Stein-Malliavin Approximations for Nonlinear Functionals of Random Eigenfunctions on $\mathbb{S}^d$ *

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Abstract

We investigate Stein-Malliavin approximations for nonlinear functionals of geometric interest of Gaussian random eigenfunctions on the unit d-dimensional sphere $\mathbb{S}^d$, $d \geq 2$. All our results are established in the high energy limit, i.e. for eigenfunctions corresponding to growing eigenvalues. More precisely, we provide an asymptotic analysis for the variance of random eigenfunctions, and also establish rates of convergence for various probability metrics for Hermite subordinated processes, arbitrary polynomials of finite order and square integral nonlinear transforms; the latter, for instance, allows to prove a quantitative Central Limit Theorem for the excursion area. Some related issues were already considered in the literature for the 2-dimensional case $\mathbb{S}^2$; our results are new or improve the existing bounds even for this special case. Proofs are based on the asymptotic analysis of moments of all order for Gegenbauer polynomials, and make extensive use of the recent literature on so-called fourth-moment theorems by Nourdin and Peccati.

- **Keywords and Phrases:** Spherical Harmonics, Gaussian Eigenfunctions, High Energy Asymptotics, Stein-Malliavin Approximations, Excursion Area
- **AMS Classification:** 60G60; 42C10, 60D05, 60B10

1 Introduction

The characterization of the asymptotic behaviour (in the high energy limit, i.e. for eigenfunctions with growing eigenvalues) of geometric functionals of Gaussian random eigenfunctions on compact manifolds is a topic which has recently drawn considerable attention. For instance, a growing literature has focussed on the investigation of the asymptotic behaviour of nodal lines, i.e. the zero sets of eigenfunctions in some random setups, or the geometry of nodal domains; in particular, much effort has been devoted to the d-dimensional torus $\mathbb{T}^d$ and the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ (see [6], [7], [10], [11], [35], [36] e.g.). Many of these papers have considered the computation of asymptotic variances in the high energy limit; Central Limit Theorem results have also been established, for instance for the so-called Defect in the two-dimensional case of the sphere $\mathbb{S}^2$ [20].

This stream of literature has been largely motivated by applications from Mathematical Physics. In particular, according to Berry's Universality conjecture [5], random Gaussian monochromatic waves (similar to e.g. random Gaussian spherical harmonics) could model deterministic eigenfunctions on a "generic" manifold with or without boundary; this heuristic has strongly motivated the analysis of nodal sets of the former. On the other hand, it is also well-known that random eigenfunctions are the Fourier components of square integrable isotropic fields on manifolds. In view of this and in light of the importance of spherical random fields in astrophysics and cosmology, the analysis of polynomial transforms or geometric functionals of random spherical harmonics is a major thread in these disciplines; these results are used for testing the adequacy of theoretical models to capture geometric features of observed data (for instance on Cosmic Microwave Background radiation, see [13], [21] or the monograph [16]).

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A CLT by itself can often provide little guidance to the actual distribution of random functionals, as it is only an asymptotic result with no information on the speed of convergence to the limiting distribution. More refined results indeed aim at the investigation of the asymptotic behaviour for various probability metrics, such as Kolmogorov, Total Variation and Wasserstein distances (to be defined below). In this respect, a major development in the last few years has been provided by the so-called *fourth-moments literature,* which is summarized in the recent monograph [26]. In short, a rapidly growing family of results is showing how it is possible to establish sharp bounds on probability distances between multiple stochastic integrals and the standard Gaussian distribution by means of the analysis of the fourth-moments/fourth cumulants alone. Such results are currently being generalized in several directions, including Poisson processes, free probability, random matrices, Markov subordinators and information theory (see [3], [12], [24], [27], [28] e.g.); in the present paper, we will stick to Gaussian subordinated circumstances, as described in §1.1.

1.1 Main results

Let us first fix some notation: for any two positive sequence a_n, b_n, we shall write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $a_n \ll b_n$ or $a_n = O(b_n)$ if the sequence $\frac{a_n}{b_n}$ is bounded. Moreover if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, then $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Also, we write as usual dx for the Lebesgue measure on the unit d-dimensional sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$, so that $\int_{\mathbb{S}^d} dx = \mu_d$ where $\mu_d := \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$. The triple $(\Omega, \mathscr{F}, \mathbb{P})$ shall denote a probability space and $\mathbb{E}$ shall stand for the expectation w.r.t $\mathbb{P}$; convergence (resp. equality) in law shall be denoted by $\to^{\mathcal{L}}$ (resp. $=^{\mathcal{L}}$) and finally, as usual, $\mathcal{N}(\mu, \sigma^2)$ shall stand for a Gaussian random variable with mean μ and variance σ^2.

Now let $\Delta_{\mathbb{S}^d}$ $(d \geq 2)$ denote as usual the spherical Laplacian operator on $\mathbb{S}^d$ and $(Y_{\ell,m;d})_{\ell,m}$ the orthonormal system of (real-valued) spherical harmonics, i.e. for $\ell \in \mathbb{N}$ the set of eigenfunctions

$$\Delta_{\mathbb{S}^d} Y_{\ell,m;d} = -\ell(\ell + d - 1) Y_{\ell,m;d}\ , \quad m = 1, 2, \dots, n_{\ell;d}\ .$$

As well-known, the spherical harmonics $(Y_{\ell,m;d})_{m=1}^{n_{\ell;d}}$ represent a family of linearly independent homogeneous polynomials of degree ℓ in $d+1$ variables restricted to $\mathbb{S}^d$ of size

$$n_{\ell;d} := \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \ \sim\ \frac{2}{(d-1)!} \ell^{d-1}\ , \quad \text{as } \ell \to +\infty\ ,$$

see e.g. [2] for further details. It is then customary to construct, for $\ell \in \mathbb{N}$, the random eigenfunction T_ℓ on $\mathbb{S}^d$ by taking

$$T_\ell(x) := \sum_{m=1}^{n_{\ell;d}} a_{\ell,m} Y_{\ell,m;d}(x)\ , \quad x \in \mathbb{S}^d\ , \tag{1.1}$$

with the coefficients $(a_{\ell,m})_{m=1}^{n_{\ell;d}}$ Gaussian i.i.d. random variables, satisfying the relation

$$\mathbb{E}[a_{\ell,m} a_{\ell,m'}] = \frac{\mu_d}{n_{\ell;d}} \delta_m^{m'}\ ,$$

where δ_a^b denotes the Kronecker delta function and $\mu_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ the hypersurface volume of the d-dimensional unit sphere as above.

It is then readily checked that $(T_\ell)_{\ell\in\mathbb{N}}$ represents a sequence of isotropic, mean-zero Gaussian random fields on $\mathbb{S}^d$, that is, for every fixed ℓ we have a collection of random variables $(T_\ell(x))_{x\in\mathbb{S}^d}$ indexed by the points of $\mathbb{S}^d$, such that the map

$$T_\ell : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}\ ; \qquad (\omega, x) \mapsto T_\ell(\omega, x)$$

is $\mathscr{F} \otimes \mathscr{B}(\mathbb{S}^d)$-measurable, where $\mathscr{B}(\mathbb{S}^d)$ denotes the Borel σ-field of $\mathbb{S}^d$. The isotropy of T_ℓ means that the probability laws of the two random fields $T_\ell(\cdot)$ and $T_\ell^g(\cdot) := T_\ell(g\,\cdot)$ are equal for every $g \in SO(d+1)$.

It is also well-known that every Gaussian and isotropic random field T on $\mathbb{S}^d$ is necessarily mean-square continuous (indeed this statement holds for every isotropic and finite variance random field on a homogeneous space of a compact group - see [17]) and satisfies in the $L^2(\Omega \times \mathbb{S}^d)$-sense the spectral representation (see [14], [16] and also [1], [4])

$$T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x) \ , \quad x \in \mathbb{S}^d \ ,$$

where $\mathbb{E}\left[T^2\right] = \sum_{\ell=1}^{\infty} c_\ell^2 < \infty$; hence the spherical Gaussian eigenfunctions $(T_\ell)_{\ell\in\mathbb{N}}$ can be viewed as the Fourier components of the field T (note that w.l.o.g. we are implicitly assuming that T is centred). Equivalently these random eigenfunctions (1.1) could be defined by their covariance function, which equals

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell;d}(\cos d(x,y)) \ , \quad x, y \in \mathbb{S}^d \ . \tag{1.2}$$

Here and in the sequel, $d(x, y)$ is the spherical distance between $x, y \in \mathbb{S}^d$, and $G_{\ell;d} : [-1, 1] \longrightarrow \mathbb{R}$ is the ℓ-th Gegenbauer polynomial, i.e. $G_{\ell;d} \equiv P_\ell^{(\frac{d}{2}-1,\frac{d}{2}-1)}$, where $P_\ell^{(\alpha,\beta)}$ are the Jacobi polynomials; as a special case, for $d = 2$, it equals $G_{\ell;2} \equiv P_\ell$, the degree-ℓ Legendre polynomial. Throughout this paper, we normalize so that $G_{\ell;d}(1) = 1$. Recall that the Jacobi polynomials $P_\ell^{(\alpha,\beta)}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(t) = (1-t)^\alpha(1+t)^\beta$ and satisfy

$$P_\ell^{(\alpha,\beta)}(1) = \begin{pmatrix} \ell+\alpha \\ \ell \end{pmatrix} \ ,$$

see [34] for more details.

The main purpose of this paper is to investigate quantitative CLTs for nonlinear functionals of Gaussian spherical eigenfunctions on $\mathbb{S}^d$. For $d = 2$ this issue was addressed in [20]; our first aim is to extend their results to arbitrary dimensions and study the asymptotic behavior, as $\ell \to \infty$, of the random variables $h_{\ell;q,d}$ defined for $\ell = 1, 2, \dots$ and $q = 0, 1, \dots$ as

$$h_{\ell;q,d} = \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx \ , \tag{1.3}$$

where H_q represent the family of Hermite polynomials ([26], [29]). The latter are defined as usual by $H_0 \equiv 1$ and for $q = 1, 2, \dots$

$$H_q(t) = (-1)^q e^{\frac{t^2}{2}} \frac{d^q}{dt^q} e^{-\frac{t^2}{2}} \ , \ t \in \mathbb{R} \ . \tag{1.4}$$

Note that, for all d

$$h_{\ell;0,d} = \mu_d \ , \qquad h_{\ell;1,d} = 0$$

a.s., and therefore it is enough to restrict our discussion to $q \geq 2$. Moreover $\mathbb{E}[h_{\ell;q,d}] = 0$ and

$$\mathrm{Var}[h_{\ell;q,d}] = q!\mu_d\mu_{d-1} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \tag{1.5}$$

(see §3 for more details). Gegenbauer polynomials satisfy the symmetry relationships

$$G_{\ell;d}(t) = (-1)^\ell G_{\ell;d}(-t) \ ,$$

whence the r.h.s. integral in (1.5) vanishes identically when both ℓ and q are odd; hence in these cases $h_{\ell;q,d} = 0$ a.s. For the remaining cases we have

$$\mathrm{Var}[h_{\ell;q,d}] = 2q!\mu_d\mu_{d-1} \int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \ . \tag{1.6}$$

Our first result, given in §3, is an upper bound for these variances, asymptotic for $\ell \to \infty$.

Proposition 1.1. *As $\ell \to \infty$, for $q, d \geq 3$,*

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta = \frac{c_{q;d}}{\ell^d}(1 + o_{q;d}(1)). \tag{1.7}$$

The constants $c_{q;d}$ are given by the formula

$$c_{q;d} = \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^{+\infty} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q\left(\frac{d}{2}-1\right)+d-1} d\psi \ , \tag{1.8}$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of order $\frac{d}{2}-1$. The r.h.s. integral in (1.8) is absolutely convergent for any pair $(d, q) \neq (3, 3)$ and conditionally convergent for $d = q = 3$.

It is well known that for $d \geq 2$, the second moment of the Gegenbauer polynomials is given by

$$\int_0^{\pi} G_{\ell;d}(\cos\vartheta)^2 (\sin\vartheta)^{d-1}\, d\vartheta = \frac{\mu_d}{\mu_{d-1}\, n_{\ell;d}} \ , \tag{1.9}$$

whence

$$\mathrm{Var}[h_{\ell;2,d}] = 2\frac{\mu_d^2}{n_{\ell;d}} \ \sim\ 4\mu_d\mu_{d-1}\frac{c_{2;d}}{\ell^{d-1}} \ , \qquad \text{as } \ell \to +\infty \ , \tag{1.10}$$

where $c_{2;d} := \frac{(d-1)!\mu_d}{4\mu_{d-1}}$. For $d = 2$ and every q, the asymptotic behaviour of these integrals was resolved in [19]. In particular, it was shown that for $q = 3$ or $q \geq 5$

$$\mathrm{Var}[h_{\ell;q,2}] = (4\pi)^2 q! \int_0^{\frac{\pi}{2}} P_\ell(\cos\vartheta)^q \sin\vartheta\, d\vartheta = (4\pi)^2 q! \frac{c_{q;2}}{\ell^2}(1 + o_q(1)) \ , \tag{1.11}$$

where

$$c_{q;2} = \int_0^{+\infty} J_0(\psi)^q \psi\, d\psi \ , \tag{1.12}$$

J_0 being the Bessel function of order 0 and the above integral being absolutely convergent for $q \geq 5$ and conditionally convergent for $q = 3$. On the other hand, for $q = 4$, as $\ell \to \infty$,

$$\mathrm{Var}[h_{\ell;4,2}] \ \sim\ 24^2 \frac{\log \ell}{\ell^2} \ . \tag{1.13}$$

Clearly for any $d, q \geq 2$, the constants $c_{q;d}$ are nonnegative and it is obvious that $c_{q;d} > 0$ for all even q. We conjecture that this strict inequality holds for every (d, q), but leave this issue as an open question for future research; also, in view of the previous discussion on the symmetry properties of Gegenbauer polynomials, to simplify the discussion in the sequel we restrict ourselves to even multipoles ℓ.

In this paper we first establish quantitative CLTs for $h_{\ell;q,d}$ (see §4) and then for other nonlinear functionals of geometric interest (see §5,6). Our results are new for $d \geq 3$; for $d = 2$ we improve the existing bounds on probability metrics that were readily established [20], and also extend this analysis to non-Hermite polynomials, and establish Breuer-Major like results with surprisingly fast convergence rates for generic nonlinear functionals, including e.g. the area of excursion sets (see the discussion below for more details).

To formulate our results we need to introduce some more notation. Denote the usual Kolmogorov d_K, Total Variation d_{TV} and Wasserstein d_W distances between random variables Z, N:

$$\begin{aligned} d_K(Z, N) &= \sup_{z\in\mathbb{R}} |\mathbb{P}(Z \leq z) - \mathbb{P}(N \leq z)| \ , \\ d_{TV}(Z, N) &= \sup_{A\in\mathscr{B}(\mathbb{R})} |\mathbb{P}(Z \in A) - \mathbb{P}(N \in A)| \ , \\ d_W(Z, N) &= \sup_{h\in\mathrm{Lip}(1)} |\mathbb{E}[h(Z)] - \mathbb{E}[h(N)]| \ , \end{aligned}$$

where $\mathscr{B}(\mathbb{R})$ denotes the Borel σ-field of $\mathbb{R}$ and $\mathrm{Lip}(1)$ the set of Lipschitz functions whose Lipschitz constant equals 1. We shall prove the following result.

Theorem 1.2. *For all $d, q = 2, 3, \ldots$, $d_{\mathcal{D}} = d_{TV}, d_W, d_K$ we have*

$$d_{\mathcal{D}}\left(\frac{h_{2\ell;q,d}}{\sqrt{Var[h_{2\ell;q,d}]}}, \mathcal{N}(0,1)\right) = O(R(\ell; q, d)) \ ,$$

where for $d = 2$

$$R(\ell; q, 2) = \begin{cases} \ell^{-\frac{1}{2}} & q = 2, 3, \\ (\log \ell)^{-1} & q = 4, \\ (\log \ell)\ell^{-\frac{1}{4}} & q = 5, 6, \\ \ell^{-\frac{1}{4}} & q \geq 7; \end{cases} \tag{1.14}$$

and for $d = 3, 4, \ldots$

$$R(\ell; q, d) = \begin{cases} \ell^{-\left(\frac{d-1}{2}\right)} & q = 2, \\ \ell^{-\left(\frac{d-5}{4}\right)} & q = 3, \\ \ell^{-\left(\frac{d-3}{4}\right)} & q = 4, \\ \ell^{-\left(\frac{d-1}{4}\right)} & q \geq 5. \end{cases} \tag{1.15}$$

The following corollary is hence immediate.

Corollary 1.3. *For all q such that $(d, q) \neq (3, 3), (3, 4), (4, 3), (5, 3)$ and $c_{q;d} > 0$, $d = 2, 3, \ldots$,*

$$\frac{h_{2\ell;q,d}}{\sqrt{Var[h_{2\ell;q,d}]}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \ , \qquad as\ \ell \to +\infty \ . \tag{1.16}$$

Remark 1.4. *For $d = 2$, the CLT in (1.16) was already provided by [20]; nevertheless Theorem 1.2 improves the existing bounds on the speed of convergence to the asymptotic Gaussian distribution. More precisely, for $d = 2, q = 2, 3, 4$ the same rate of convergence as in (1.14) was given in their Proposition 3.4; however for arbitrary q the total variation rate was only shown to satisfy (up to logarithmic terms) $d_{TV} = O(\ell^{-\delta_q})$, where $\delta_4 = \frac{1}{10}$, $\delta_5 = \frac{1}{7}$, and $\delta_q = \frac{q-6}{4q-6} < \frac{1}{4}$ for $q \geq 7$.*

Remark 1.5. *The cases not included in Corollary 1.3 correspond to the pairs where $q = 4$ and $d = 3$, or $q = 3$ and $d = 3, 4, 5$; in these circumstances the bounds we establish on fourth-order cumulants are not sufficient to ensure that the CLT holds. Most probably, these four special cases can be dealt with ad hoc arguments based on the explicit evaluations of multiple integrals of spherical harmonics by means of so-called Clebsch-Gordan coefficients, following the steps of Lemma 3.3 in [20], see also [15], [16]. Such computations, however, seem of limited interest for the present paper, and we therefore omit the investigation of these special cases for brevity's sake.*

The random variables $h_{\ell;q,d}$ defined in (1.3) are the basic building blocks for the analysis of any square integrable nonlinear transforms of Gaussian spherical eigenfunctions on $\mathbb{S}^d$. Indeed, let us consider generic polynomial functionals of the form

$$Z_\ell = \sum_{q=0}^{Q} b_q \int_{\mathbb{S}^d} T_\ell(x)^q \, dx \ , \qquad Q \in \mathbb{N}, \ b_q \in \mathbb{R}, \tag{1.17}$$

which include, for instance, the so-called polyspectra of isotropic random fields defined on $\mathbb{S}^d$. Note

$$Z_\ell = \sum_{q=0}^{Q} \beta_q h_{2\ell;q,d} \tag{1.18}$$

for some $\beta_q \in \mathbb{R}$. It is easy to establish CLTs for generic polynomials (1.18) from convergence results on $h_{2\ell;q,d}$, see e.g. [30]. It is more difficult to investigate the speed of convergence in the CLT in terms of the probability metrics we introduced earlier; indeed, in §5 we establish the following.

Theorem 1.6. *As $\ell \to \infty$,*

$$d_{\mathcal{D}}\left(\frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O(R(Z_\ell; d)) \ ,$$

where $d_{\mathcal{D}} = d_{TV}, d_W, d_K$ and for $d = 2, 3, \dots$

$$R(Z_\ell; d) = \begin{cases} \ell^{-\left(\frac{d-1}{2}\right)} & \text{if} \quad \beta_2 \neq 0 \ , \\ \max_{q=3,\dots,Q \,:\, \beta_q, c_{q;d} \neq 0} R(\ell; q, d) & \text{if} \quad \beta_2 = 0 \ . \end{cases}$$

All the results as above can be summarized as follows: for polynomials of Hermite rank 2 (i.e. their projection against $H_2(T_\ell)$ $\beta_2 \neq 0$, does not vanish), the asymptotic behaviour of Z_ℓ is dominated by the term $h_{\ell;2,d}$, whose variance is of order ℓ^{-d+1} rather than $O(\ell^{-d})$ as for the other terms. On the other hand, when $\beta_2 = 0$, the convergence rate to the asymptotic Gaussian distribution for a generic polynomial is the slowest among the rates for the Hermite components into which Z_ℓ can be decomposed.

The fact that the bound for generic polynomials is of the same order as for the Hermite case (and not slower) is indeed rather unexpected; it can be shown to be due to the cancellation of some cross-product terms, which are dominating in the general Nourdin-Peccati framework, while they vanish in the framework of spherical eigenfunctions of arbitrary dimension (see (5.2) and Remark 5.1). An inspection of our proof will reveal that this result is a by-product of the orthogonality of eigenfunctions corresponding to different eigenvalues; it is plausible that similar ideas may be exploited in many related circumstances, for instance random eigenfunction on generic compact manifolds.

Theorem 1.6 shows that the asymptotic behaviour of arbitrary polynomials of Hermite rank 2 is of particularly simple nature. Our result below will show that this feature holds in much greater generality, at least as far as the Wasserstein distance is concerned. Indeed, we shall consider the case of functionals of the form

$$S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x))\, dx \ , \tag{1.19}$$

where $M : \mathbb{R} \to \mathbb{R}$ is any square integrable, measurable nonlinear function. It is well known that for such transforms the following expansion holds in $L^2(\Omega)$-sense

$$M(T_\ell) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} H_q(T_\ell), \quad \mathbb{E}[M(T_\ell)^2] < \infty, \quad J_q(M) := \mathbb{E}[M(T_\ell) H_q(T_\ell)] \ . \tag{1.20}$$

Therefore the asymptotic analysis, as $\ell \to \infty$, of $S_\ell(M)$ in (1.19) directly follows from the Gaussian approximation for $h_{\ell;q,d}$ and their polynomial transforms Z_ℓ. More precisely, in §6 we prove the following result.

Theorem 1.7. *For functions M in (1.19) such that $\mathbb{E}[M(Z) H_2(Z)] = J_2(M) \neq 0$, we have*

$$d_W\left(\frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{Var[S_{2\ell}(M)]}}, \mathcal{N}(0,1)\right) = O(\ell^{-\frac{1}{2}}) \ , \qquad \text{as } \ell \to \infty \ , \tag{1.21}$$

in particular

$$\frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{\mathrm{Var}[S_{2\ell}(M)]}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \ . \tag{1.22}$$

Theorem 1.7 provides a Breuer-Major like result on nonlinear functionals, in the high-frequency limit (compare for instance [25]). While the CLT in (1.22) is somewhat expected, the square-root speed of convergence (1.21) to the limiting distribution may be considered quite remarkable; it is mainly due to some specific features in the chaos expansion of random eigenfunctions, which is dominated by a single term at $q = 2$. Note that the function M need not be smooth in any meaningful sense; indeed our main motivating rationale here is the analysis of the asymptotic behaviour of the empirical measure for excursion sets, where $M(\cdot) = M_z(\cdot) = \mathbb{I}(\cdot \leq z)$ is the indicator function of the interval $(-\infty, z]$. Therefore, in words, $S_\ell(z) := S_\ell(M_z)$ is the (random) measure of an excursion set, i.e. T_ℓ lies above a given level $z \in \mathbb{R}$; an application of Theorem 1.7 yields a quantitative CLT for $S_\ell(z)$, $z \neq 0$.

2 Background

In a number of recent papers summarized in the monograph [26], a beautiful connection has been established between Malliavin calculus and the so-called Stein method to prove Berry-Esseen bounds and quantitative CLTs on functionals of Gaussian subordinated random fields. In this section, we first briefly review some notation and the main results in this area, which we shall deeply exploit in the sequel of the paper.

2.1 Stein-Malliavin Normal approximations

Let us consider the measure space $(X, \mathcal{X}, \mu)$, where X is a Polish space, $\mathcal{X}$ is the σ-field on X and μ is a positive, σ-finite and non-atomic measure on $(X, \mathcal{X})$. Denote $H = L^2(X, \mathcal{X}, \mu)$ the real (separable) Hilbert space of square integrable functions on X w.r.t. μ, with inner product $\langle f, g\rangle_H = \int_X f(x)g(x)\,d\mu(x)$. Let us recall the construction of an isonormal Gaussian field on H. First consider a Gaussian white noise on X, i.e. a centered Gaussian family W

$$W = \{W(A) : A \in \mathcal{X}, \mu(A) < +\infty\}$$

such that for $A, B \in \mathcal{X}$ of finite measure, we have

$$\mathbb{E}[W(A)W(B)] = \int_X \mathbb{I}(A \cap B)\, d\mu \ .$$

We define a Gaussian random field T on H as follows. For each $f \in H$, let

$$T(f) = \int_X f(x)\, dW(x) \ , \tag{2.1}$$

i.e. the Wiener-Ito integral of f with respect to W. The random field T is the isonormal Gaussian field on H; indeed

$$\mathrm{Cov}\,(T(f), T(g)) = \langle f, g\rangle_H \ .$$

Let us recall now the notion of Wiener chaoses. Define the space of constants $C_0 := \mathbb{R} \subseteq L^2(\Omega)$, and for $q \geq 1$, let C_q be the closure in $L^2(\Omega)$ of the linear subspace generated by random variables of the form

$$H_q(T(f)) \ , \qquad f \in H,\ \|f\|_H = 1 \ ,$$

where H_q is the q-th Hermite polynomial (1.4). C_q is called the q-th Wiener chaos. The following, well-known property will be useful in the sequel: let $Z_1, Z_2 \sim \mathcal{N}(0,1)$ be jointly Gaussian; then, for all $q_1, q_2 \geq 0$

$$\mathbb{E}[H_{q_1}(Z_1)H_{q_2}(Z_2)] = q_1!\ \mathbb{E}[Z_1 Z_2]^{q_1}\, \delta_{q_2}^{q_1} \ . \tag{2.2}$$

Moreover the following chaotic Wiener-Ito expansion holds:

$$L^2(\Omega) = \bigoplus_{q=0}^{+\infty} C_q \ ,$$

the above sum being orthogonal from (2.2). Equivalently, each random variable $F \in L^2(\Omega)$ admits a unique decomposition in the $L^2(\Omega)$-sense of the form

$$F = \sum_{q=0}^{\infty} J_q(F) \ , \tag{2.3}$$

where $J_q : L^2(\Omega) \longrightarrow C_q$ is the orthogonal projection operator. Remark that $J_0(F) = \mathbb{E}[F]$.

We denote by $H^{\otimes q}$ and $H^{\odot q}$ the q-th tensor product and the q-th symmetric tensor product of H respectively. In particular $H^{\otimes q} = L^2(X^q, \mathcal{X}^q, \mu^q)$ and $H^{\odot q} = L^2_s(X^q, \mathcal{X}^q, \mu^q)$ where by L^2_s we mean the square integrable and symmetric functions. Note that for $(x_1, x_2, \ldots, x_q) \in X^q$ and $f \in H$, we have

$$f^{\otimes q}(x_1, x_2, \ldots, x_q) = f(x_1)f(x_2)\ldots f(x_q)\ .$$

Now for $q \geq 1$ define the map I_q as

$$I_q(f^{\otimes q}) := H_q(T(f))\ , \qquad f \in H\ , \tag{2.4}$$

which can be extended to a linear isometry between $H^{\odot q}$ equipped with the modified norm $\sqrt{q!}\|\cdot\|_{H^{\odot q}}$ and the q-th Wiener chaos C_q. Moreover for $q = 0$, set $I_0(c) = c \in \mathbb{R}$. Under the new notation the equality (2.3) becomes

$$F = \sum_{q=0}^{\infty} I_q(f_q)\ , \tag{2.5}$$

where $f_0 = \mathbb{E}[F]$ and for $q \geq 1$, the kernels $f_q \in H^{\odot q}$ are uniquely determined.

In our case, it is well known that for $h \in H^{\odot q}$, $I_q(h)$ coincides with the multiple Wiener-Ito integral of h with respect to the Gaussian measure W, i.e.

$$I_q(h) = \int_{X^q} h(x_1, x_2, \ldots x_q)\, dW(x_1)dW(x_2)\ldots dW(x_q) \tag{2.6}$$

and, in words, F in (2.5) can be seen as a series of (multiple) stochastic integrals.

For every $p, q \geq 1$, $f \in H^{\otimes p}, g \in H^{\otimes q}$ and $r = 1, 2, \ldots, p \wedge q$, the so-called *contraction* of f and g of order r is the element $f \otimes_r g \in H^{\otimes p+q-2r}$ defined as

$$\begin{aligned} &(f \otimes_r g)(x_1, \ldots, x_{p+q-2r}) = \\ &= \int_{X^r} f(x_1, \ldots, x_{p-r}, y_1, \ldots, y_r) g(x_{p-r+1}, \ldots, x_{p+q-2r}, y_1, \ldots, y_r)\, d\mu(y_1)\ldots d\mu(y_r)\ . \end{aligned} \tag{2.7}$$

For $p = q = r$, we have $f \otimes_r g = \langle f, g \rangle_{H^{\otimes r}}$ and for $r = 0$, $f \otimes_0 g = f \otimes g$. Denote by $f \widetilde{\otimes}_r g$ the canonical symmetrization of $f \otimes_r g$. The following multiplication formula is well-known; for $p, q = 1, 2, \ldots,$ $f \in H^{\odot p}, g \in H^{\odot q}$, we have

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g)\ .$$

We now briefly recall some basic Malliavin calculus formulas for this setting. For $q, r \geq 1$, the r-th Malliavin derivative of a random variable $F = I_q(f) \in C_q$ where $f \in H^{\odot q}$, can be identified as the element $D^r F : \Omega \to H^{\odot r}$ given by

$$D^r F = \frac{q!}{(q-r)!} I_{q-r}(f)\ , \tag{2.8}$$

for $r \leq q$, and $D^r F = 0$ for $r > q$. So that, the r-th Malliavin derivative of the random variable F in (2.5) could be written as

$$D^r F = \sum_{q=r}^{+\infty} \frac{q!}{(q-r)!} I_{q-r}(f_q)\ .$$

For simplicity of notation, we shall write D instead of D^1. We say that F as in (2.5) belongs to $\mathbb{D}^{r,q}$ if

$$\|F\|_{\mathbb{D}^{r,q}} := \big(\ \mathbb{E}[|F|^q] + \ldots\ \mathbb{E}[\|D^r F\|^q_{H^{\odot r}}]\big)^{\frac{1}{q}} < +\infty\ ;$$

it is easy to check that $F \in \mathbb{D}^{1,2}$ if and only if

$$\mathbb{E}[\|DF\|^2_H] = \sum_{q=1}^{\infty} q \|J_q(F)\|^2_{L^2(\Omega)} < +\infty\ .$$

We need to introduce also the generator of the Ornstein-Uhlenbeck semigroup, defined as

$$L = -\sum_{q=0}^{\infty} q J_q \ ,$$

where J_q is the orthogonal projection operator on C_q, as in (2.3). The domain of L is $\mathbb{D}^{2,2}$, equivalently the space of Gaussian subordinated random variables F such that

$$\sum_{q=1}^{+\infty} q^2 \|J_q(F)\|^2_{L^2(\Omega)} < +\infty \ .$$

The pseudo-inverse operator of L is defined as

$$L^{-1} = -\sum_{q=1}^{\infty} \frac{1}{q} J_q$$

and satisfies for each $F \in L^2(\Omega)$

$$LL^{-1}F = F - \mathbb{E}[F] \ .$$

The connection between stochastic calculus and probability metrics is summarized in the following celebrated result (see e.g. [26], Theorem 5.1.3), which will provide the basis for most of our results to follow.

Proposition 2.1. *Let $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}[F] = 0$, $\mathbb{E}[F^2] = \sigma^2 < +\infty$. Then we have*

$$d_W(F, \mathcal{N}(0,1)) \le \sqrt{\frac{2}{\sigma^2 \pi}} \mathbb{E}[\left|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H\right|] \ .$$

Also, assuming in addition that F has a density

$$\begin{aligned} d_{TV}(F, \mathcal{N}(0,1)) &\le \frac{2}{\sigma^2} \mathbb{E}[\left|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H\right|] \ , \\ d_K(F, \mathcal{N}(0,1)) &\le \frac{1}{\sigma^2} \mathbb{E}[\left|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H\right|] \ . \end{aligned}$$

Moreover if $F \in \mathbb{D}^{1,4}$, we have also

$$\mathbb{E}[\left|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H\right|] \le \sqrt{\mathrm{Var}[\langle DF, -DL^{-1}F\rangle_H]} \ .$$

Furthermore, in the special case where $F = I_q(f)$ for $f \in H^{\odot q}$, then from [26], Theorem 5.2.6

$$\mathbb{E}[\left|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H\right|] \le \sqrt{\frac{1}{q^2} \sum_{r=1}^{q-1} r^2 r!^2 \binom{q}{r}^4 (2q-2r)! \|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}}} \ . \tag{2.9}$$

Note that in (2.9) we can replace $\|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}}$ with the norm of the unsymmetryzed contraction $\|f \otimes_r f\|^2_{H^{\otimes 2q-2r}}$ for the upper bound, because $\|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}} \le \|f \otimes_r f\|^2_{H^{\otimes 2q-2r}}$ by the triangular inequality.

2.2 Polynomial transforms in Wiener chaoses

As mentioned earlier in §1.1, we shall be concerned first with random variables $h_{\ell;q,d}$, $\ell \ge 1$, $q, d \ge 2$

$$h_{\ell;q,d} = \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx \ ,$$

and their (finite) linear combinations

$$Z_\ell = \sum_{q=2}^{Q} \beta_q h_{\ell;q,d} \ , \qquad \beta_q \in \mathbb{R}, Q \in \mathbb{N} \ . \tag{2.10}$$

Our first objective is to represent (2.10) as a (finite) sum of (multiple) stochastic integrals as in (2.5), in order to apply the results recalled in §2.1. More explicitly, we shall first provide the isonormal representation (2.1) on $L^2(\mathbb{S}^d)$ for the Gaussian random eigenfunctions T_ℓ, $\ell \ge 1$ i.e., we shall show that the following identity in law holds:

$$T_\ell(x) \stackrel{\mathcal{L}}{=} \int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,y))\, dW(y) \ , \qquad x \in \mathbb{S}^d \ ,$$

where W is a Gaussian white noise on $\mathbb{S}^d$. To compare with (2.1), $T_\ell(x) = T(f_x)$, where T is the isonormal Gaussian field on $L^2(\mathbb{S}^d)$ and $f_x(\cdot) := \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,\cdot))$. Moreover we have immediately that

$$\mathbb{E}\left[\int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,y))\, dW(y)\right] = 0 \ ,$$

and by the reproducing formula for Gegenbauer polynomials ([34])

$$\mathbb{E}\left[\int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x_1,y_1))\, dW(y_1) \int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x_2,y_2))\, dW(y_2)\right] =$$
$$= \frac{n_{\ell;d}}{\mu_d} \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,y)) G_{\ell;d}(\cos d(x_2,y))dy = G_{\ell;d}(\cos d(x_1,x_2)) \ .$$

Note that by (2.4), we also have

$$H_q(T_\ell(x)) = I_q(f_x^{\otimes q}) =$$
$$= \int_{(\mathbb{S}^d)^q} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q/2} G_{\ell;d}(\cos d(x,y_1)) \dots G_{\ell;d}(\cos d(x,y_q))\, dW(y_1)...dW(y_q) \ ,$$

so that

$$h_{\ell;q,d} \stackrel{\mathcal{L}}{=} \int_{(\mathbb{S}^d)^q} g_{\ell;q}(y_1,...,y_q)\, dW(y_1)...dW(y_q) \ ,$$

where

$$g_{\ell;q}(y_1,...,y_q) := \int_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q/2} G_{\ell;d}(\cos d(x,y_1)) \dots G_{\ell;d}(\cos d(x,y_q))\, dx \ . \tag{2.11}$$

Thus we just established that $h_{\ell;q,d} \stackrel{\mathcal{L}}{=} I_q(g_{\ell;q})$ and therefore

$$Z_\ell \stackrel{\mathcal{L}}{=} \sum_{q=2}^{Q} I_q(\beta_q\, g_{\ell;q}) \ , \tag{2.12}$$

as required. It should be noted that for such random variables Z_ℓ, the conditions of the Proposition 2.1 are trivially satisfied.

3 On the variance of $h_{\ell;q,d}$

In this section we study the variance of $h_{\ell;q,d}$ defined in (1.3). By (2.2) and the definition of Gaussian random eigenfunctions (1.2), it follows that (1.5) hold at once:

$$\mathrm{Var}[h_{\ell;q,d}] = \mathbb{E}\left[\left(\int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx\right)^2\right] = \int_{(\mathbb{S}^d)^2} \mathbb{E}[H_q(T_\ell(x_1))H_q(T_\ell(x_2))]\, dx_1 dx_2 =$$
$$= q! \int_{(\mathbb{S}^d)^2} \mathbb{E}[T_\ell(x_1)T_\ell(x_2)]^q\, dx_1 dx_2 = q! \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,x_2))^q\, dx_1 dx_2 =$$
$$= q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta.$$

Now we prove Proposition 1.1, inspired by the proof of [19], Lemma 5.2.

3.1 Proof Proposition 1.1

Proof. By the Hilb's asymptotic formula for Jacobi polynomials (see [34], Theorem 8.21.12), we have uniformly for $\ell \geq 1$, $\vartheta \in [0, \frac{\pi}{2}]$

$$(\sin\vartheta)^{\frac{d}{2}-1} G_{\ell;d}(\cos\vartheta) = \frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}} \left(a_{\ell,d} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{1}{2}} J_{\frac{d}{2}-1}(L\vartheta) + \delta(\vartheta) \right) ,$$

where $L = \ell + \frac{d-1}{2}$,

$$a_{\ell,d} = \frac{\Gamma(\ell+\frac{d}{2})}{(\ell+\frac{d-1}{2})^{\frac{d}{2}-1}\ell!} \ \sim\ 1 \quad \text{as } \ell \to \infty, \tag{3.1}$$

and the remainder is

$$\delta(\vartheta) \ll \begin{cases} \sqrt{\vartheta}\,\ell^{-\frac{3}{2}} & \ell^{-1} < \vartheta < \frac{\pi}{2} \ , \\ \vartheta^{\left(\frac{d}{2}-1\right)+2}\,\ell^{\frac{d}{2}-1} & 0 < \vartheta < \ell^{-1} \ . \end{cases}$$

Therefore, in light of (3.1) and $\vartheta \to \frac{\vartheta}{\sin\vartheta}$ being bounded,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1} d\vartheta &= \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}} J^q_{\frac{d}{2}-1}(L\vartheta)(\sin\vartheta)^{d-1} d\vartheta \ + \\ &+ O\left(\frac{1}{\ell^{q(\frac{d}{2}-1)}} \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \delta(\vartheta)(\sin\vartheta)^{d-1} d\vartheta \right) ,\end{aligned} \tag{3.2}$$

where we used

$$\binom{\ell+\frac{d}{2}-1}{\ell} \ll \frac{1}{\ell^{\frac{d}{2}-1}}$$

(note that we readily neglected the smaller terms, corresponding to higher powers of $\delta(\vartheta)$). We rewrite (3.2) as

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1} d\vartheta = N + E \ , \tag{3.3}$$

where

$$N = N(d,q;\ell) := \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(L\vartheta)^q (\sin\vartheta)^{d-1} d\vartheta \tag{3.4}$$

and

$$E = E(d,q;\ell) \ll \frac{1}{\ell^{q(\frac{d}{2}-1)}} \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \delta(\vartheta)(\sin\vartheta)^{d-1} d\vartheta \ . \tag{3.5}$$

To bound the error term E we split the range of the integration in (3.5) and write

$$\begin{aligned} E \ll & \frac{1}{\ell^{q(\frac{d}{2}-1)}} \int_0^{\frac{1}{\ell}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \vartheta^{\left(\frac{d}{2}-1\right)+2} \ell^{\frac{d}{2}-1} (\sin\vartheta)^{d-1}\, d\vartheta + \\ &+ \frac{1}{\ell^{q(\frac{d}{2}-1)}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \sqrt{\vartheta}\,\ell^{-\frac{3}{2}} (\sin\vartheta)^{d-1}\, d\vartheta \ . \end{aligned} \tag{3.6}$$

For the first integral in (3.6) recall that $J_{\frac{d}{2}-1}(z) \sim z^{\frac{d}{2}-1}$ as $z \to 0$, so that as $\ell \to \infty$,

$$\frac{1}{\ell^{(q-1)(\frac{d}{2}-1)}} \int_0^{\frac{1}{\ell}} \left(\frac{\vartheta}{\sin\vartheta}\right)^{q(\frac{d}{2}-1)-d+1} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \vartheta^{-(q-1)\left(\frac{d}{2}-1\right)+d+1}\, d\vartheta \ll$$

$$\ll \int_0^{\frac{1}{\ell}} \vartheta^{d+1}\, d\vartheta = \frac{1}{\ell^{d+2}} \ , \tag{3.7}$$

which is enough for our purposes. Furthermore, since for z big $|J_{\frac{d}{2}-1}(z)| = O(z^{-\frac{1}{2}})$ (and keeping in mind that L is of the same order of magnitude as ℓ), we may bound the second integral in (3.6) as

$$\ll \frac{1}{\ell^{q(\frac{d}{2}-1)+\frac{3}{2}}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} \left(\frac{\vartheta}{\sin\vartheta}\right)^{q(\frac{d}{2}-1)-d+1} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \vartheta^{-q(\frac{d}{2}-1)+d-\frac{1}{2}}\, d\vartheta \ll$$

$$\ll \frac{1}{\ell^{q(\frac{d}{2}-1)+\frac{3}{2}}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} (\ell\vartheta)^{-\frac{q-1}{2}} \vartheta^{-q(\frac{d}{2}-1)+d-\frac{1}{2}}\, d\vartheta = \frac{1}{\ell^{q(\frac{d}{2}-\frac{1}{2})+2}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} \vartheta^{-q(\frac{d}{2}-\frac{1}{2})+d}\, d\vartheta \ll$$

$$\ll \frac{1}{\ell^{(d+2)\wedge\left(q\left(\frac{d}{2}-\frac{1}{2}\right)+1\right)}} = o(\ell^{-d}) \ , \tag{3.8}$$

where the last equality in (3.8) holds for $q \geq 3$. From (3.7) (bounding the first integral in (3.6)) and (3.8) (bounding the second integral in (3.6)) we finally find that the error term in (3.3) is

$$E = o(\ell^{-d}) \tag{3.9}$$

for $q \geq 3$, admissible for our purposes.

Therefore, substituting (3.9) into (3.3) we have

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta =$$

$$= \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(L\vartheta)^q (\sin\vartheta)^{d-1} d\vartheta + o(\ell^{-d}) =$$

$$= \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \frac{1}{L} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi + o(\ell^{-d}) \ , \tag{3.10}$$

where in the last equality we transformed $\psi/L = \vartheta$; it then remains to evaluate the first term in (3.10), which we denote by

$$N_L := \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \frac{1}{L} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \ .$$

Now recall that as $\ell \to \infty$

$$\binom{\ell+\frac{d}{2}-1}{\ell} \ \sim \ \frac{\ell^{\frac{d}{2}-1}}{(\frac{d}{2}-1)!} \ ;$$

moreover (3.1) holds, therefore we find of course that as $L \to \infty$

$$N_L \ \sim \ \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \ . \tag{3.11}$$

In order to finish the proof of Proposition 1.1, it is enough to check that, as $L \to \infty$

$$L^d \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \longrightarrow c_{q;d} \ ,$$

actually from (3.10) and (3.11), we have

$$\lim_{\ell\to+\infty} \ell^d \int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta =$$

$$= \lim_{L\to+\infty} L^d \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \ .$$

Now we write

$$\frac{\psi/L}{\sin\psi/L} = 1 + O\left(\psi^2/L^2\right) ,$$

so that

$$L^d \frac{\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \Big(\frac{\psi/L}{\sin\psi/L}\Big)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi =$$

$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} \Big(\frac{\psi/L}{\sin\psi/L}\Big)^{q(\frac{d}{2}-\frac{1}{2})-d+1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi =$$

$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} \left(1 + O\left(\psi^2/L^2\right)\right)^{q(\frac{d}{2}-\frac{1}{2})-d+1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi =$$

$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi +$$

$$+ O\left(\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi\right) .$$

Note that as $L \to +\infty$, the first term of the previous summation converges to $c_{q;d}$ defined in (1.8), i.e.

$$\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi \to c_{q;d} . \tag{3.12}$$

It remains to bound the remainder

$$\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} |J_{\frac{d}{2}-1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi = O(1) + \frac{1}{L^2}\int_1^{L\frac{\pi}{2}} |J_{\frac{d}{2}-1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi .$$

Now for the second term on the r.h.s.

$$\int_1^{L\frac{\pi}{2}} |J^q_{\frac{d}{2}-1}(\psi)| \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi \ll \int_1^{L\frac{\pi}{2}} \psi^{-q(\frac{d}{2}-\frac{1}{2})+d+1}\, d\psi =$$

$$= O(1 + L^{-q(\frac{d}{2}-\frac{1}{2})+d+2}) .$$

Therefore we obtain

$$\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi + O\left(\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi\right) =$$

$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi + O(L^{-2} + L^{-q(\frac{d}{2}-\frac{1}{2})+d}) ,$$

so that we have just checked the statement of the present proposition for $q > \frac{2d}{d-1}$. This is indeed enough for each $q \geq 3$ when $d \geq 4$.

It remains to investigate separately just the case $d = q = 3$. Recall that for $d = 3$ we have an explicit formula for the Bessel function of order $\frac{d}{2} - 1$ ([34]), that is

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z) ,$$

and hence the integral in (1.8) is indeed convergent for $q = d = 3$ by integrations by parts.

We have hence to study the convergence of the following integral

$$\frac{8}{\pi^{\frac{3}{2}}}\int_0^{L\frac{\pi}{2}} \left(\frac{\psi/L}{\sin\psi/L}\right) \frac{\sin^3\psi}{\psi}\, d\psi .$$

To this aim, let us consider a large parameter $K \gg 1$ and divide the integration range into $[0, K]$ and $[K, \frac{\pi}{2}]$; the main contribution comes from the first term, whence we have to prove that the latter vanishes. Note that

$$\int_K^{L\frac{\pi}{2}} \left(\frac{\psi/L}{\sin\psi/L}\right) \frac{\sin^3\psi}{\psi}\, d\psi \ll \frac{1}{K} \ , \tag{3.13}$$

where we use integration by part with the bounded function $I(T) = \int_0^T \sin^3 z\, dz$. On $[0, K]$, we write

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^K \left(\frac{\psi/L}{\sin\psi/L}\right) \frac{\sin^3\psi}{\psi}\, d\psi = \frac{8}{\pi^{\frac{3}{2}}} \int_0^K \frac{\sin^3\psi}{\psi}\, d\psi + O\left(\frac{1}{L^2}\int_0^K \psi\sin^3\psi\, d\psi\right) =$$

$$= \frac{8}{\pi^{\frac{3}{2}}} \int_0^K \frac{\sin^3\psi}{\psi}\, d\psi + O\left(\frac{K^2}{L^2}\right).$$

Consolidating the latter with (3.13) we find that

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^{L\frac{\pi}{2}} \left(\frac{\psi/L}{\sin\psi/L}\right) \frac{\sin^3\psi}{\psi}\, d\psi = \frac{8}{\pi^{\frac{3}{2}}} \int_0^K \frac{\sin^3\psi}{\psi}\, d\psi + O\left(\frac{1}{K} + \frac{K^2}{L^2}\right).$$

Now as $K \to +\infty$,

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^K \frac{\sin^3\psi}{\psi}\, d\psi \to c_{3;3} \ ;$$

to conclude the proof, it is then enough to choose $K = K(L) \to \infty$ sufficiently slowly, i.e. $K = \sqrt{L}$. □

4 The quantitative Central Limit Theorem for $h_{\ell;q,d}$

In this section we prove Theorem 1.2 with the help of Proposition 2.1 and (2.9) in particular. The identifications of §2.2 lead to some very explicit expressions for the contractions (2.7), as in the following result.

For $\ell \geq 1, q \geq 2$, let $g_{\ell;q}$ be defined as in (2.11).

Lemma 4.1. *For all $q_1, q_2 \geq 2$, $r = 1, ..., q_1 \wedge q_2 - 1$, we have the identities*

$$\|g_{\ell;q_1} \otimes_r g_{\ell;q_2}\|^2_{H^{\otimes n}} =$$

$$= \int_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1, x_2)) G^{q_1\wedge q_2 - r}_{\ell;d}(\cos d(x_2, x_3)) G^r_{\ell;d}(\cos d(x_3, x_4)) G^{q_1\wedge q_2 - r}_{\ell;d}(\cos d(x_1, x_4))\, d\underline{x} \ ,$$

where we set $d\underline{x} := dx_1 dx_2 dx_3 dx_4$ and $n := q_1 + q_2 - 2r$.

Proof. Assume w.l.o.g. $q_1 \leq q_2$ and set for simplicity of notation $d\underline{t} := dt_1 \dots dt_r$. The contraction (2.7) here takes the form

$$(g_{\ell;q_1} \otimes_r g_{\ell;q_2})(y_1, ..., y_n) =$$

$$= \int_{(\mathbb{S}^d)^r} g_{\ell;q_1}(y_1, \dots, y_{q_1 - r}, t_1, \dots, t_r) g_{\ell;q_2}(y_{q_1-r+1}, \dots, y_n, t_1, \dots, t_r)\, d\underline{t} =$$

$$= \int_{(\mathbb{S}^d)^r} \int_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_1/2} G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_1, t_r))\, dx_1 \times$$

$$\times \int_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_2/2} G_{\ell;d}(\cos d(x_2, y_{q_1-r+1})) \dots G_{\ell;d}(\cos d(x_2, t_r))\, dx_2\, d\underline{t} =$$

$$= \int_{(\mathbb{S}^d)^2} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{n/2} G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_1, y_{q_1-r})) \times$$

$$\times G_{\ell;d}(\cos d(x_2, y_{q_1-r+1})) \dots G_{\ell;d}(\cos d(x_2, y_n)) G^r_{\ell;d}(\cos d(x_1, x_2))\, dx_1 dx_2 \ ,$$

where in the last equality we have repeatedly used the reproducing property of Gegenbauer polynomials ([34]). Now set $d\underline{y} := dy_1 \dots dy_n$. It follows at once that

$$\|g_{\ell;q_1} \otimes_r g_{\ell;q_2}\|^2_{H^{\otimes n}} =$$

$$= \int_{(\mathbb{S}^d)^n} (g_{\ell;q_1} \otimes_r g_{\ell;q_2})^2 (y_1, \dots, y_n)\, d\underline{y} =$$

$$= \int_{(\mathbb{S}^d)^n} \int_{(\mathbb{S}^d)^2} \left(\frac{n_{\ell;d}}{\mu_d}\right)^n G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_2, y_n)) G^r_{\ell;d}(\cos d(x_1, x_2)) dx_1 dx_2 \times$$

$$\times \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_4, y_1)) \dots G_{\ell;d}(\cos d(x_3, y_n)) G^r_{\ell;d}(\cos d(x_3, x_4))\, dx_3 dx_4\, d\underline{y} =$$

$$= \int_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1, x_2)) G^{q_1-r}_{\ell;d}(\cos d(x_2, x_3)) G^r_{\ell;d}(\cos d(x_3, x_4)) G^{q_1-r}_{\ell;d}(\cos d(x_1, x_4))\, d\underline{x}\ ,$$

as claimed. □

We need now to introduce some further notation, i.e. for $q \geq 2$ and $r = 1, \dots, q-1$

$$\mathcal{K}_\ell(q; r) := \int_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1, x_2)) G^{q-r}_{\ell;d}(\cos d(x_2, x_3)) \times$$
$$\times G^r_{\ell;d}(\cos d(x_3, x_4)) G^{q-r}_{\ell;d}(\cos d(x_1, x_4))\, dx_1 dx_2 dx_3 dx_4,$$

Lemma 4.1 asserts that

$$\mathcal{K}_\ell(q; r) = \|g_{\ell;q} \otimes_r g_{\ell;q}\|^2_{H^{\otimes 2q-2r}}\ ; \tag{4.1}$$

it is immediate to check that

$$\mathcal{K}_\ell(q; r) = \mathcal{K}_\ell(q; q-r)\ . \tag{4.2}$$

In the following two propositions we bound each term of the form $\mathcal{K}(q; r)$ (from (4.2) it is enough to consider $r = 1, \dots, \left[\frac{q}{2}\right]$). As noted in §1.1, these bounds improve the existing literature even for the case $d = 2$, from which we start our analysis.

For $d = 2$, as previously recalled, Gegenbauer polynomials become standard Legendre polynomials P_ℓ, for which it is well-known that (see (1.9))

$$\int_{\mathbb{S}^2} P_\ell(\cos d(x_1, x_2))^2\, dx_1 = O\left(\frac{1}{\ell}\right)\ ; \tag{4.3}$$

also, from [20], Lemma 3.2 we have that

$$\int_{\mathbb{S}^2} P_\ell(\cos d(x_1, x_2))^4\, dx_1 = O\left(\frac{\log \ell}{\ell^2}\right)\ . \tag{4.4}$$

Finally, it is trivial to show that

$$\int_{\mathbb{S}^2} |P_\ell(\cos d(x_1, x_2))|\, dx_1 \leq \sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_1, x_2))^2\, dx_1} = O\left(\frac{1}{\sqrt{\ell}}\right) \tag{4.5}$$

and

$$\int_{\mathbb{S}^2} |P_\ell(\cos d(x_2, x_3))|^3\, dx_2 \leq \sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_2, x_3))^2\, dx_2} \sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_1, x_2))^4\, dx_1} = O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right)\ . \tag{4.6}$$

Proposition 4.2. *For all $r = 1, 2, \dots, q-1$, we have*

$$\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^5}\right) \text{ for } q = 3\ , \tag{4.7}$$

$$\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^4}\right) \text{ for } q = 4\ , \tag{4.8}$$

$$\mathcal{K}_\ell(q; r) = O\left(\frac{\log \ell}{\ell^{9/2}}\right) \text{ for } q = 5, 6 \tag{4.9}$$

and

$$\mathcal{K}_\ell(q;1)=\mathcal{K}_\ell(q;q-1)=O\left(\frac{1}{\ell^{9/2}}\right)\ ,\ \mathcal{K}_\ell(q;r)=O\left(\frac{1}{\ell^{5}}\right)\ ,\ r=2,...,q-2,\ \textit{for}\ q\geq 7\ . \qquad (4.10)$$

Proof. The bounds (4.7), (4.8) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner's 3j and 6j coefficients, see [15], [16], [20]. The bounds in (4.9),(4.10) derives from a simple improvement in the proof of Proposition 2.2 in [20], which can be obtained when focussing only on a subset of the terms (the circulant ones) considered in that reference. In the proof to follow, we exploit repeatedly (4.3), (4.4), (4.5) and (4.6).

Let us start investigating the case $q=5$:

$$\begin{aligned}\mathcal{K}_\ell(5;1)&=\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^4\left|P_\ell(\cos d(x_2,x_3))\right|\times\\
&\times\left|P_\ell(\cos d(x_3,x_4))\right|^4\left|P_\ell(\cos d(x_4,x_1))\right|dx_1dx_2dx_3dx_4\leq\\
&\leq\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^4\left|P_\ell(\cos d(x_2,x_3))\right|\left|P_\ell(\cos d(x_3,x_4))\right|^4dx_1dx_2dx_3dx_4\leq\\
&\leq\int_{(\mathbb{S}^2)^3}\left|P_\ell(\cos d(x_1,x_2))\right|^4\left|P_\ell(\cos d(x_2,x_3))\right|\left\{\int_{\mathbb{S}^2}\left|P_\ell(\cos d(x_3,x_4))\right|^4dx_4\right\}dx_1dx_2dx_3\leq\\
&\leq O\left(\frac{\log\ell}{\ell^2}\right)\times\int_{\mathbb{S}^2\times\mathbb{S}^2}\left|P_\ell(\cos d(x_1,x_2))\right|^4\left\{\int_{\mathbb{S}^2}\left|P_\ell(\cos d(x_2,x_3))\right|dx_3\right\}dx_1dx_2\leq\\
&\leq O\left(\frac{\log\ell}{\ell^2}\right)\times O\left(\frac{1}{\sqrt{\ell}}\right)\times\int_{\mathbb{S}^2\times\mathbb{S}^2}\left|P_\ell(\cos d(x_1,x_2))\right|^4dx_1dx_2\leq\\
&\leq O\left(\frac{\log\ell}{\ell^2}\right)\times O\left(\frac{1}{\sqrt{\ell}}\right)\times O\left(\frac{\log\ell}{\ell^2}\right)=O\left(\frac{\log^2\ell}{\ell^{9/2}}\right)\ ;\end{aligned}$$

$$\begin{aligned}\mathcal{K}_\ell(5;2)&=\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^3\left|P_\ell(\cos d(x_2,x_3))\right|^2\times\\
&\times\left|P_\ell(\cos d(x_3,x_4))\right|^3\left|P_\ell(\cos d(x_4,x_1))\right|^2dx_1dx_2dx_3dx_4\leq\\
&\leq\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^3\left|P_\ell(\cos d(x_2,x_3))\right|^2\left|P_\ell(\cos d(x_3,x_4))\right|^3dx_1dx_2dx_3dx_4\leq\\
&\leq\int_{(\mathbb{S}^2)^3}\left|P_\ell(\cos d(x_1,x_2))\right|^3\left|P_\ell(\cos d(x_2,x_3))\right|^2\left\{\int_{\mathbb{S}^2}\left|P_\ell(\cos d(x_3,x_4))\right|^3dx_4\right\}dx_1dx_2dx_3\leq\\
&\leq O\left(\sqrt{\frac{\log\ell}{\ell^3}}\right)\times\int_{\mathbb{S}^2\times\mathbb{S}^2}\left|P_\ell(\cos d(x_1,x_2))\right|^3\left\{\int_{\mathbb{S}^2}\left|P_\ell(\cos d(x_2,x_3))\right|^2dx_3\right\}dx_1dx_2\leq\\
&\leq O\left(\sqrt{\frac{\log\ell}{\ell^3}}\right)\times O\left(\frac{1}{\ell}\right)\times\int_{\mathbb{S}^2\times\mathbb{S}^2}\left|P_\ell(\cos d(x_1,x_2))\right|^3dx_1dx_2\leq\\
&\leq O\left(\sqrt{\frac{\log\ell}{\ell^3}}\right)\times O\left(\frac{1}{\ell}\right)\times O\left(\sqrt{\frac{\log\ell}{\ell^3}}\right)=O\left(\frac{\log\ell}{\ell^4}\right)\ .\end{aligned}$$

For $q=6$ and $r=1$ we simply note that $\mathcal{K}_\ell(6;1)\leq\mathcal{K}_\ell(5;1)$, actually

$$\begin{aligned}\mathcal{K}_\ell(6;1)&=\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^5\left|P_\ell(\cos d(x_2,x_3))\right|\times\\
&\times\left|P_\ell(\cos d(x_3,x_4))\right|^5\left|P_\ell(\cos d(x_4,x_1))\right|dx_1dx_2dx_3dx_4\leq\\
&\leq\int_{(\mathbb{S}^2)^4}\left|P_\ell(\cos d(x_1,x_2))\right|^4\left|P_\ell(\cos d(x_2,x_3))\right|\times\\
&\times\left|P_\ell(\cos d(x_3,x_4))\right|^4\left|P_\ell(\cos d(x_4,x_1))\right|dx_1dx_2dx_3dx_4=\mathcal{K}_\ell(5;1)=O\left(\frac{\log^2\ell}{\ell^{9/2}}\right)\ .\end{aligned}$$

Then we find with analogous computations as for $q = 5$ that

$$\mathcal{K}_\ell(6;2) = \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))|^2 \times$$
$$\times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4 \le$$
$$\le \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))|^2 \times$$
$$\times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4 \le$$
$$\le \int_{\mathbb{S}^2 \times \mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^4 \, dx_1 \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))|^2 \, dx_2 \right\} \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3,x_4))|^4 \, dx_4 \right\} dx_3 =$$
$$= O\left(\frac{\log \ell}{\ell^2}\right) \times O\left(\frac{1}{\ell}\right) \times O\left(\frac{\log \ell}{\ell^2}\right) = O\left(\frac{\log^2 \ell}{\ell^5}\right)$$

and likewise

$$\mathcal{K}_\ell(6;3) = \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^3 \, |P_\ell(\cos d(x_2,x_3))|^3 \times$$
$$\times |P_\ell(\cos d(x_3,x_4))|^3 \, |P_\ell(\cos d(x_4,x_1))|^3 \, dx_1 dx_2 dx_3 dx_4 \le$$
$$\le \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^3 \, |P_\ell(\cos d(x_2,x_3))|^3 \, |P_\ell(\cos d(x_3,x_4))|^3 \, dx_1 dx_2 dx_3 dx_4 =$$
$$= O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) \times O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) \times O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) = O\left(\frac{\log^{3/2} \ell}{\ell^{9/2}}\right) \ .$$

Finally for $q = 7$

$$\mathcal{K}_\ell(7;1) = \int_{\mathbb{S}^2 \times \ldots \times S^2} |P_\ell(\cos d(x_1,x_2))|^6 \, |P_\ell(\cos d(x_2,x_3))| \times$$
$$\times |P_\ell(\cos d(x_3,x_4))|^6 \, |P_\ell(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4 \le$$
$$\le \int_{\mathbb{S}^2 \times S^2} |P_\ell(\cos d(x_1,x_2))|^6 \, dx_1 \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))| \, dx_3 \right\} \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3,x_4))|^6 \, dx_4 \right\} dx_2 =$$
$$= O\left(\frac{1}{\ell^2}\right) \times O\left(\frac{1}{\ell^{1/2}}\right) \times O\left(\frac{1}{\ell^2}\right) = O\left(\frac{1}{\ell^{9/2}}\right)$$

and repeating the same argument we obtain

$$\mathcal{K}_\ell(7;2) = O\left(\frac{1}{\ell^5}\right) \qquad \text{and} \qquad \mathcal{K}_\ell(7;3) = O\left(\frac{\log^{9/2} \ell}{\ell^{11/2}}\right) \ .$$

From (4.2), we have indeed computed the bounds for $\mathcal{K}_\ell(q;r)$, $q = 1, \ldots, 7$ and $r = 1, \ldots, q-1$.

To conclude the proof we note that, for $q > 7$

$$\max_{r=1,\ldots,q-1} \mathcal{K}_\ell(q;r) = \max_{r=1,\ldots,\left[\frac{q}{2}\right]} \mathcal{K}_\ell(q;r) \le \max_{r=1,\ldots,3} \mathcal{K}_\ell(6;r) = O\left(\frac{1}{\ell^{9/2}}\right) \ .$$

Moreover in particular

$$\max_{r=2,\ldots,\left[\frac{q}{2}\right]} \mathcal{K}_\ell(q;r) \le \mathcal{K}_\ell(7;2) \vee \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^5}\right) \ ,$$

so that the dominant terms are of the form $\mathcal{K}_\ell(q;1)$. □

We can now move to the higher-dimensional case, as follows. Let us start with the bounds for all order moments of Gegenbauer polynomials. From (1.9)

$$\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^2 dx_1 = O\left(\frac{1}{\ell^{d-1}}\right) \ ; \tag{4.11}$$

also, from Proposition 1.1, we have that if $q = 2p$, $p = 2, 3, 4...$,

$$\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1, x_2))^q dx_1 = O\left(\frac{1}{\ell^d}\right) \ . \tag{4.12}$$

Finally, it is trivial to show that

$$\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2, x_3))| \, dx_2 \leq \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2, x_3))^2 dx_2} = O\left(\frac{1}{\sqrt{\ell^{d-1}}}\right) \ , \tag{4.13}$$

$$\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2, x_3))|^3 \, dx_2 \leq \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2, x_3))^2 dx_2} \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1, x_2))^4 dx_1} = O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right) \tag{4.14}$$

and for $q \geq 5$ odd,

$$\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2, x_3))|^q \, dx_2 \leq \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2, x_3))^4 dx_2} \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1, x_2))^{2(q-2)} dx_1} = O\left(\frac{1}{\ell^d}\right) \ . \tag{4.15}$$

Analogously to the 2-dimensional case, we have the following.

Proposition 4.3. *For all* $r = 1, 2, ...q - 1$,

$$\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^{2d+\frac{d-5}{2}}}\right) \ \textit{for} \ q = 3 \ , \tag{4.16}$$

$$\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^{2d+\frac{d-3}{2}}}\right) \ \textit{for} \ q = 4 \ , \tag{4.17}$$

and

$$\mathcal{K}_\ell(q; 1) = \mathcal{K}_\ell(q; q-1) = O\left(\frac{1}{\ell^{2d+\frac{d-1}{2}}}\right) \ , \ \mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^{3d-1}}\right) \ , \ r = 2, ..., q-2, \ \textit{for} \ q \geq 5 \ . \tag{4.18}$$

Proof. The proof relies on the same argument of the proof of Proposition 4.2, therefore we shall omit some calculations. In what follows we exploit repeatedly the inequalities (4.12), (4.13), (4.14) and (4.15).

For $q = 3$ we immediately have

$$\begin{aligned}
\mathcal{K}_\ell(3; 1) = \int_{(\mathbb{S}^d)^4} &|G_{\ell;d}(\cos d(x_1, x_2))|^2 \, |G_{\ell;d}(\cos d(x_2, x_3))| \times \\
&\times |G_{\ell;d}(\cos d(x_3, x_4))|^2 \, |G_{\ell;d}(\cos d(x_4, x_1))| \, dx_1 dx_2 dx_3 dx_4 \leq \\
\leq \int_{(\mathbb{S}^d)^4} &|G_{\ell;d}(\cos d(x_1, x_2))|^2 \, |G_{\ell;d}(\cos d(x_2, x_3))| \, |G_{\ell;d}(\cos d(x_3, x_4))|^2 \, dx_1 dx_2 dx_3 dx_4 = \\
&= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\sqrt{\ell^{d-1}}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{5}{2}}}\right) \ .
\end{aligned}$$

Likewise for $q = 4$

$$\begin{aligned}
\mathcal{K}_\ell(4; 1) = \int_{(\mathbb{S}^d)^4} &|G_{\ell;d}(\cos d(x_1, x_2))|^3 \, |G_{\ell;d}(\cos d(x_2, x_3))| \times \\
&\times |G_{\ell;d}(\cos d(x_3, x_4))|^3 \, |G_{\ell;d}(\cos d(x_4, x_1))| \, dx_1 dx_2 dx_3 dx_4 \leq \\
\leq \int_{(\mathbb{S}^d)^4} &|G_{\ell;d}(\cos d(x_1, x_2))|^3 \, |G_{\ell;d}(\cos d(x_2, x_3))| \, |G_{\ell;d}(\cos d(x_3, x_4))|^3 \, dx_1 dx_2 dx_3 dx_4 = \\
&= O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{\frac{d}{2}-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{3}{2}}}\right)
\end{aligned}$$

and moreover

$$\mathcal{K}_\ell(4;2) = \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 \times$$

$$\times |G_{\ell;d}(\cos d(x_2,x_3))|^2 |G_{\ell;d}(\cos d(x_3,x_4))|^2 |G_{\ell;d}(\cos d(x_4,x_1))|^2 \, dx_1dx_2dx_3dx_4 \le$$

$$\le \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 |G_{\ell;d}(\cos d(x_2,x_3))|^2 |G_{\ell;d}(\cos d(x_3,x_4))|^2 \, dx_1dx_2dx_3dx_4 =$$

$$= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{3d-3}}\right) .$$

Similarly, for $q=5$ we get the bounds

$$\mathcal{K}_\ell(5;1) = \int_{\mathbb{S}^d \times \ldots \times S^d} |G_{\ell;d}(\cos d(x_1,x_2))|^4 \times$$

$$\times |G_{\ell;d}(\cos d(x_2,x_3))| \, |G_{\ell;d}(\cos d(x_3,x_4))|^4 \, |G_{\ell;d}(\cos d(x_4,x_1))| \, dx_1dx_2dx_3dx_4 \le$$

$$\le \int_{\mathbb{S}^d \times \ldots \times S^d} |G_{\ell;d}(\cos d(x_1,x_2))|^4 \, |G_{\ell;d}(\cos d(x_2,x_3))| \, |G_{\ell;d}(\cos d(x_3,x_4))|^4 \, dx_1dx_2dx_3dx_4 =$$

$$= O\left(\frac{1}{\ell^{d}}\right) \times O\left(\frac{1}{\ell^{\frac{d}{2}-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{d}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right)$$

and

$$\mathcal{K}_\ell(5;2) = O\left(\frac{1}{\ell^{3d-2}}\right) .$$

It is immediate to check that

$$\mathcal{K}_\ell(6;1) = \mathcal{K}_\ell(7;1) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right) , \quad \mathcal{K}_\ell(6;2) = \mathcal{K}_\ell(7;2) = O\left(\frac{1}{\ell^{2d+d-1}}\right) ,$$

whereas

$$\mathcal{K}_\ell(6;3) = O\left(\frac{1}{\ell^{2d+d-\frac{3}{2}}}\right) \quad \text{and} \quad \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^{2d+d-\frac{1}{2}}}\right) .$$

The remaining terms are indeed bounded thanks to (4.2).

In order to finish the proof, it is enough to note, as for that for $q > 7$

$$\max_{r=1,\ldots,q-1} \mathcal{K}_\ell(q;r) = \max_{r=1,\ldots,\left[\frac{q}{2}\right]} \mathcal{K}_\ell(q;r) \le \max_{r=1,\ldots,3} \mathcal{K}_\ell(6;r) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right) . \tag{4.19}$$

In particular we have

$$\max_{r=2,\ldots,\left[\frac{q}{2}\right]} \mathcal{K}_\ell(q;r) \le \mathcal{K}_\ell(7;2) \vee \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^{3d-1}}\right) , \tag{4.20}$$

so that the dominant terms are again of the form $\mathcal{K}_\ell(q;1)$. □

Exploiting the results in this section and §3, we have the following.

Proof Theorem 1.2. For the case $q=2$ the standard CLT applies. For $q \ge 3$, from Proposition 2.1 and (2.9), for $d_{\mathcal{D}} = d_K, d_{TV}, d_W$

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}}, \mathcal{N}(0,1)\right) = O\left(\sup_r \sqrt{\frac{\mathcal{K}_\ell(q;r)}{\mathrm{Var}[h_{\ell;q,d}]^2}}\right) . \tag{4.21}$$

The proof is an immediate consequence of the previous equality and the results in Proposition 1.1, Proposition 4.2 and Proposition 4.3. □

5 General polynomials

We show how the previous results can be extended to establish quantitative CLTs, with no loss to the case of general, nonHermite polynomials. To this aim, we need to introduce some more notation, namely (for Z_ℓ defined as in (2.10))

$$\mathcal{K}(Z_\ell) := \max_{q:\beta_q\neq 0} \max_{r=1,\dots,q-1} \mathcal{K}_\ell(q;r) \ ,$$

$$R(Z_\ell) = \begin{cases} \frac{1}{\ell^{\frac{d-1}{2}}} \ , & \text{for } \beta_2 \neq 0 \ , \\ \max_{q=3,,,,Q:\beta_q\neq 0} R(\ell;q,d) \ , & \text{for } \beta_2 = 0 \ . \end{cases}$$

In words, $\mathcal{K}(Z_\ell)$ is the largest contraction term among those emerging from the analysis of the different Hermite components, and $R(Z_\ell)$ is the slowest convergence rate of the same components. The next result is stating that these are the only quantities to look at when considering the general case.

Proof Theorem 1.6. We apply Proposition 2.1. In our case

$$\mathrm{Var}[\langle DZ_\ell, -DL^{-1}Z_\ell\rangle_H] = \mathrm{Var}\left[\langle \sum_{q_1=2}^{Q} \beta_{q_1} Dh_{\ell;q_1,d}, -\sum_{q_21=2}^{Q} \beta_{q_2} DL^{-1}h_{\ell;q_2,d}\rangle_H\right] =$$
$$= \mathrm{Var}\left[\sum_{q_1=2}^{Q}\sum_{q_1=2}^{Q} \beta_{q_1}\beta_{q_2}\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right] \ .$$

From §2.1 recall that for $q_1 \neq q_2$

$$E[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H] = 0 \ ,$$

whence we write

$$\mathrm{Var}\left[\sum_{q_1=2}^{Q}\sum_{q_2=2}^{Q} \beta_{q_1}\beta_{q_2}\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right] =$$
$$= \sum_{q_1=2}^{Q}\sum_{q_2=2}^{Q} \beta_{q_1}^2\beta_{q_2}^2 \mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right) +$$
$$+ \sum_{q_1=2}^{Q}\sum_{q_2\neq q_1}^{Q}\sum_{q_3=2}^{Q}\sum_{q_4\neq q_3}^{Q} \beta_{q_1}\beta_{q_2}\beta_{q_3}\beta_{q_4} \mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right) \ .$$

Now of course we have

$$\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right) \leq$$
$$\leq \left(\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H\right] \mathrm{Var}\left[\langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right]\right)^{1/2} ,$$

$$\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right) \leq$$
$$\leq \left(\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right] \mathrm{Var}\left[\langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right]\right)^{1/2} .$$

Applying [26], Lemma 6.2.1 it is immediate to show that

$$\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H\right] \leq$$
$$\leq q_1^2 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \binom{q_1-1}{r-1}^4 (2q_1-2r)! \, \|g_{\ell;q_1} \otimes_r g_{\ell;q_1}\|^2_{H^{\otimes 2q_1-2r}} =$$
$$= q_1^2 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \binom{q_1-1}{r-1}^4 (2q_1-2r)! \mathcal{K}_\ell(q_1;r) \ .$$

Also, for $q_1 < q_2$

$$\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right] =$$
$$= q_1^2 \sum_{r=1}^{q_1} ((r-1)!)^2 \binom{q_1-1}{r-1}^2 \binom{q_2-1}{r-1}^2 (q_1+q_2-2r)! \left\| g_{\ell;q_1} \widetilde{\otimes}_r g_{\ell;q_2} \right\|^2_{H^{\otimes(q_1+q_2-2r)}} =$$
$$= q_1^2((q_1-1)!)^2 \binom{q_2-1}{q_1-1}^2 (2q_1-2r)! \left\| g_{\ell;q_1} \widetilde{\otimes}_{q_1} g_{\ell;q_2} \right\|^2_{H^{\otimes(q_2-q_1)}} +$$
$$+ q_1^2 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \binom{q_1-1}{r-1}^2 \binom{q_2-1}{r-1}^2 (q_1+q_2-2r)! \left\| g_{\ell;q_1} \widetilde{\otimes}_r g_{\ell;q_2} \right\|^2_{H^{\otimes(q_1+q_2-2r)}} =: A + B\ .$$

Let us focus on the first summand A, which includes terms that, from Lemma 4.1, take the form

$$\left\| g_{\ell;q_1} \widetilde{\otimes}_{q_1} g_{\ell;q_2} \right\|^2_{H^{\otimes(q_2-q_1)}} \le \left\| g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2} \right\|^2_{H^{\otimes(q_2-q_1)}} =$$
$$= \int_{(\mathbb{S}^d)^{q_2-q_1}} \int_{(\mathbb{S}^d)^2} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_2-q_1} G_{\ell;d}(\cos d(x_2,y_1))...G_{\ell;d}(\cos d(x_2,y_{q_2-q_1}))G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\, dx_1 dx_2 \times$$
$$\times \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_3,y_1))...G_{\ell;d}(\cos d(x_3,y_{q_2-q_1}))G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\, dx_3 dx_4\, d\underline{y} =: I\ ,$$

where for the sake of simplicity we have set $d\underline{y} := dy_1...dy_{q_2-q_1}$. Applying q_2-q_1 times the reproducing formula for Gegenbauer polynomials ([34]) we get

$$I = \int_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\, d\underline{x}\ . \tag{5.1}$$

In graphical terms, these contractions correspond to the diagrams such that all q_1 edges corresponding to vertex 1 are linked to vertex 2, vertex 2 and 3 are connected by q_2-q_1 edges, vertex 3 and 4 by q_1 edges, and no edges exist between 1 and 4, i.e. the diagram has no proper loop.

Now immediately we write

$$(5.1) = \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\, dx_1 \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\, dx_4 \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\, dx_2 dx_3 =$$
$$= \frac{1}{(q_1!)^2} \mathrm{Var}[h_{\ell;q_1,d}]^2 \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\, dx_2 dx_3\ .$$

Moreover we have

$$\int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\, dx_2 dx_3 = 0\ , \quad \text{if } q_2-q_1 = 1 \tag{5.2}$$

and from (1.9) if $q_2 - q_1 \ge 2$

$$\int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\, dx_2 dx_3 \le \mu_d \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x,y))^2\, dx = O\left(\frac{1}{\ell^{d-1}}\right)\ .$$

It follows that

$$\left\| g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2} \right\|^2_{H^{\otimes(q_2-q_1)}} = O\left(\mathrm{Var}[h_{\ell;q_1,d}]^2 \frac{1}{\ell^{d-1}}\right) \tag{5.3}$$

always. For the second term, still from [26], Lemma 6.2.1 we have

$$B \le \frac{q_1^2}{2} \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left(\begin{array}{c} q_1-1 \\ r-1 \end{array}\right)^2 \left(\begin{array}{c} q_2-1 \\ r-1 \end{array}\right)^2 (q_1+q_2-2r)! \times$$
$$\times \left(\left\| g_{\ell;q_1} \otimes_{q_1-r} g_{\ell;q_1} \right\|^2_{H^{\otimes 2r}} + \left\| g_{\ell;q_2} \otimes_{q_2-r} g_{\ell;q_2} \right\|^2_{H^{\otimes 2r}} \right) =$$

$$= \frac{q_1^2}{2} \sum_{r=1}^{q_1-1} ((r-1)!)^2 \begin{pmatrix} q_1 - 1 \\ r-1 \end{pmatrix}^2 \begin{pmatrix} q_2 - 1 \\ r-1 \end{pmatrix}^2 (q_1+q_2-2r)! \left(\mathcal{K}_\ell(q_1;r) + \mathcal{K}_\ell(q_2;r)\right) \ , \tag{5.4}$$

where the last step follows from Lemma 4.1.

Let us first investigate the case $d=2$. From (1.10), (1.11) and (1.13) it is immediate that

$$\mathrm{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \mathrm{Var}[h_{\ell;q}] = \begin{cases} O(\ell^{-1}) \ , & \text{for } \beta_2 \neq 0 \\ O(\ell^{-2}\log \ell) \ , & \text{for } \beta_2 = 0 \ , \ \beta_4 \neq 0 \\ O(\ell^{-2}) \ , & \text{otherwise.} \end{cases} \tag{5.5}$$

Hence we have that for $\beta_2 \neq 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(2;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\ell^{-1/2}\right) \ ;$$

for $\beta_2 = 0$, $\beta_4 \neq 0$,

$$d_{TV}(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)) = O(\frac{\sqrt{\mathcal{K}_\ell(4;r)}}{\mathrm{Var}[Z_\ell]}) = O\left(\frac{1}{\log \ell}\right)$$

and for $\beta_2 = \beta_4 = 0$, $\beta_5 \neq 0$ and $c_5 > 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(5;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{\log \ell}{\ell^{1/4}}\right) \ .$$

and analogously we deal with the remaining cases, so that we obtain the claimed result for $d=2$.

For $d \geq 3$ from (1.9) and Proposition 1.1, it holds

$$\mathrm{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \mathrm{Var}(h_{\ell;q,d}) = \begin{cases} O(\ell^{-d+1}) \ , & \text{for } \beta_2 \neq 0 \ , \\ O(\ell^{-d}) \ , & \text{otherwise} \ . \end{cases}$$

Hence we have for $\beta_2 \neq 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(2;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-1}{2}}}\right) \ .$$

Likewise for $\beta_2 = 0$, $\beta_3, c_{3;d} \neq 0$,

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(3;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-5}{4}}}\right)$$

and for $\beta_2 = \beta_3 = 0$, $\beta_4 \neq 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(4;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-3}{2}}}\right) \ .$$

Finally if $\beta_2 = \beta_3 = \beta_4 = 0$, $\beta_q, c_{q;d} \neq 0$ for some q, then

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(q;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\sqrt{\frac{\ell^{2d}}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}}\right) = O\left(\frac{1}{\ell^{\frac{d-1}{4}}}\right) \ .$$

□

Remark 5.1. *To compare our result in these specific circumstances with the general bound obtained by Nourdin and Peccati, we note that for (5.1), these authors are exploiting the inequality*

$$\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} \le \|g_{\ell;q_1}\|^2_{H^{\otimes q_1}} \, \|g_{\ell;q_2} \otimes_{q_2-q_1} g_{\ell;q_2}\|_{H^{\otimes 2q_1}} \ ,$$

see [26], Lemma 6.2.1. *In the special framework we consider here (i.e., orthogonal eigenfunctions), this provides, however, a less efficient bound than (5.3): indeed from (5.1), repeating the same argument as in Lemma 4.1, one obtains*

$$\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} = \int_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1} \, d\underline{x} \le$$

$$\le \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} \, dx_1 dx_2 \times$$

$$\times \sqrt{\int_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1} G_{\ell;d}(\cos d(x_1,x_4))^{q_2-q_1} \, d\underline{x}} =$$

$$= O\left(\mathrm{Var}[h_{\ell;q_1,d}] \sqrt{\mathcal{K}_\ell(q_2,q_1)}\right) \ ,$$

yielding a bound of order

$$O\left(\sqrt{\frac{\mathrm{Var}[h_{\ell;q_1,d}]\sqrt{\mathcal{K}_\ell(q_2,q_1)}}{\mathrm{Var}[h_{\ell;q_1,d}]^2}}\right) = O\left(\frac{\sqrt[4]{\mathcal{K}_\ell(q_2,q_1)}}{\sqrt{\mathrm{Var}[h_{\ell;q_1,d}]}}\right) \tag{5.6}$$

rather than

$$O\left(\sqrt{\frac{\mathcal{K}_\ell(q_2,q_1)}{\mathrm{Var}[h_{\ell;q_1,d}]^2}}\right) \ ; \tag{5.7}$$

for instance, for $d = 2$ note that (5.6) is typically $= O(\ell \times \ell^{-9/8}) = O(\ell^{-1/8})$, while we have established for (5.7) bounds of order $O(\ell^{-1/4})$.

Remark 5.2. *Clearly the fact that $\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} = 0$ for $q_2 = q_1 + 1$ entails that the contraction $g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}$ is identically null. Indeed repeating the same argument as in Lemma 4.1*

$$g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_1+1} =$$

$$= \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,y)) G_{\ell;d}(\cos d(x_1,x_2))^{q_1} \, dx_1 dx_2 =$$

$$= \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,y)) \, dx_1 \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} \, dx_2 = 0 \ ,$$

as expected.

6 General nonlinear functionals and excursion sets

The techniques and results developed in §4, 5 are restricted to finite-order polynomials. In the special case of the Wasserstein distance, we shall show below how they can indeed be extended to general nonlinear functionals of the form (1.19)

$$S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x)) dx \ ;$$

here $M : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\mathbb{E}[M(T_\ell)^2] < \infty$ and $J_2(M) \neq 0$, where we recall that $J_q(M) := \mathbb{E}[M(T_\ell) H_q(T_\ell)]$.

Remark 6.1. *Without loss of generality, the first two coefficients $J_0(M), J_1(M)$ can always be taken to be zero in the present framework. Indeed, $J_0(M) := \mathbb{E}[M(T_\ell)] = 0$, assuming we work with centred variables and moreover as we noted earlier $h_{\ell;1,d} = \int_{\mathbb{S}^d} T_\ell(x)\, dx = 0$.*

Proof Theorem 1.7. As in [18], from (1.20) we write the expansion

$$S_\ell(M) = \int_{\mathbb{S}^d} \sum_{q=2}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx \ .$$

Precisely, we write for $d = 2$

$$S_\ell(M) = \frac{J_2(M)}{2} h_{\ell;2,2} + \frac{J_3(M)}{3!} h_{\ell;3,2} + \frac{J_4(M)}{4!} h_{\ell;4,2} + \int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx \ , \tag{6.1}$$

whereas for $d \geq 3$

$$S_\ell(M) = \frac{J_2(M)}{2} h_{\ell;2,d} + \int_{\mathbb{S}^d} \sum_{q=3}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx \ . \tag{6.2}$$

Let us first investigate the case $d = 2$. Set for the sake of simplicity

$$S_\ell(M;1) := \frac{J_2(M)}{2} h_{\ell;2,2} + \frac{J_3(M)}{3!} h_{\ell;3,2} + \frac{J_4(M)}{4!} h_{\ell;4,2} \ ,$$

$$S_\ell(M;2) := \int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx \ .$$

Hence from (6.1) and the triangular inequality

$$d_W\left(\frac{S_\ell(M)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}(0,1)\right) \leq$$

$$\leq d_W\left(\frac{S_\ell(M)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}\right) + d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right)\right) +$$

$$+ d_W\left(\mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right), \mathcal{N}(0,1)\right) \leq$$

$$\leq \frac{1}{\sqrt{\mathrm{Var}[S_\ell(M)]}} \mathbb{E}\left[\left(\int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx\right)^2\right]^{1/2} +$$

$$+ d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]})\right) + d_W\left(\mathcal{N}(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}), \mathcal{N}(0,1)\right) \ .$$

Let us bound the first term of the previous summation. Of course

$$\mathrm{Var}[S_\ell(M)] = \mathrm{Var}[S_\ell(M;1)] + \mathrm{Var}[S_\ell(M;2)] \ ;$$

now we have (see [18])

$$\mathrm{Var}[S_\ell(M;1)] = \frac{J_2^2(M)}{2} \mathrm{Var}[h_{\ell;2,2}] + \frac{J_3^2(M)}{6} \mathrm{Var}[h_{\ell;3,2}] + \frac{J_4^2(M)}{4!} \mathrm{Var}[h_{\ell;4,2}]$$

and moreover

$$\mathrm{Var}[S_\ell(M;2)] = \mathbb{E}\left[\left(\int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx\right)^2\right] = \sum_{q=5}^{\infty} \frac{J_q^2(M)}{q!} \mathrm{Var}[h_{\ell;q,2}] \ll \frac{1}{\ell^2} \sum_{q=5}^{\infty} \frac{J_q^2(M)}{q!} \ll \frac{1}{\ell^2} \ ,$$

where the last bounds follows from (1.11) and (1.12). Therefore recalling also (1.10) and (1.13)

$$\frac{1}{\mathrm{Var}[S_\ell(M)]} \mathbb{E}\left[\left(\int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx\right)^2\right] \ll \frac{1}{\ell} \ .$$

On the other hand, from Theorem 1.6

$$d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}},\mathcal{N}\left(0,\frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right)\right)=O\left(\frac{1}{\sqrt{\ell}}\right)$$

and finally, using Proposition 3.6.1 in [26],

$$d_W\left(\mathcal{N}\left(0,\frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right),\mathcal{N}(0,1)\right)\quad\le\quad\sqrt{\frac{2}{\pi}}\left|\frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}-1\right|=O\left(\frac{1}{\ell}\right)\ ,$$

so that the proof for $d=2$ is completed.

The proof in the general case $d\geq 3$ is indeed analogous, just setting

$$S_\ell(M;1):=\frac{J_2(M)}{2}h_{\ell;2,d}\ ,$$

$$S_\ell(M;2):=\int_{\mathbb{S}^2}\sum_{q=3}^{\infty}\frac{J_q(M)H_q(T_\ell(x))}{q!}dx$$

and recalling from (1.9) that $\mathrm{Var}[h_{\ell;2,d}]=O(\frac{1}{\ell^{d-1}})$ whereas for $q\geq 3$, $\mathrm{Var}[h_{\ell;q,d}]=O(\frac{1}{\ell^d})$ from Proposition 1.1. □

A remarkable special case is obtained for the excursion sets, which for any fixed $z\in\mathbb{R}$ can be defined as

$$S_\ell(z):=S_\ell(\mathbb{I}(\cdot\leq z))=\int_{\mathbb{S}^d}\mathbb{I}(T_\ell(x)\leq z)dx\ ,$$

where $\mathbb{I}(\cdot\leq z)$ is the indicator function of the interval $(-\infty,z]$. Note that $\mathbb{E}[S_\ell(z)]=\mu_d\Phi(z)$, where $\Phi(z)$ is the cdf of the standard Gaussian law, and in this case we have $M=M_z:=\mathbb{I}(\cdot\leq z)$, $J_2(M_z)=z\phi(z)$, ϕ denoting the standard Gaussian density. The following corollary is then immediate:

Corollary 6.2. *If $z\neq 0$, as $\ell\to\infty$, we have that*

$$d_W\left(\frac{S_\ell(z)-\mu_d\Phi(z)}{\sqrt{\mathrm{Var}[S_\ell(z)]}},\mathcal{N}(0,1)\right)=O\left(\frac{1}{\sqrt{\ell}}\right)\ .$$

Remark 6.3. *It should be noted that the rate obtained here is much sharper than the one provided by [31] for the Euclidean case with $d=2$. The asymptotic setting we consider is rather different from his, in that we consider the case of spherical eigenfunction with diverging eigenvalues, whereas he focusses on functionals evaluated on increasing domains $[0,T]^d$ for $T\to\infty$. However the contrast in the converging rates is not due to these different settings, indeed [8] establish rates of convergence analogous to those by [31] for spherical random fields with more rapidly decaying covariance structure than the one we are considering here. The main point to notice is that the slow decay of Gegenbauer polynomials entails some form of long range dependent behaviour on random spherical harmonics; in this sense, hence, our results may be closer in spirit to the work by [9] on empirical processes for long range dependent stationary processes on $\mathbb{R}$.*

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