, because in this setting we are also covering the case where the dimension d grows to infinity.

Proof of Theorem 16. We note first that

$$\begin{aligned}
&d_{Kol}(h_{\alpha,\ell}(X), Z) \\
&\quad = \sup_u \left| \int_{S^d} 1_{(-\infty,u]} \left(h_{\alpha,\ell}(x) \right) \mathrm{d}x - \Phi(u) \right| \\
&\quad = \sup_u \left| \int_{S^d} \left(1_{(-\infty,u]} \left(h_{\alpha,\ell}(x) \right) - 1_{(-\infty,x]} \left(T_\ell(x) \right) \right) \mathrm{d}x \right. \\
&\qquad\qquad \left. + \int_{S^d} 1_{(-\infty,u]} \left(T_\ell(x) \right) \mathrm{d}x - \Phi(u) \right| \\
&\quad \leq \sup_u \left| \int_{S^d} \left(1_{(-\infty,u]} \left(h_{\alpha,\ell}(x) \right) - 1_{(-\infty,u]} \left(T_\ell(x) \right) \right) \mathrm{d}x \right| \\
&\qquad\qquad + \sup_u \left| \int_{S^d} 1_{(-\infty,u]} \left(T_\ell(x) \right) \mathrm{d}x - \Phi(u) \right| ,
\end{aligned}$$

where Φ denotes the standard Gaussian cumulative distribution function. Now

$$\begin{aligned}
&\mu_\ell \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,u]} \left(h_{\alpha,\ell}(x) \right) - \Phi(u) \right) \mathrm{d}x \right| > \varepsilon_\ell \right\} \\
&\quad = \mu_\ell^* \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,ru]} \left(r_\ell h_{\alpha,\ell}(x) \right) - \Phi(u) \right) \mathrm{d}x \right| > \varepsilon_\ell \right\} \\
&\quad = \mu_\ell^* \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,ru]} \left(T_\ell(x) \right) - \Phi(u) \right) \mathrm{d}x \right| > \varepsilon_\ell \right\} \\
&\quad = \mu_\ell^* \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,ru]} \left(T_\ell(x) \right) - \Phi(ur_\ell) + \Phi(ur_\ell) - \Phi(u) \right) \mathrm{d}x \right| > \varepsilon_\ell \right\} \\
&\quad \leq \mu_l^* \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,u]} \left(T_\ell(x) \right) - \Phi(u) \right) \mathrm{d}x \right| > \frac{\varepsilon_\ell}{2} \right\} \\
&\qquad\qquad + \mu_\ell^* \left\{ \sup_u \left| \Phi(ur_\ell) - \Phi(u) \right| > \frac{\varepsilon_\ell}{2} \right\} ,
\end{aligned}$$

where we used the inclusion

$$\left\{(x,y)\in\mathbb{R}^2\colon\ |x-y|>\varepsilon\right\}\subseteq\left\{(x,y)\in\mathbb{R}^2\colon\ |x-z|>\frac{\varepsilon}{2}\right\}\cup\left\{(x,y)\in\mathbb{R}^2\colon\ |y-z|>\frac{\varepsilon}{2}\right\},$$

which is valid for any $z\in\mathbb{R}$.

By concavity of Φ and a Taylor argument, we obtain for the second term

$$\sup_u |\Phi(ur_\ell)-\Phi(u)|\leq \sup_u |\phi(u)u|\,|r_\ell-1|\,.$$

Hence it holds that

$$\begin{aligned}
\mu_\ell^*\left\{\sup_u|\Phi(ur_\ell)-\Phi(u)|>\frac{\varepsilon_\ell}{2}\right\} &= \nu_\ell\left\{\sup_u|\Phi(ur_\lambda)-\Phi(u)|>\frac{\varepsilon_\ell}{2}\right\}\\
&\leq \nu_\ell\left\{|r_\ell-1|>\varepsilon_\ell\sqrt{\frac{\pi}{2}}\right\}\\
&\leq \frac{2\,\mathrm{E}_{\nu_\ell}\left[(R_\ell-1)^2\right]}{\varepsilon_\ell^2\pi}\\
&\leq \frac{K}{n_{\ell d}\varepsilon_\ell^2}
\end{aligned}$$

For the first term, we follow an argument similar to the standard proof of uniform convergence for Glivenko-Cantelli theorems (see for instance [12]). In particular, let us choose an array of refining partitions $\{u_{k;\ell}\}$ such that $\Phi(u_{k;\ell})-\Phi(u_{k-1;\ell})<\varepsilon_\ell/6$. Then, given u, we can find k such that $u_{k-1}\leq u\leq u_k$. As cumulative distribution functions are increasing, we thus have that

$$\begin{aligned}
\int_{S^d}\left(1_{(-\infty,u]}\left(T_\ell(x)\right)-\Phi(u)\right)dx &\leq \int_{S^d}\left(1_{(-\infty,u_k]}\left(T_\ell(x)\right)-\Phi(u_{k-1})\right)dx\\
&\leq \int_{S^d}\left(1_{(-\infty,u_k]}\left(T_\ell(x)\right)-\Phi(u_k)+\frac{\varepsilon_\ell}{6}\right)dx
\end{aligned}$$

Likewise, we have that

$$\begin{aligned}
&\int_{S^d}\left(1_{(-\infty,u]}\left(T_\ell(x)\right)-\Phi(u)\right)dx\\
&\qquad\geq \int_{S^d}\left(1_{(-\infty,u_{k-1}]}\left(T_\ell(x)\right)-\Phi(u_{k-1})-\frac{\varepsilon_\ell}{6}\right)dx
\end{aligned}$$

and therefore it holds that

$$\begin{aligned}
&\sup_u\left|\int_{S^d}\left(1_{(-\infty,u]}\left(T_\ell(x)\right)-\Phi(u)\right)dx\right|\\
&\qquad\leq \max_k\left|\int_{S^d}\left(1_{(-\infty,u_k]}\left(T_\ell(x)\right)-\Phi(u_k)\right)dx\right|+\frac{\varepsilon_\ell}{6}.
\end{aligned}$$

Hence, choosing a sequence of partitions $\{u_{k;\ell}\}$, we get that

$$\begin{aligned}
&\mu_\ell^* \left\{ \sup_u \left| \int_{S^d} \left(1_{(-\infty,u]} \left(T_\ell(x) \right) - \Phi(u) \right) dx \right| > \frac{\varepsilon_\ell}{2} \right\} \\
&\leq \mu_\ell^* \left\{ \sum_k \left| \int_{S^d} \left(1_{(-\infty,u_{k;\ell}]} \left(T_\ell(x) \right) - \Phi(u_{k;\ell}) \right) dx \right| > \frac{\varepsilon_\ell}{3} \right\} \\
&\leq 9\# \left\{ u_{k;\ell} \right\} \frac{\max_k \mathbb{E} \left[\left| \int_{S^d} \left(1_{(-\infty,u_{k;\ell}]} \left(T_\ell(x) \right) - \Phi(u_{k;\ell}) \right) dx \right| \right]^2}{\varepsilon_\ell^2},
\end{aligned}$$

where $\#\left\{u_{\ell;k}\right\}$ denotes the cardinality of the partition, which clearly is of order $1/\varepsilon_\ell$.

Now we recall from [17] that

$$\max_k \mathbb{E} \left[\left| \int_{S^d} \left(1_{(-\infty,u_{k;\ell}]} \left(T_\ell(x) \right) - \Phi(u_{k;\ell}) \right) dx \right| \right]^2 \leq \frac{K}{n_{\ell d}},$$

where $K \leq \sup_u \left(\Phi(u)(1-\Phi(u)) \right) = 1/4$, which finishes the proof of the first statement of the Theorem. The proof of the second statement can be given by exactly the same argument - we need only establish a uniform bound on the behaviour of the variance of the excursion volume, for ℓ fixed and as d goes to infinity. The proof of this bound is given in Lemma 28 in the appendix. □

Proof of Corollary 17. We consider the case where ℓ is fixed and d diverges to infinity; the proof for the other statement in the Corollary is identical.

For fixed ℓ, it holds that

$$\begin{aligned}
&\mathbb{E} \left[d_{Kol}(h_{\alpha,\ell}(X), Z) \right] \\
&\leq \mu_\ell \left\{ d_{Kol}(h_{\alpha,\ell}(X), Z) < \kappa_\ell) \right\} \times \kappa_\ell + \int_{\kappa_\ell}^\infty \mu_\ell \left\{ h_{\alpha,\ell}(X), Z) > u \right\} du \\
&\leq \kappa_\ell + C \int_{\kappa_\ell}^\infty \frac{1}{n_{\ell d} u^3} du,
\end{aligned}$$

where the constant C is uniform in d, in view of Lemma 29 in the appendix and Chebyshev's inequality. It is then easy to conclude that

$$\mathbb{E} \left[d_{Kol}(h_{\alpha,\ell}(X), Z) \right] \leq \kappa_\ell + \frac{C}{n_{\ell d} \kappa_\ell^2}$$

and choosing $\kappa_\ell = 1/n_{\ell d}^{1/3}$ together with the asymptotics $n_{\ell d} \approx d^\ell$ for $d \to \infty$ and $n_{\ell d} \approx \ell^{d-1}$ (see the introduction) finishes the proof. □

Proof of Proposition 18. The same argument as in the proof of Theorem 12.iii works here as well. □

5. Appendix: Technical lemmas

The proofs collected in this final Section are all rather standard, and they have been included mainly for completeness.

Lemma 27. *For all $a > 1$, there exist constants $K, a' > 0$ such that*

$$\mu_\ell^* \left\{ R_\ell^2 \geq a \right\} + \mu_\ell^* \left\{ R_\ell^2 \leq a^{-1} \right\} \leq K \exp\left(-\ell a' \right).$$

Proof. Let (X_n) be a sequence of i.i.d. random variables with common law μ, whose Laplace transform $\hat{\mu}$ is finite in a neighborhood of the origin. Then it is well known that the sequence $(\overline{X}_n)$ satisfies the large deviation principle

$$\limsup_{n\to+\infty} \frac{1}{n} \log \Pr(\overline{X}_n \in A) \leq - \inf_{x\in\overline{A}} \Lambda^*(x)$$

$$\liminf_{n\to+\infty} \frac{1}{n} \log \Pr(\overline{X}_n \in A) \geq - \inf_{x\in A^o} \Lambda^*(x)$$

for any Borel set A. Here, Λ^* is the Cramér transform

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}} (\theta x - \Lambda(\theta)) \ ,$$

where Λ is the logarithm of the Laplace transform $\hat{\mu}$. Hence we need the Cramér transform for chi-square distributions. In particular, we have that $X_1 \sim \Gamma(\frac{1}{2}, \frac{1}{2})$, so that its Laplace transform is for $\theta < \frac{1}{2}$

$$\widehat{\mu}(\theta) = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2} - \theta\right)^{\frac{1}{2}}}$$

and still for $\theta < \frac{1}{2}$

$$\Lambda(\theta) = \log \widehat{\mu}(\theta) = -\frac{1}{2} \log (1 - 2\theta) \ .$$

Now let us compute its Cramér transform

$$\Lambda^*(x) = \sup_{\theta<\frac{1}{2}} \left(\theta x + \frac{1}{2} \log (1 - 2\theta) \right) \ .$$

Note that if $x \leq 0$, then

$$\Lambda^*(x) = +\infty \ .$$

Consider $x > 0$, then the function $\theta \mapsto \theta x + \frac{1}{2} \log (1 - 2\theta)$ has unique maximum at the critical point θ^*

$$x - \frac{1}{1 - 2\theta^*} = 0 \quad \Rightarrow \quad \theta^* = -\frac{1}{2x} + \frac{1}{2} \ .$$

Therefore

$$\begin{aligned} \Lambda^*(x) &= \left(-\frac{1}{2x} + \frac{1}{2}\right)x + \frac{1}{2} \log\left(\frac{1}{x}\right) \\ &= \frac{1}{2}(x - 1 - \log x) \end{aligned}$$

Note that the above function is convex, has a unique minimum (equal to 0) at $x = 1$ (indeed 1 is the expected value of a chi-square random variable with one degree of freedom), for $0 < x < 1$ is strictly decreasing and for $x > 1$ is strictly increasing. Therefore we have (in our notation R_ℓ^2 is the empirical mean of $2\ell + 1$ i.i.d. chi-square random variables with one degree of freedom)

$$\lim_{\ell\to\infty} \frac{1}{2\ell + 1} \log \Pr\left(R_\ell^2 \geq a\right) = - \inf_{x\geq a} \Lambda^*(x)$$

where

$$\inf_{x\geq a} \Lambda^*(x) = \begin{cases} 0 \ , \text{ for } a \leq 1 \\ \Lambda^*(a) \ , \text{ for } a > 1 \end{cases} \ .$$

It hence follows immediately that for all $\delta > 0$, $a > 1$, ℓ large enough we have

$$\Pr\left(R_\ell^2 \geq a\right\} \leq K \exp\left\{-\ell\left[\frac{1}{2}(a-1-\log a)-\delta\right]\right) .$$

Likewise

$$\lim_{\ell\to\infty} \frac{1}{2\ell+1}\log \Pr\left(R_\ell^2 \leq \frac{1}{a}\right) = -\inf_{x\leq a^{-1}} \Lambda^*(x) = \begin{cases} 0 \text{ , for } a \leq 1 \\ -\Lambda^*(a^{-1}) \text{ , for } a > 1 \end{cases} ,$$

so that for all $\delta > 0$, $a < 1$, ℓ large enough we have

$$\Pr\left(R_\ell^2 \leq a^{-1}\right) \leq K \exp\left(-\ell\left[\frac{1}{2}(a^{-1}-1+\log a)-\delta\right]\right) .$$

It then suffices to take δ such that

$$a' = \min\left\{\frac{1}{2}(a^{-1}-1+\log a)-\delta, \frac{1}{2}(a-1-\log a)-\delta\right\} > 0 ,$$

and the proof is completed. □

5.1. **Verification of excursion functional properties for Example 9.** For the following lemmas, we adopt the setting and notation of Example 9.

Lemma 28. *As $\ell \to \infty$, we have*

$$\operatorname{Var}\left(\int_{S^d}\left(1_{[u,\infty)}\left(T_\ell(x)\right)-\Phi(u_\ell)\right)dx\right) = O\left(\frac{1}{n_{\ell,d}}\right) ,$$

uniformly over u.

Proof. The L^2 expansion for the excursion volume is given by (see [12, **?**, 17])

$$\int_{S^d}\left(1_{[u,\infty)}\left(T_\ell(x)\right)-\Phi(u_\ell)\right)dx = \int_{S^d}\sum_{q=2}^{\infty}\frac{J_q(u)}{q!}H_q(T_\ell(x))dx,$$

where

$$J_q(u) := \Phi^{(q)}(u) .$$

Hence we have the uniform bound

$$\begin{aligned} \operatorname{Var}\left(\int_{S^d}\sum_{q=2}^{\infty}\frac{J_q(u)}{q!}H_q(T_\ell(x))dx\right) &= \sum_{q=2}^{\infty}\frac{J_q^2(u)}{q!}\int_{S^d}\int_{S^d}\frac{G_{\ell;d}(\langle x,y\rangle)}{G_{\ell;d}(1)}dxdy \\ &\leq \frac{K}{n_{\ell,d}}\sum_{q=2}^{\infty}\frac{J_q^2(u)}{q!} \leq \frac{K}{n_{\ell,d}}\Phi(u)\left(1-\Phi(u)\right) \\ &\leq \frac{K}{4n_{\ell,d}}, \end{aligned}$$

because

$$\sum_{q=2}^{\infty}\frac{J_q^2(u)}{q!} = \operatorname{Var}\left(1_{[u,\infty)}\left(T_\ell(x)\right)\right) = \Phi(u)\left(1-\Phi(u)\right)$$

and

$$\sup_u \Phi(u)\left(1-\Phi(u)\right) = \Phi(0)(1-\Phi(0)) = \frac{1}{4} .$$

□

Lemma 29. *Let $M : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[M(Z)^2] < +\infty$, $Z \sim \mathcal{N}(0,1)$ and define*

$$S_\ell(M) = \int_{S^d} M(T_\ell(x))\, dx.$$

Then it holds that

$$\mathrm{Var}(S_\ell(M)) \sim \frac{1}{n_{\ell d}}$$

for both, $\ell \to \infty$ and $d \to \infty$.

Proof. For $\ell \to \infty$, the result was given in [17]. Similarly, for $d \to \infty$, it holds that

$$S_\ell(M) = \int_{S^d} M(T_\ell(x))\, dx = \sum_{q=0}^{+\infty} \frac{J_q(M)}{q!} \underbrace{\int_{S^d} H_q(T_\ell(x))\, dx}_{:=h_{\ell;q,d}} .$$

Therefore

$$\mathrm{Var}(S_\ell(M)) = \sum_{q=2}^{+\infty} \frac{J_q(M)^2}{(q!)^2} \mathrm{Var}(h_{\ell;q,d}) .$$

Simple computations give

$$\mathrm{Var}(h_{\ell;q,d}) = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta$$

where μ_d is the Lebesgue measure of the hyperspherical surface.

If we normalize the hypersphere, we get

$$\mathrm{Var}(h_{\ell;q,d}) = q! \frac{\mu_d \mu_{d-1}}{\mu_d^2} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta .$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(S_\ell(M)) &= \sum_{q=2}^{+\infty} \frac{J_q(M)^2}{q!} \frac{\mu_d \mu_{d-1}}{\mu_d^2} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \\
&\le \frac{\mu_d \mu_{d-1}}{\mu_d^2} \int_0^\pi G_{\ell;d}(\cos\vartheta)^2 (\sin\vartheta)^{d-1} d\vartheta \underbrace{\sum_{q=2}^{+\infty} \frac{J_q(M)^2}{q!}}_{\sim \mathrm{Var}[M(Z)] < +\infty} \\
&= \frac{\mu_d \mu_{d-1}}{\mu_d^2} \frac{\mu_d}{\mu_{d-1} n_{\ell;d}} \underbrace{\sum_{q=2}^{+\infty} \frac{J_q(M)^2}{q!}}_{\sim \mathrm{Var}[M(Z)] < +\infty} \sim \frac{1}{n_{\ell;d}}
\end{aligned}$$

as $\ell \to \infty$ and also as $d \to \infty$. □

Lemma 30. *For all $u \in \mathbb{R}$, $r \geq 0$, $f : S^2 \to \mathbb{R}$, $f \in C^2(S^2)$, we have*

$$\mathcal{N}^c(f_\ell; u) = \mathcal{N}^c(r f_\ell; r u) .$$

Proof. It is obvious that for all constant $r > 0$

$$\int_{S^2} \left|\nabla^2 h_{\alpha,\ell}(x)\right| \delta(\|\nabla h_{\alpha,\ell}(x)\| \, \mathrm{d}x = \int_{S^2} \left|\nabla^2 r h_{\alpha,\ell}(x)\right| \delta(\|\nabla r h_{\alpha,\ell}(x)\|) \mathrm{d}x \ ,$$

whence the result follows from Kac-Rice formula. □

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