Non-Universality of Nodal Length Distribution for Arithmetic Random Waves

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Abstract

"Arithmetic random waves" are the Gaussian Laplace eigenfunctions on the two-dimensional torus [R-W, K-K-W]. In this paper we find that their nodal length converges to a non-universal (non-Gaussian) limiting distribution, depending on the angular distribution of lattice points lying on circles.

Our argument has two main ingredients. An explicit derivation of the Wiener-Itô chaos expansion for the nodal length shows that it is dominated by its 4th order chaos component (in particular, somewhat surprisingly, the second order chaos component vanishes). The rest of the argument relies on the precise analysis of the fourth order chaotic component.

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1 Introduction and main results

1.1 Arithmetic random waves

Let $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus and Δ the Laplacian on $\mathbb{T}$. We are interested in the (totally discrete) spectrum of Δ i.e., eigenvalues $E > 0$ of the Schrödinger equation

$$\Delta f + E f = 0. \tag{1.1}$$

Let

$$S = \{n \in \mathbb{Z} : n = a^2 + b^2 \ \text{ for some } a,\, b \in \mathbb{Z}\}$$

be the collection of all numbers expressible as a sum of two squares. Then, the eigenvalues of (1.1) (also called *energy levels* of the torus) are all numbers of the form $E_n = 4\pi^2 n$ with $n \in S$.

In order to describe the Laplace eigenspace corresponding to E_n, denote by Λ_n the set of *frequencies*:

$$\Lambda_n := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \lambda_1^2 + \lambda_2^2 = n\}$$

whose cardinality

$$\mathcal{N}_n := |\Lambda_n| = r_2(n) \tag{1.2}$$

equals the number of ways to express n as a sum of two squares. (Geometrically, Λ_n is the collection of all standard lattice points lying on the centred circle with radius $\sqrt{n}$.) For $\lambda \in \Lambda_n$ denote the complex exponential associated to the frequency λ

$$e_\lambda(x) = \exp(2\pi i \langle \lambda, x\rangle)$$

with $x = (x_1, x_2) \in \mathbb{T}$. The collection

$$\{e_\lambda(x)\}_{\lambda \in \Lambda_n}$$

of the complex exponentials corresponding to the frequencies $\lambda \in \Lambda_n$, is an L^2-orthonormal basis of the eigenspace $\mathcal{E}_n$ of Δ corresponding to the eigenvalue E_n. In particular, the dimension of $\mathcal{E}_n$ is

$$\dim \mathcal{E}_n = \mathcal{N}_n = |\Lambda_n|$$

(cf. (1.2)). The number $\mathcal{N}_n$ is subject to large and erratic fluctuations; it grows [La] *on average* as $\sqrt{\log n}$, but could be as small as 8 for (an infinite sequence of) prime numbers $p \equiv 1 \mod 4$, or as large as a power of $\log n$.

Following [R-W] and [K-K-W], we define the *arithmetic random waves* (also called *random Gaussian toral Laplace eigenfunctions*) to be the random fields

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(x), \quad x \in \mathbb{T}, \tag{1.3}$$

where the coefficients a_λ are standard complex-Gaussian random variables verifying the following properties: a_λ is stochastically independent of a_γ whenever $\gamma \notin \{\lambda, -\lambda\}$, and

$$a_{-\lambda} = \overline{a_\lambda}$$

(ensuring that the T_n are real-valued).[1] By the definition (1.3), T_n is a stationary (i.e. the law of T_n is invariant under all the translations

$$f(\cdot) \mapsto f(x' + \cdot),$$

$x' \in \mathbb{T}$), centered Gaussian random field with covariance function

$$r_n(x, x') = r_n(x - x') := \mathbb{E}[T_n(x)\overline{T_n(x')}] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} e_\lambda(x - x') = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos\left(2\pi \langle x - x', \lambda \rangle\right),$$

$x, x' \in \mathbb{T}$ (by the standard abuse of notation for stationary fields). Note that $r_n(0) = 1$, i.e. T_n has unit variance.

1.2 Nodal length: mean and variance

Consider the total *nodal length* of the random eigenfunctions, i.e. the collection $\{\mathcal{L}_n\}_{n \in S}$ of all random variables with the form

$$\mathcal{L}_n := \mathrm{length}(T_n^{-1}(0)). \tag{1.4}$$

The expected value of $\mathcal{L}_n$ was computed in [R-W] to be

$$\mathbb{E}[\mathcal{L}_n] = \frac{1}{2\sqrt{2}} \sqrt{E_n}, \tag{1.5}$$

consistent with Yau's conjecture [Ya, D-F]. The more challenging question of the asymptotic behaviour of the variance $\mathrm{Var}(\mathcal{L}_n)$ of $\mathcal{L}_n$ was addressed in [R-W], and fully resolved in [K-K-W] as follows.

[1] From now on, we assume that every random object considered in this paper is defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}$ denoting mathematical expectation with respect to $\mathbb{P}$.

Given $n \in S$, define a probability measure μ_n on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$ supported on angles corresponding to lattice points in Λ_n:

$$\mu_n := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\frac{\lambda}{\sqrt{n}}}.$$

It is known [E-H] that for a density 1 sequence of numbers $\{n_j\} \subseteq S$ the angles of lattice points in Λ_n tend to be equidistributed, in the sense that

$$\mu_{n_j} \Rightarrow \frac{d\phi}{2\pi} \tag{1.6}$$

(where $\Rightarrow$ indicates weak-$*$ convergence of probability measures, and $d\phi$ stands for the Lebesgue measure on $\mathcal{S}^1$). However the sequence $\{\mu_n\}_{n\in S}$ has other weak-$*$ adherent points [Ci, K-K-W] (called *attainable measures*), partially classified in [K-W].

It was proved in [K-K-W] that one has

$$\mathrm{Var}(\mathcal{L}_n) = c_n \frac{E_n}{\mathcal{N}_n^2}(1 + o_{\mathcal{N}_n \to \infty}(1)), \tag{1.7}$$

where

$$c_n = \frac{1 + \widehat{\mu_n}(4)^2}{512}, \tag{1.8}$$

and, for a measure μ on $\mathcal{S}^1$,

$$\widehat{\mu}(k) = \int_{\mathcal{S}^1} z^{-k}\, d\mu(z), \quad k \in \mathbb{Z},$$

are the Fourier coefficients of μ on the unit circle. As

$$|\widehat{\mu_n}(4)| \le 1$$

by the triangle inequality, the result (1.7) shows that the true order of magnitude of $\mathrm{Var}(\mathcal{L}_n)$ is $\frac{E_n}{\mathcal{N}_n^2}$: this is of smaller order than what would be a natural guess, namely $\frac{E_n}{\mathcal{N}_n}$; this situation (customarily called *arithmetic Berry's cancellation*, see [K-K-W]) is similar to the *cancellation phenomenon* observed by Berry in a different setting, see [Be, W1].

In addition, (1.7) shows that, in order for $\mathrm{Var}(\mathcal{L}_n)$ to exhibit an asymptotic law (equivalent to $\{c_n\}$ in (1.8) being convergent along a subsequence) we need to pass to a subsequence $\{n_j\} \subseteq S$ such that the limit

$$\lim_{j\to\infty} |\widehat{\mu}_{n_j}(4)|$$

exists. For example, if $\{n_j\} \subseteq S$ is a subsequence such that $\mu_{n_j} \Rightarrow \mu$ for some probability measure μ on $\mathcal{S}^1$, then (1.7) reads (under the usual extra-assumption $\mathcal{N}_{n_j} \to \infty$)

$$\mathrm{Var}(\mathcal{L}_{n_j}) \sim c(\mu) \frac{E_{n_j}}{\mathcal{N}_{n_j}^2} \tag{1.9}$$

with

$$c(\mu) = \frac{1 + \widehat{\mu}(4)^2}{512},$$

where, here and for the rest of the paper, we write $a_n \sim b_n$ to indicate that the two positive sequences $\{a_n\}$ and $\{b_n\}$ are such that $a_n/b_n \to 1$, as $n \to \infty$. Here, the set of the possible values for the 4th Fourier coefficient $\widehat{\mu}(4)$ attains the whole interval $[-1, 1]$ (see [K-K-W, K-W]). This implies in particular that the possible values of the asymptotic constant $c(\mu)$ attain the whole interval $\left[\frac{1}{512}, \frac{1}{256}\right]$; the above is a complete classification of the asymptotic behaviour of $\mathrm{Var}(\mathcal{L}_n)$.

1.3 Statement of the main results: asymptotic distribution of the nodal length

Our main goal is the study of the fine asymptotic behaviour, as $\mathcal{N}_n \to \infty$, of the distributions of the sequence of normalised random variables

$$\widetilde{\mathcal{L}}_n := \frac{\mathcal{L}_n - \mathbb{E}[\mathcal{L}_n]}{\sqrt{\mathrm{Var}(\mathcal{L}_n)}}, \quad n \in S, \tag{1.10}$$

(this is equivalent to studying $\widetilde{\mathcal{L}}_{n_j}$ along subsequences $\{n_j\}_{j\geq 1} \subseteq S$ satisfying $\mathcal{N}_{n_j} \to \infty$; note that it is possible to choose a full density subsequence in S as above). Since the variance (1.7) diverges to infinity, it seems reasonable to expect a central limit result, that is, that the sequence $\{\widetilde{\mathcal{L}}_n\}$ converges in distribution to a standard Gaussian random variable. Our findings not only contradict this (somewhat naive) prediction, but also classify all the weak-$*$ adherent points of the probability distributions associated with the collection of random variables $\left\{\widetilde{\mathcal{L}}_n : n \in S\right\}$ (where the adherent points are in the sense of weak-$*$ convergence of probability measures). In particular, we will show that such a set of weak-$*$ adherent points coincides with the collection of probability distributions associated with a family of linear combinations of two independent squared Gaussian random variables; these linear combinations are parameterized by the adherent points of the sequence $\{|\widehat{\mu_n}(4)|\}$ of real non-negative numbers ≤ 1. This will show the remarkable fact that the angular distribution of Λ_n (or, more specifically, the 4th Fourier coefficient of μ_n) does not only prescribe the leading term of the nodal length variance $\mathrm{Var}(\mathcal{L}_n)$, but, in addition, it prescribes the asymptotic distribution of $\widetilde{\mathcal{L}}_n$.

To state our results formally, we will need some more notation. For $\eta \in [0,1]$, let $\mathcal{M}_\eta$ be the random variable

$$\mathcal{M}_\eta := \frac{1}{2\sqrt{1+\eta^2}}(2 - (1+\eta)X_1^2 - (1-\eta)X_2^2), \tag{1.11}$$

where $X = (X_1, X_2)$ are independent standard Gaussians. Note that for $\eta_1 \neq \eta_2$ the distributions of $\mathcal{M}_{\eta_1}$ and $\mathcal{M}_{\eta_2}$ are genuinely different; this follows for example from the observation that the support of the distribution of $\mathcal{M}_\eta$ is

$$\left(-\infty, \frac{1}{\sqrt{1+\eta^2}}\right].$$

Our first main result establishes a limiting law for the nodal length distribution for subsequences $\{n_j\}_{j\geq 1} \subseteq S$ provided that the numerical sequence

$$\left\{\left|\widehat{\mu}_{n_j}(4)\right| : j \geq 1\right\}$$

of non-negative numbers is convergent. As it was mentioned above, for some full density subsequence $\{n_j\}_{j\geq 1} \subseteq S$ the corresponding lattice points Λ_{n_j} are asymptotically equidistributed (1.6), so that for this subsequence, in particular,

$$\widehat{\mu}_{n_j}(4) \to 0.$$

More generally, if for some subsequence $\{n_j\}_{j\geq 1} \subseteq S$ the angular distribution of the corresponding lattice points converges to μ, i.e. $\mu_{n_j} \Rightarrow \mu$, where μ is some probability measure on $\mathcal{S}^1$, then

$$\widehat{\mu}_{n_j}(4) \to \widehat{\mu}(4).$$

From now on, we use the symbol $\xrightarrow{\mathrm{d}}$ to denote convergence in distribution of random variables; similarly, we will write $X \overset{\mathrm{d}}{=} Y$ to indicate that the random variables X and Y have the same distribution.

Theorem 1.1. *Let $\{n_j\} \subseteq S$ be a subsequence of S satisfying $\mathcal{N}_{n_j} \to \infty$, such that the sequence $\left\{ \left| \widehat{\mu_{n_j}}(4) \right| : j \geq 1 \right\}$ of non-negative numbers converges, that is:*

$$|\widehat{\mu}_{n_j}(4)| \to \eta,$$

for some $\eta \in [0,1]$. Then

$$\widetilde{\mathcal{L}}_{n_j} \xrightarrow{\mathrm{d}} \mathcal{M}_\eta, \tag{1.12}$$

where $\mathcal{M}_\eta$ was defined in (1.11).

Since [K-K-W, K-W] showed that the set of adherent points of $\{\widehat{\mu}_n(4)\}_{n\in S}$ is all of $[-1,1]$, the result above clearly implies that $\widetilde{\mathcal{L}}_n$ does not converge in distribution for $\mathcal{N}_n \to \infty$; in particular, if the sequence $\{|\widehat{\mu}_{n_j}(4)|\}$ does not converge, then the set of probability distributions associated with the random variables $\{\widetilde{\mathcal{L}}_{n_j}\}$ has at least two different adherent points in the topology of weak-$*$ convergence. It would be desirable to formulate a uniform asymptotic result a la (1.12) with no separation of the full sequence S into subsequences according to the angular distribution of Λ_n (still as $\mathcal{N}_n \to \infty$). This has two subtleties though.

First, since there is no convergence in distribution, we need to couple the random variables on the same probability space and work with some metric on the space of probability measures; we choose to work with the L^p-metrics, $p \in (0,2)$. Second, as, given a number $n \in S$, there is no limiting value η of $\widehat{\mu}_n(4)$, for each $n \in S$ the candidate $\mathcal{M}_\eta$ for the limiting random variable will bear

$$\eta = \eta_n = |\widehat{\mu}_n(4)|$$

rather than its limiting value.

Theorem 1.2. *On some auxiliary probability space $(A, \mathscr{A}, \widetilde{\mathbb{P}})$ for every $n \in S$ there exists a coupling of the random variables $\widetilde{\mathcal{L}}_n$ and $\mathcal{M}_{|\widehat{\mu}_n(4)|}$ such that, as $\mathcal{N}_n \to \infty$,*

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left| \widetilde{\mathcal{L}}_n - \mathcal{M}_{|\widehat{\mu}_n(4)|} \right|^p \right] \to 0, \tag{1.13}$$

for every $p \in (0,2)$, and

$$\widetilde{\mathcal{L}}_n - \mathcal{M}_{|\widehat{\mu}_n(4)|} \to 0, \quad a.s. - \widetilde{\mathbb{P}}. \tag{1.14}$$

Relation (1.14) is equivalent to saying that, for every sequence $\{n_j\} \subseteq S$ such that $\mathcal{N}_{n_j} \to \infty$, $\widetilde{\mathbb{P}}\big(\widetilde{\mathcal{L}}_{n_j} - \mathcal{M}_{|\widehat{\mu}_{n_j}(4)|} \to 0\big) = 1$. Since, under the most natural coupling of the family of variables $\{\mathcal{M}_\eta\}_{\eta\in[0,1]}$ we have

$$\mathbb{E}\left[|\mathcal{M}_{\eta_1} - \mathcal{M}_{\eta_2}|\right] \leq c|\eta_1 - \eta_2|,$$

for all $\eta_1, \eta_2 \in [0,1]$ (with $c > 0$ an absolute constant) it is clear that Theorem 1.2 implies the result of Theorem 1.1. In fact, by the triangle inequality and an immediate computation, Theorem 1.2 implies the stronger, L^p-convergence, $p \in (0,2)$ to suitably coupled $\mathcal{M}_\eta$ in (1.12).

1.4 On the proofs of the main results

In Proposition 3.2 we compute the *Wiener-Itô chaos expansion* for the nodal length $\mathcal{L}_n$ (1.4), i.e. a series converging in $L^2(\mathbb{P})$ of the form

$$\mathcal{L}_n = \sum_{q=0}^{\infty} \mathrm{proj}(\mathcal{L}_n | C_q) = \sum_{q=0}^{\infty} \mathcal{L}_n[q]. \tag{1.15}$$

Here C_q, $q = 0, 1, \dots$ are the so-called *Wiener chaoses* (see §2.1), namely the orthogonal components of the L^2-space of those random variables that are functionals of some Gaussian white noise on $\mathbb{T}$ – while $\mathcal{L}_n[q] := \mathrm{proj}(\mathcal{L}_n | C_q)$ denotes the orthogonal projection of $\mathcal{L}_n$ onto the q-th chaos.

The decomposition (1.15) is of independent interest, and entails in particular the vanishing of all the odd-order chaotic components and the term of order two, i.e. $\mathcal{L}_n[q] = 0$ if $q = 2m+1, m = 0, 1, \dots$ or $q = 2$. The precise analysis of the asymptotic behavior of the fourth-order projection in Proposition 2.2 will allow us to show that its variance is asymptotic to the total variance of the nodal length (see Proposition 2.3); since the different components are orthogonal by construction, this will imply that all the projections other than the one on the fourth chaos are negligible. We notice that it is relatively easy to show that the contribution to the nodal length variance of each of the chaotic projections of order $q \neq 4$ is negligible. It is in principle also possible to directly bound the total contribution to the variance of the sum of all these projections, thus establishing relation (1.7) independently. However, this task seems to be technically demanding, and would make our argument significantly longer. Since the asymptotic result (1.7) is already available from [K-K-W], we do not pursue such a strategy in the present manuscript.

As a consequence, to study the asymptotic behavior of $\mathcal{L}_n$ it will be sufficient to focus on the above-mentioned fourth-order component; Proposition 2.2 shows that along subsequences $\{n_j\}$ satisfying the same hypothesis as in Theorem 1.1, we have

$$\frac{\mathcal{L}_{n_j}[4]}{\sqrt{\mathrm{Var}(\mathcal{L}_{n_j}[4])}} \xrightarrow{\mathrm{d}} \mathcal{M}_\eta,$$

where $\mathcal{M}_\eta$ is as in (1.11).

We are then able to prove Theorem 1.1 thanks to Proposition 2.3 and Proposition 2.2. Finally, Theorem 1.2 will follow from Theorem 1.1 and some standard arguments for uniformly bounded sequences in L^2.

1.5 Plan of the paper

In §2.1 we recall Wiener-Itô chaotic expansions, which we then exploit throughout the whole paper to prove the main results, given in §2.2; §3 is devoted to the proof of the chaotic expansion for the nodal length (Proposition 3.2), whereas in §4 we prove Proposition 2.2 and Proposition 2.3. Finally, in §5 we collect the technical proofs of auxiliary lemmas for the results given in §4.

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2 Proofs of the main results

The proofs of our results rely on a pervasive use of *Wiener-Itô chaotic expansions* for non-linear functionals of Gaussian fields; this notion is presented below in a form that is adapted to the random functions considered in the present paper (see e.g. [N-P, P-T] for an exhaustive discussion).

2.1 Wiener Chaos

Denote by $\{H_k\}_{k\ge 0}$ the usual Hermite polynomials on $\mathbb{R}$. These are defined recursively as follows: $H_0 \equiv 1$, and, for $k \ge 1$,

$$H_k(t) = \delta H_{k-1}(t),$$

where

$$\delta f(t) = tf(t) - f'(t).$$

Recall that $\mathbb{H} := \{[k!]^{-1/2} H_k : n \ge 0\}$ constitutes a complete orthonormal system in

$$L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma(t)dt) := L^2(\gamma),$$

where $\gamma(t) = (2\pi)^{-1/2} e^{-t^2/2}$ is the standard Gaussian density on the real line.

The arithmetic random waves (1.3) considered in this work are a by-product of a family of complex-valued Gaussian random variables $\{a_\lambda : \lambda \in \mathbb{Z}^2\}$, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and satisfying the following properties: **(a)** every a_λ has the form $x_\lambda + iy_\lambda$, where x_λ and y_λ are two independent real-valued Gaussian random variables with mean zero and variance 1/2; **(b)** a_λ and a_τ are stochastically independent whenever $\lambda \notin \{\tau, -\tau\}$, and **(c)** $a_\lambda = \overline{a_{-\lambda}}$. Define the space $\mathbf{A}$ to be the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables ξ of the form

$$\xi = z\, a_\lambda + \overline{z}\, a_{-\lambda},$$

where $\lambda \in \mathbb{Z}^2$ and $z \in \mathbb{C}$. The space $\mathbf{A}$ is a real centered Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

Definition 2.1. For an integer $q \ge 0$ the q-th *Wiener chaos* associated with $\mathbf{A}$, written C_q, is the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables of the form

$$H_{p_1}(\xi_1) \cdot H_{p_2}(\xi_2) \cdots H_{p_k}(\xi_k)$$

for $k \ge 1$, where the integers $p_1, ..., p_k \ge 0$ satisfy $p_1 + \cdots + p_k = q$, and $(\xi_1, ..., \xi_k)$ is a standard real Gaussian vector extracted from $\mathbf{A}$ (note that, in particular, $C_0 = \mathbb{R}$).

Using the orthonormality and completeness of $\mathbb{H}$ in $L^2(\gamma)$, together with a standard monotone class argument (see e.g. [N-P, Theorem 2.2.4]), it is not difficult to show that $C_q \perp C_m$ (where the orthogonality holds in the sense of $L^2(\mathbb{P})$) for every $q \neq m$, and moreover

$$L^2(\Omega, \sigma(\mathbf{A}), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q;$$

that is, every real-valued functional F of $\mathbf{A}$ can be (uniquely) represented in the form

$$F = \sum_{q=0}^{\infty} \mathrm{proj}(F \,|\, C_q) = \sum_{q=0}^{\infty} F[q], \tag{2.16}$$

where as before $F[q] := \mathrm{proj}(F \,|\, C_q)$ stands for the the projection onto C_q, and the series converges in $L^2(\mathbb{P})$. Plainly, $F[0] = \mathrm{proj}(F \,|\, C_0) = \mathbb{E}[F]$.

A straightforward differentiation of the definition (1.3) of T_n yields, for $j = 1, 2$

$$\partial_j T_n(x) = \frac{2\pi i}{\sqrt{\mathcal{N}_n}} \sum_{(\lambda_1,\lambda_2)\in\Lambda_n} \lambda_j a_\lambda e_\lambda(x), \tag{2.17}$$

(here $\partial_j = \frac{\partial}{\partial x_j}$). Hence the random fields $T_n, \partial_1 T_n, \partial_2 T_n$ viewed as collections of Gaussian random variables indexed by $x \in \mathbb{T}$ are all lying in $\mathbf{A}$, i.e. for every $x \in \mathbb{T}$ we have

$$T_n(x),\ \partial_1 T_n(x),\ \partial_2 T_n(x) \in \mathbf{A}.$$

2.2 Proof of Theorem 1.1

We apply the Wiener chaos decomposition (2.16) on the nodal length

$$\mathcal{L}_n = \sum_{q=0}^{\infty} \mathcal{L}_n[q], \tag{2.18}$$

in $L^2(\mathbb{P})$. The following proposition is a reformulation of Theorem 1.1 with the projection $\mathcal{L}_n[4]$ of the nodal length $\mathcal{L}_n$ onto the 4th order chaos in place of replacing $\mathcal{L}_n$ and it will be proven in §4.2.

Proposition 2.2. *Let $\{n_j\} \subseteq S$ be a subsequence of S satisfying $\mathcal{N}_{n_j} \to \infty$, such that the sequence $\left\{\left|\widehat{\mu_{n_j}}(4)\right| : j \geq 1\right\}$ of non-negative numbers converges, that is,*

$$\left|\widehat{\mu_{n_j}}(4)\right| \to \eta,$$

for some $\eta \in [0, 1]$. Then, the corresponding sequences of random variables converges in distribution to $\mathcal{M}_\eta$ as defined in (1.11), that is,

$$\frac{\mathcal{L}_{n_j}[4]}{\sqrt{\mathrm{Var}(\mathcal{L}_{n_j}[4])}} \xrightarrow{\mathrm{d}} \mathcal{M}_\eta. \tag{2.19}$$

Moreover,

$$\mathrm{Var}\left(\mathcal{L}_{n_j}[4]\right) \sim \frac{1+\eta^2}{512} \frac{E_{n_j}}{\mathcal{N}_{n_j}^2}. \tag{2.20}$$

The next proposition, whose proof is given in §4.2, entails that the fourth-order chaotic component gives the leading term in the expansion, i.e. its behaviour asymptotically dominates the nodal length on the torus.

Proposition 2.3. *For every* $\{n_j : j \geq 1\} \subseteq S$ *subsequence of* S *such that* $\lim_{j\to\infty} \mathcal{N}_{n_j} = \infty$ *and the sequence* $\left\{ \left|\widehat{\mu_{n_j}}(4)\right| : j \geq 1 \right\}$ *of non-negative numbers converges,*

$$\mathrm{Var}\left(\mathcal{L}_{n_j} - \mathcal{L}_{n_j}[4]\right) = o\left(\frac{E_{n_j}}{\mathcal{N}_{n_j}^2}\right). \tag{2.21}$$

Equivalently, under the above assumptions we have that

$$\mathrm{Var}\left(\mathcal{L}_{n_j}\right) \sim \mathrm{Var}\left(\mathcal{L}_{n_j}[4]\right). \tag{2.22}$$

Proof of Theorem 1.1 assuming Proposition 2.2 and Proposition 2.3. The chaotic expansion (2.18) and Proposition 2.3 entail that, as $j \to +\infty$,

$$\widetilde{\mathcal{L}}_{n_j} = \widetilde{\mathcal{L}}_{n_j}[4] + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability. Actually, by linearity we have

$$\widetilde{\mathcal{L}}_{n_j}[4] = \frac{\mathcal{L}_{n_j}[4]}{\sqrt{\mathrm{Var}(\mathcal{L}_{n_j})}}. \tag{2.23}$$

It hence follows that $\widetilde{\mathcal{L}}_{n_j}$ and the random variable $\widetilde{\mathcal{L}}_{n_j}[4]$ have the same asymptotic distribution. Proposition 2.3 together with (2.19) and (2.23) allow to conclude the proof, i.e. they immediately imply (1.12). □

2.3 Proof of Theorem 1.2

Having established Theorem 1.1, the proof of Theorem 1.2 follows by a rather standard coupling argument. To achieve the goal of this section, we need to recall the definition of the *Prohorov distance* between two probability measures $\mathbb{P}_0, \mathbb{P}_1$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$:

$$\rho\left(\mathbb{P}_0, \mathbb{P}_1\right) := \inf\left\{\varepsilon > 0 : \mathbb{P}_0(A) \leq \varepsilon + \mathbb{P}_1(A^\varepsilon),\ \text{ for every Borel set } A \subset \mathbb{R}\right\},$$

where $A^\varepsilon := \{x : |x-y| < \varepsilon,\ \text{for some } y \in A\}$. It is a well-known fact that the distance ρ metrizes the weak-$*$ convergence between probability measures, that is: for any collection $\{\mathbb{P}_\infty, \mathbb{P}_n : n \geq 1\}$ of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, one has that $\rho(\mathbb{P}_n, \mathbb{P}_\infty) \to 0$ if and only if $\mathbb{P}_n \Rightarrow \mathbb{P}_\infty$ (see e.g. [D, Chapter 11]). For every pair of random variables X_1, X_2, we write $\rho(X_1, X_2)$ for the quantity $\rho(\mathbf{D}(X_1), \mathbf{D}(X_2))$, where $\mathbf{D}(X_i)$ denotes the probability distribution on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induced by X_i.

Proof. Let us first show that there exists a probability space $(A, \mathscr{A}, \widetilde{\mathbb{P}})$, such that for every $n \in S$ we have a coupling of the random variables $\widetilde{\mathcal{L}}_n$ and $\mathcal{M}_{|\widehat{\mu}_n(4)|}$ and (1.14) holds. Set $I := [0, 1)$. Following the proof of [D, Theorem 11.7.1], it is possible to prove that there exists a measurable space $(Z, \mathscr{Z})$, as well as

(a) a collection of probability measures $\{\widetilde{\mathbb{P}}_t : t \in I\}$ on $(Z, \mathscr{Z})$, and

(b) a set of measurable mappings $X_n, Y_n : Z \to \mathbb{R}$, $n \in S$, enjoying the following properties (i)–(iii):

(i) the mapping

$$C \mapsto \int_I \int_Z \mathbf{1}_C(t,z)\widetilde{\mathbb{P}}_t(dz)\,dt =: \widetilde{\mathbb{P}}(C)$$

is a well-defined probability measure on $(A, \mathscr{A}) := (I \times Z, \mathscr{B}(I) \otimes \mathscr{Z})$;

(ii) under $\widetilde{\mathbb{P}}$, one has that $X_n \overset{\mathrm{d}}{=} \widetilde{\mathcal{L}}_n$ and $Y_n \overset{\mathrm{d}}{=} \mathcal{M}_{|\widehat{\mu}_n(4)|}$ for every $n \in S$, that is, for every Borel set B

$$\widetilde{\mathbb{P}}(X_n \in B) = \int_I \widetilde{\mathbb{P}}_t(X_n \in B)dt = \mathbb{P}(\widetilde{\mathcal{L}}_n \in B), \quad \text{and}$$
$$\widetilde{\mathbb{P}}(Y_n \in B) = \int_I \widetilde{\mathbb{P}}_t(Y_n \in B)dt = \mathbb{P}(\mathcal{M}_{|\widehat{\mu}_n(4)|} \in B);$$

(iii) if $t < 1 - (\alpha_n + |n|^{-1})$ for some $n \in S$, then

$$\widetilde{\mathbb{P}}_t(|X_n - Y_n| \le \alpha_n + |n|^{-1}) = 1,$$

where we set, for $n \in S$,

$$\alpha_n := \rho(\widetilde{\mathcal{L}}_n, \mathcal{M}_{|\widehat{\mu}_n(4)|}) \tag{2.24}$$

i.e., the Prohorov distance between the law of $\widetilde{\mathcal{L}}_n$ and the law of $\mathcal{M}_{|\widehat{\mu}_n(4)|}$.

Since (i) and (ii) give the coupling construction, it remains to prove (1.14). To this end it is enough to show that α_n as defined in (2.24), vanishes as $\mathcal{N}_n \to +\infty$. For every subsequence $\{n_j\} \subseteq \{n\}$, since $|\widehat{\mu}_{n_j}(4)| \le 1$ for every j, there exists a subsubsequence $\{n'_j\} \subseteq \{n_j\}$ such that $|\widehat{\mu}_{n'_j}(4)| \to \eta$ for some $\eta \in [0,1]$, yielding that

$$\rho\big(\widetilde{\mathcal{L}}_{n'_j}, \mathcal{M}_{|\widehat{\mu}_{n'_j}(4)|}\big) \le \rho\big(\widetilde{\mathcal{L}}_{n'_j}, \mathcal{M}_\eta\big) + \rho\big(\mathcal{M}_{|\widehat{\mu}_{n'_j}(4)|}, \mathcal{M}_\eta\big) \to 0,$$

where we have applied Theorem 1.1, as well as the fact that, if $\eta_j \to \eta$, then $\mathcal{M}_{\eta_j} \overset{\mathrm{d}}{\to} \mathcal{M}_\eta$. This immediately shows that $\alpha_n = \rho\big(\widetilde{\mathcal{L}}_n, \mathcal{M}_{|\widehat{\mu}_n(4)|}\big) \to 0$, thus yielding the equality

$$\widetilde{\mathbb{P}}\big(|X_n - Y_n| \to 0\big) = \int_I \widetilde{\mathbb{P}}_t\big(|X_n - Y_n| \to 0\big)\,dt = 1,$$

where we have used (i) and the fact that, by virtue of property (iii) above as well as of the relation $\alpha_n \to 0$, for every $t \in I$ there exists an integer n_t such that

$$\widetilde{\mathbb{P}}_t(|X_n - Y_n| \le \alpha_n + |n|^{-1},\, \forall n > n_t) = 1,$$

implying that, for every $t \in I$, $\widetilde{\mathbb{P}}_t\big(|X_n - Y_n| \to 0\big) = 1$.

Now we are going to prove (1.13). The sequence of random variables $\{\widetilde{\mathcal{L}_n} - \mathcal{M}_n\}$ is bounded in L^2, actually for every $n \in S$ we have $\mathbb{E}_{\widetilde{\mathbb{P}}}[|\widetilde{\mathcal{L}}_n - \mathcal{M}_n|^2] \le 2(\mathbb{E}_{\widetilde{\mathbb{P}}}[|\widetilde{\mathcal{L}}_n|^2] + \mathbb{E}_{\widetilde{\mathbb{P}}}[|\mathcal{M}_n|^2]) = 4$. This implies (see e.g. [VdV, Theorem 2.20]) that for each $p \in (0,2)$, the sequence $\{|\widetilde{\mathcal{L}_n} - \mathcal{M}_n|^p\}$ is uniformly integrable, i.e.

$$\lim_{M \to +\infty} \sup_n \mathbb{E}_{\widetilde{\mathbb{P}}}\left[|\widetilde{\mathcal{L}}_n - \mathcal{M}_n|^p 1_{|\widetilde{\mathcal{L}_n} - \mathcal{M}_n|^p > M}\right] = 0, \tag{2.25}$$

where, as usual, 1_A denotes the indicator function of some set A. Moreover, we have just shown that (1.14) holds, so that, as $\mathcal{N}_n \to +\infty$,

$$|\widetilde{\mathcal{L}}_n - \mathcal{M}_n|^p \longrightarrow 0, \quad \widetilde{\mathbb{P}} - a.s. \tag{2.26}$$

We can hence apply [D, Theorem 10.3.6] to the sequence $\{|\widetilde{\mathcal{L}_n} - \mathcal{M}_n|^p\}$: (2.25) together with (2.26) implies the L^p-convergence stated in (1.13). The proof is thus concluded.

□

3 Chaotic expansion of $\mathcal{L}_n$

In order to prove Proposition 2.2 and Proposition 2.3 we need to compute the Wiener-Itô chaotic expansion (2.18) of the random variable $\mathcal{L}_n$.

3.1 Statement

Let us introduce some more notation to properly state the main result of this section. The nodal length (1.4) can be formally written as

$$\mathcal{L}_n = \int_{\mathbb{T}} \delta_0(T_n(\theta))\|\nabla T_n(\theta)\|\, d\theta, \tag{3.27}$$

where δ_0 denotes the Dirac delta function and $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^2$ (see [R-W, Lemma 3.1] and §3.2.1).

We shall often use the following easy result from [R-W]:

Lemma 3.1 ([R-W], (4.1))**.** *For $j = 1, 2$ we have that*

$$\mathrm{Var}[\partial_j T_n(x)] = \frac{4\pi^2}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_j^2 = 4\pi^2 \frac{n}{2},$$

where the derivatives $\partial_j T_n(x)$ are as in (2.17)*.*

Accordingly, for $x = (x_1, x_2) \in \mathbb{T}$ and $j = 1, 2$, we will denote by $\partial_j \widetilde{T}_n(x)$ the normalized derivative

$$\partial_j \widetilde{T}_n(x) := \frac{1}{2\pi}\sqrt{\frac{2}{n}} \frac{\partial}{\partial x_j} T_n(x) = \sqrt{\frac{2}{n}} \frac{i}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j\, a_\lambda e_\lambda(x). \tag{3.28}$$

In view of convention (3.28), we formally rewrite (3.27) as

$$\mathcal{L}_n = \sqrt{\frac{4\pi^2 n}{2}} \int_{\mathbb{T}} \delta_0(T_n(x)) \sqrt{\partial_1 \widetilde{T}_n(x)^2 + \partial_2 \widetilde{T}_n(x)^2}\, dx.$$

We also introduce two collections of coefficients $\{\alpha_{2n,2m} : n, m \ge 1\}$ and $\{\beta_{2l} : l \ge 0\}$, that are related to the (formal) Hermite expansions of the norm $\|\cdot\|$ in $\mathbb{R}^2$ and the Dirac mass $\delta_0(\cdot)$ respectively. These are given by

$$\beta_{2l} := \frac{1}{\sqrt{2\pi}} H_{2l}(0), \tag{3.29}$$

where H_{2l} denotes the $2l$-th Hermite polynomial, and

$$\alpha_{2n,2m} = \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m}} p_{n+m}\left(\frac{1}{4}\right), \tag{3.30}$$

where for $N = 0, 1, 2, \dots$ and $x \in \mathbb{R}$

$$p_N(x) := \sum_{j=0}^{N} (-1)^j \cdot (-1)^N \binom{N}{j} \frac{(2j+1)!}{(j!)^2} x^j,$$

$\frac{(2j+1)!}{(j!)^2}$ being the so-called *swinging factorial* restricted to odd indices.

We are now ready to state the main result of this section. It illustrates the cancellations that occur for the components of the chaotic expansion (2.18) of $\mathcal{L}_n$ (precisely, odd terms and the second-order one). Consistent to Proposition 2.3, computing the fourth-order component only is sufficient to establish the asymptotic behavior of the nodal length. However, we believe that the complete expansion is of clear independent interest; for instance, (a) it gives the basic building block to extend our results to other random fields on the torus and (b) it sheds some light on the Berry's cancellation phenomenon [Be, W1, W2].

More precisely, as far as point (b) is concerned, we note that the nodal length $\mathcal{L}_\ell$ of Gaussian Laplace eigenfunctions T_ℓ, $\ell \in \mathbb{N}$, on the two-dimensional sphere have the same qualitative behavior. Indeed, on one hand in the chaotic expansion of $\mathcal{L}_\ell$, the odd terms and the second chaotic projection vanish and the fourth-order component exhibits the same asymptotic variance as the full nodal length (see [R]). On the other hand, it is also shown in [R] that the second chaotic projection in the Wiener-Itô expansion of the length of level curves $T_\ell^{-1}(u)$, $u \in \mathbb{R}$ vanishes if and only if $u = 0$. These results explain why the asymptotic variance of the length of level curves is consistent to the natural scaling, except for the nodal case [W1, W2]. Finally, we note that an analogous cancellation phenomenon occurs for the excursion area and the Euler-Poincaré characteristic of excursion sets for spherical eigenfunctions, see [M-W, M-R, CMW].

Proposition 3.2 (**Chaotic expansion of $\mathcal{L}_n$**)**.** (a) *For $q = 2$ or $q = 2m+1$ odd ($m \geq 1$),*

$$\mathcal{L}_n[q] \equiv 0,$$

that is, the corresponding chaotic projection vanishes.

(b) *For $q \geq 2$*

$$\begin{aligned}
&\mathcal{L}_n[2q] \\
&= \sqrt{\frac{4\pi^2 n}{2}} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{2k,2u-2k}\beta_{2q-2u}}{(2k)!(2u-2k)!(2q-2u)!} \times \\
&\qquad\qquad \times \int_{\mathbb{T}} H_{2q-2u}(T_n(x)) H_{2k}(\partial_1 \widetilde{T}_n(x)) H_{2u-2k}(\partial_2 \widetilde{T}_n(x))\, dx.
\end{aligned} \tag{3.31}$$

Consolidating the above, the Wiener-Itô chaotic expansion of $\mathcal{L}_n$ is

$$\begin{aligned}
\mathcal{L}_n = \mathbb{E}\mathcal{L}_n + \sqrt{\frac{4\pi^2 n}{2}} \sum_{q=2}^{+\infty} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{2k,2u-2k}\beta_{2q-2u}}{(2k)!(2u-2k)!(2q-2u)!} \times \\
\times \int_{\mathbb{T}} H_{2q-2u}(T_n(x)) H_{2k}(\partial_1 \widetilde{T}_n(x)) H_{2u-2k}(\partial_2 \widetilde{T}_n(x))\, dx,
\end{aligned}$$

in $L^2(\mathbb{P})$.

3.2 Proof of Proposition 3.2

Let us start with an approximating result in $L^2(\mathbb{P})$ for the nodal length $\mathcal{L}_n$.

3.2.1 Approximating the nodal length

Consider the family of random variables $\{\mathcal{L}_n^\varepsilon, \varepsilon > 0\}$ defined as

$$\mathcal{L}_n^\varepsilon = \frac{1}{2\varepsilon}\int_{\mathbb{T}} 1_{[-\varepsilon,\varepsilon]}(T_n(x))\|\nabla T_n(x)\|dx, \tag{3.32}$$

where $1_{[-\varepsilon,\varepsilon]}$ is the indicator function of the interval $[-\varepsilon,\varepsilon]$, and $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^2$.

In view of the convention (3.28) we rewrite (3.32) as

$$\mathcal{L}_n^\varepsilon = \sqrt{\frac{4\pi^2 n}{2}}\frac{1}{2\varepsilon}\int_{\mathbb{T}} 1_{[-\varepsilon,\varepsilon]}(T_n(x))\sqrt{\partial_1\widetilde{T}_n(x)^2 + \partial_2\widetilde{T}_n(x)^2}\,dx.$$

In [R-W, Lemma 3.1] it was shown that, a.s.

$$\mathcal{L}_n = \lim_{\varepsilon\to 0}\mathcal{L}_n^\varepsilon, \tag{3.33}$$

(a rigorous manifistation of (3.27)), and moreover, by [R-W, Lemma 3.2], $\mathcal{L}_n^\varepsilon$ is uniformly bounded, that is:

$$\mathcal{L}_n^\varepsilon \le 12\sqrt{E_n}. \tag{3.34}$$

Applying the Dominated Convergence Theorem to (3.33) while bearing in mind the uniform bound (3.34) implies that the convergence in (3.33) is in $L^2(\mathbb{P})$, i.e. the following result:

Lemma 3.3. *For every $n\in S$, we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}[|\mathcal{L}_n^\varepsilon - \mathcal{L}_n|^2] = 0.$$

3.2.2 Proof of Proposition 3.2: technical computations

In view of Lemma 3.3, we first compute the chaotic expansion of $\mathcal{L}_n^\varepsilon$ and then deduce Proposition 3.2 by letting $\varepsilon\to 0$. Let us start by expanding the function $\frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(\cdot)$ into Hermite polynomials, as defined in §2.1.

Lemma 3.4. *The following decomposition holds in $L^2(\gamma)$ (where, as before, γ is the standard Gaussian density on $\mathbb{R}$):*

$$\frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(\cdot) = \sum_{l=0}^{+\infty}\frac{1}{l!}\beta_l^\varepsilon\, H_l(\cdot),$$

where, for $l\ge 1$

$$\beta_l^\varepsilon = -\frac{1}{2\varepsilon}\gamma(\varepsilon)\left(H_{l-1}(\varepsilon) - H_{l-1}(-\varepsilon)\right),$$

while for $l=0$

$$\beta_0^\varepsilon = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\gamma(t)\,dt.$$

Proof. Using the completeness and orthonormality of the set $\mathbb{H}$ in $L^2(\gamma)$, one has that $\beta_0^\varepsilon = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\gamma(t)\,dt$, and, for $l\ge 1$,

$$\beta_l^\varepsilon = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\gamma(t)H_l(t)\,dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\gamma(t)(-1)^l\gamma^{-1}(t)\frac{d^l}{dt^l}\gamma(t)\,dt =$$
$$= \frac{1}{2\varepsilon}(-1)^l\left(\frac{d^{l-1}}{dt^{l-1}}\gamma(\varepsilon) - \frac{d^{l-1}}{dt^{l-1}}\gamma(-\varepsilon)\right) = -\frac{1}{2\varepsilon}\gamma(\varepsilon)\left(H_{l-1}(\varepsilon) - H_{l-1}(-\varepsilon)\right).$$

Now, if l is odd, then H_{l-1} is an even function, and therefore $\beta_l^\varepsilon = 0$: it follows that

$$\frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(\cdot) = \beta_0^\varepsilon + \sum_{l=1}^{+\infty} \frac{1}{(2l)!}\left(-\frac{1}{\varepsilon}\gamma(\varepsilon)\, H_{2l-1}(\varepsilon)\right) H_{2l}(\cdot).$$

Using the notation (3.29), we have that, for all $l \geq 0$,

$$\lim_{\varepsilon} \beta_{2l}^\varepsilon = -\frac{1}{\sqrt{2\pi}}(2l-1)!\frac{(-1)^{l-1}}{(l-1)!2^{l-1}} = \frac{1}{\sqrt{2\pi}}H_{2l}(0) = \beta_{2l}. \tag{3.35}$$

□

Note that, setting $\beta_l = 0$ for l odd, the set $\{\beta_l : l = 0, 1, 2, ...\}$ can be interpreted as the sequence of the coefficients appearing in the formal Hermite expansion of the Dirac mass δ_0.

Now fix $x \in \mathbb{T}$, and recall that the coordinates of the vector

$$\nabla\widetilde{T}_n(x) := (\partial_1\widetilde{T}_n(x), \partial_2\widetilde{T}_n(x)),$$

are unit variance centered independent Gaussian random variables (see i.e., [K-K-W]). Now, since the random variable $\|\nabla\widetilde{T}_n(x)\|$ is square-integrable, it can be expanded into an (infinite) series of Hermite polynomials, as detailed in the following statement.

Lemma 3.5. *For (Z_1, Z_2) a standard Gaussian bivariate vector, we have the L^2-expansion*

$$\|(Z_1, Z_2)\| = \sum_{n=0}^{+\infty}\sum_{m=0}^{n} \frac{\alpha_{2n,2n-2m}}{(2n)!(2n-2m)!} H_{2n}(Z_1)H_{2n-2m}(Z_2),$$

where the $\alpha_{2n,2n-2m}$ are as in (3.30).

Proof. We may expand

$$\|(Z_1, Z_2)\| = \sum_{u=0}^{+\infty}\sum_{m=0}^{u} \frac{\alpha_{u,u-m}}{u!(u-m)!} H_u(Z_1)H_{u-m}(Z_2),$$

where

$$\alpha_{n,n-m} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \sqrt{y^2+z^2}H_n(y)H_{n-m}(z)\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz.$$

Our aim is to compute $\alpha_{n,n-m}$ as explicitly as possible. First of all, we observe that, if n or $n-m$ is odd, then the above integral vanishes (since the two mappings $z \mapsto \sqrt{y^2+z^2}$ and $y \mapsto \sqrt{y^2+z^2}$ are even). It follows therefore that

$$\|(Z_1, Z_2)| = \sum_{n=0}^{+\infty}\sum_{m=0}^{n} \frac{\alpha_{2n,2n-2m}}{(2n)!(2n-2m)!} H_{2n}(Z_1)H_{2n-2m}(Z_2).$$

We are therefore left with the task of showing that the integrals

$$\alpha_{2n,2n-2m} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \sqrt{y^2+z^2}H_{2n}(y)H_{2n-2m}(z)\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz,$$

where $n \geq 0$ and $m = 0, \ldots, n$, are given by (3.30). One elegant way for dealing with this task is to use the following Hermite polynomial expansion (see e.g. [N-P, Proposition 1.4.2])

$$\mathrm{e}^{\lambda y - \frac{\lambda^2}{2}} = \sum_{a=0}^{+\infty} H_a(y)\frac{\lambda^a}{a!}, \quad \lambda \in \mathbb{R}. \tag{3.36}$$

Let us consider the integral

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}\mathrm{e}^{\lambda y-\frac{\lambda^2}{2}}\mathrm{e}^{\mu z-\frac{\mu^2}{2}}\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz=\frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}\mathrm{e}^{-\frac{(y-\lambda)^2+(z-\mu)^2}{2}}\,dydz.$$

This integral coincides with the expected value of the random variable $W:=\sqrt{Y^2+Z^2}$ where (Y,Z) is a vector of independent Gaussian random variables with variance one and mean λ and μ, respectively. Since $W^2=Y^2+Z^2$ has a non-central χ^2-distribution (more precisely, $Y^2+Z^2\sim\chi^2(2,\lambda^2+\mu^2)$) it is easily checked that the density of W is given by

$$f_W(t)=\sum_{j=0}^{+\infty}\mathrm{e}^{-(\lambda^2+\mu^2)/2}\frac{((\lambda^2+\mu^2)/2)^j}{j!}f_{2+2j}(t^2)\,2t\,1_{\{t>0\}},\tag{3.37}$$

where f_{2+2j} is the density function of a χ^2_{2+2j}-distributed random variable. The expected value of W is therefore

$$\mathbb{E}[W]=2\sum_{j=0}^{+\infty}\mathrm{e}^{-(\lambda^2+\mu^2)/2}\frac{((\lambda^2+\mu^2)/2)^j}{j!}\int_0^{+\infty}f_{2+2j}(t^2)\,t^2\,dt.\tag{3.38}$$

From the definition of f_{2+2j} we have

$$\begin{aligned}\int_0^{+\infty}f_{2+2j}(t^2)\,t^2\,dt &= \frac{1}{2^{1+j}\Gamma(1+j)}\int_0^{+\infty}t^{2j+2}\mathrm{e}^{-t^2/2}\,dt\\ &= \frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{2^{1+j}\Gamma(1+j)}.\end{aligned}\tag{3.39}$$

Substituting (3.39) into (3.38) we have

$$\mathbb{E}[W]=2\mathrm{e}^{-(\lambda^2+\mu^2)/2}\sum_{j=0}^{+\infty}\frac{((\lambda^2+\mu^2)/2)^j}{j!}\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{2^{1+j}\Gamma(1+j)}=:F(\lambda,\mu).\tag{3.40}$$

Applying Newton's binomial formula to $((\lambda^2+\mu^2)/2)^j$, we may expand the function F in (3.40) as follows:

$$\begin{aligned}F(\lambda,\mu)&=2\sum_{a=0}^{+\infty}\frac{(-1)^a\lambda^{2a}}{2^a a!}\sum_{b=0}^{+\infty}\frac{(-1)^b\mu^{2b}}{2^b b!}\sum_{j=0}^{+\infty}\frac{1}{j!}\sum_{l=0}^{j}\binom{j}{l}\lambda^{2l}\mu^{2j-2l}\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{2^{1+2j}\Gamma(1+j)}=\\ &=\sum_{a,b=0}^{+\infty}\frac{(-1)^a}{2^a a!}\frac{(-1)^b}{2^b b!}\sum_{j=0}^{+\infty}\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{j!2^{2j}\Gamma(1+j)}\sum_{l=0}^{j}\binom{j}{l}\lambda^{2l+2a}\mu^{2j+2b-2l}.\end{aligned}$$

Setting $n:=l+a$ and $m:=j+b-l$, we also have that

$$\begin{aligned}F(\lambda,\mu)&=\sum_{a,b=0}^{+\infty}\frac{(-1)^a}{2^a a!}\frac{(-1)^b}{2^b b!}\sum_{j=0}^{+\infty}\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{j!2^{2j}\Gamma(1+j)}\sum_{l=0}^{j}\binom{j}{l}\lambda^{2l+2a}\mu^{2j+2b-2l}\\ &=\sum_{n,m}\sum_{j}\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{j!2^{2j}\Gamma(1+j)}\sum_{l=0}^{j}\frac{(-1)^{(n-l)}}{2^{n-l}(n-l)!}\frac{(-1)^{m+l-j}}{2^{m+l-j}(m+l-j)!}\binom{j}{l}\lambda^{2n}\mu^{2m}.\end{aligned}\tag{3.41}$$

Since $F(\lambda,\mu)=\mathbb{E}[W]$ from (3.40), on the other hand (3.36) yields

$$\begin{aligned}
F(\lambda,\mu) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}\mathrm{e}^{\lambda y-\frac{\lambda^2}{2}}\mathrm{e}^{\mu z-\frac{\mu^2}{2}}\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}\sum_{a=0}^{+\infty}H_a(y)\frac{\lambda^a}{a!}\sum_{b=0}^{+\infty}H_b(z)\frac{\mu^b}{b!}\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz \\
&= \sum_{a,b=0}^{+\infty}\left(\underbrace{\frac{1}{a!b!2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}H_a(y)H_b(z)\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz}_{:=d(a,b)}\right)\lambda^a\mu^b. \qquad (3.42)
\end{aligned}$$

By the same reasoning as above, if a or b is odd, then $d(a,b)$ in (3.42) must vanish. By combining the expansions in (3.41) and (3.42), we have

$$\begin{aligned}
\alpha_{2n,2m} &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}H_{2n}(y)H_{2m}(z)\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz \\
&= (2n)!(2m)!\frac{(-1)^{m+n}}{2^{n+m}}\sum_j\frac{(-1)^j\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{2^j j!\Gamma(1+j)}\sum_{l=0}^{j}\frac{\binom{j}{l}}{(n-l)!(m+l-j)!}. \qquad (3.43)
\end{aligned}$$

The equality (3.30) now follows from (3.43) and some computations:

$$\begin{aligned}
\alpha_{2n,2m} &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\sqrt{y^2+z^2}H_{2n}(y)H_{2m}(z)\mathrm{e}^{-\frac{y^2+z^2}{2}}\,dydz \\
&= (2n)!(2m)!\frac{(-1)^{m+n}}{2^{n+m}}\sum_j(-1)^j\frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\frac{\pi}{2}}}{2^j j!\Gamma(1+j)}\sum_{l=0}^{j}\frac{\binom{j}{l}}{(n-l)!(m+l-j)!} \\
&= (2n)!(2m)!\frac{(-1)^{m+n}}{2^{n+m}}\sum_j(-1)^j\frac{(2j+1)!!\sqrt{\frac{\pi}{2}}}{2^j (j!)^2}\sum_{l=0}^{j}\frac{\binom{j}{l}}{(n-l)!(m+l-j)!} \\
&= \frac{(2n)!(2m)!}{n!m!}\frac{(-1)^{m+n}}{2^{n+m}}\sum_{j=0}^{n+m}(-1)^j\frac{(2j+1)!!\sqrt{\frac{\pi}{2}}}{2^j j!}\sum_{l=0}^{j}\binom{n}{l}\binom{m}{j-l} \\
&= \sqrt{\frac{\pi}{2}}\frac{(2n)!(2m)!}{n!m!}\frac{(-1)^{m+n}}{2^{n+m}}\sum_{j=0}^{n+m}(-1)^j\frac{(2j+1)!!}{2^j j!}\binom{n+m}{j} \\
&= \sqrt{\frac{\pi}{2}}\frac{(2n)!(2m)!}{n!m!}\frac{(-1)^{m+n}}{2^{n+m}}\sum_{j=0}^{n+m}(-1)^j\frac{(2(j+1))!}{2^{j+1}2^j j!(j+1)!}\binom{n+m}{j} \\
&= \sqrt{\frac{\pi}{2}}\frac{(2n)!(2m)!}{n!m!}\frac{(-1)^{m+n}}{2^{n+m}}\sum_{j=0}^{n+m}(-1)^j\frac{(2j+1)!}{2^{2j}(j!)^2}\binom{n+m}{j}.
\end{aligned}$$

□

Proof of Proposition 3.2. In view of Definition 2.1, the computations in Lemma 3.4 and Lemma 3.5 (together with the fact that the three random variables $T_n(x)$, $\partial_1\widetilde{T}_n(x)$ and $\partial_2\widetilde{T}_n(x)$ are stochastically independent, as recalled above) show that, for fixed $x\in\mathbb{T}$, the projection of the random variable

$$\frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(T_n(x))\sqrt{\partial_1\widetilde{T}_n(x)^2+\partial_2\widetilde{T}_n(x)^2}$$

onto each odd chaos vanishes, whereas the projection onto the chaos C_{2q}, for $q \geq 1$, equals

$$\sum_{u=0}^{q}\sum_{m=0}^{u} \frac{\alpha_{2m,2u-2m}\beta^{\varepsilon}_{2q-2u}}{(2m)!(2u-2m)!(2q-2u)!} H_{2q-2u}(T_n(x))H_{2m}(\partial_1 \widetilde{T}_n(x))H_{2u-2m}(\partial_2 \widetilde{T}_n(x)).$$

Since $\int_{\mathbb{T}} dx < \infty$, standard arguments based on Jensen's inequality and dominated convergence yield that $\mathcal{L}^{\varepsilon}_n[q] = 0$ if q is odd and for every $q \geq 1$,

$$\begin{aligned}
&\mathcal{L}^{\varepsilon}_n[2q] \\
&= \sqrt{\frac{4\pi^2 n}{2}} \sum_{u=0}^{q}\sum_{m=0}^{u} \frac{\alpha_{2m,2u-2m}\beta^{\varepsilon}_{2q-2u}}{(2m)!(2u-2m)!(2q-2u)!} \times \\
&\qquad\qquad \times \int_{\mathbb{T}} H_{2q-2u}(T_n(x))H_{2m}(\partial_1 \widetilde{T}_n(x))H_{2u-2m}(\partial_2 \widetilde{T}_n(x))\, dx.
\end{aligned}$$

In view of Lemma 3.3 and (2.18) one has that for every $q \geq 0$, as $\varepsilon \to 0$, $\mathcal{L}^{\varepsilon}_n[q]$ necessarily converge to $\mathcal{L}_n[q]$ in L^2. We just proved that

$$\mathcal{L}^{\varepsilon}_n[q] = 0$$

for $q = 2m+1$ as stated in part (a) of Proposition 3.2. Moreover, using (3.35), we deduce from this fact that representation (3.31) in part (b) of Proposition 3.2 is valid. To complete the proof of part (a) of Proposition 3.2, we need first to show that that $\mathcal{L}_n[2] = 0$. From the previous discussion we deduce that $\mathcal{L}_n[2]$ equals

$$\begin{aligned}
\mathcal{L}_n[2] = \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\Bigg(&\frac{\alpha_{0,0}\beta_2}{2}\int_{\mathbb{T}} H_2(T_n(x))\, dx + \frac{\alpha_{0,2}\beta_0}{2}\int_{\mathbb{T}} H_2(\partial_2 \widetilde{T}_n(x))\, dx \\
&+ \frac{\alpha_{2,0}\beta_0}{2}\int_{\mathbb{T}} H_2(\partial_1 \widetilde{T}_n(x))\, dx \Bigg).
\end{aligned} \tag{3.44}$$

Since $H_2(x) = x^2 - 1$, we may write

$$\begin{aligned}
\int_{\mathbb{T}} H_2(T_n(x))\, dx &= \int_{\mathbb{T}} \left(T_n(x)^2 - 1\right)\, dx = \int_{\mathbb{T}} \left(\frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} a_\lambda \overline{a}_{\lambda'} e_{\lambda-\lambda'}(x) - 1 \right) dx \\
&= \frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} a_\lambda \overline{a}_{\lambda'} \underbrace{\int_{\mathbb{T}} e_{\lambda-\lambda'}(x)\, dx}_{\delta_\lambda^{\lambda'}} - 1 = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1),
\end{aligned} \tag{3.45}$$

where $\delta_\lambda^{\lambda'}$ is the Kronecker symbol. (Observe that $\mathbb{E}[|a_\lambda|^2] = 1$, hence the expected value of the integral $\int_{\mathbb{T}} H_2(T_n(x))\, dx$ is 0, as expected.) Analogously, for $j = 1, 2$ we have

$$\begin{aligned}
\int_{\mathbb{T}} H_2(\partial_j \widetilde{T}_n(x))\, dx &= \int_{\mathbb{T}} \left(\frac{2}{n}\frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} \lambda_j \lambda'_j a_\lambda \overline{a}_{\lambda'} e_{\lambda-\lambda'}(x) - 1 \right) dx \\
&= \frac{2}{n}\frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_j^2 |a_\lambda|^2 - 1 = \frac{1}{\mathcal{N}_n}\frac{2}{n} \sum_{\lambda \in \Lambda_n} \lambda_j^2 (|a_\lambda|^2 - 1),
\end{aligned} \tag{3.46}$$

where the used Lemma 3.1 to establish the last equality.

Since $\alpha_{2n,2m} = \alpha_{2m,2n}$, and in light of (3.45) and (3.46), we may rewrite (3.44) as

$$\begin{aligned}
\mathcal{L}_n[2] &= \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\left(\frac{\alpha_{0,0}\beta_2}{2}\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1)+\frac{\alpha_{0,2}\beta_0}{2}\frac{1}{\mathcal{N}_n}\frac{2}{n}\sum_{\lambda\in\Lambda_n}\underbrace{(\lambda_1^2+\lambda_2^2)}_{=n}(|a_\lambda|^2-1)\right)\\
&= \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\frac{1}{2\mathcal{N}_n}\left(\alpha_{0,0}\beta_2\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1)+2\alpha_{0,2}\beta_0\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1)\right)\\
&= \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\frac{1}{2\mathcal{N}_n}\left(\alpha_{0,0}\beta_2+2\alpha_{0,2}\beta_0\right)\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1).
\end{aligned} \tag{3.47}$$

Since

$$\alpha_{0,0} = \sqrt{\frac{\pi}{2}}, \quad \alpha_{0,2} = \alpha_{2,0} = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \quad \beta_0 = \frac{1}{\sqrt{2\pi}}, \quad \beta_2 = -\frac{1}{\sqrt{2\pi}}$$

by some explicit computations, (3.47) reads

$$\begin{aligned}
\mathcal{L}_n[2] &= \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\frac{1}{2\mathcal{N}_n}\left(-\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}+2\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}\right)\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1)\\
&= \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\frac{1}{2\mathcal{N}_n}\left(-\frac{1}{2}+\frac{1}{2}\right)\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1) = 0.
\end{aligned}$$

The proof of Proposition 3.2 is hence concluded, in view of (2.18). □

4 Proofs of Proposition 2.2 and Proposition 2.3

One of the main findings of the present paper is that, for any sequence $\{n_j\}$ such that $\mathcal{N}_{n_j} \to \infty$ and $|\widehat{\mu}_{n_j}(4)|$ converges, the distribution of the normalised sequence $\{\widetilde{\mathcal{L}}_{n_j}\}$ in (1.10) is asymptotic to one of its fourth-order chaotic projections. The aim of this section is a precise analysis of the asymptotic behavior of the sequence

$$\frac{\mathcal{L}_{n_j}[4]}{\sqrt{\mathrm{Var}(\mathcal{L}_{n_j}[4])}}, \quad j \geq 1.$$

which will allow us to prove Proposition 2.2 and Proposition 2.3.

4.1 Preliminary results

Here we state the key tools for our proofs: first an explicit formula for $\mathcal{L}_{n_j}[4]$ and then a Central Limit Theorem for some of its ingredients. First we need some intermediate results, whose proofs follow immediately from the fact that, for every $n \in S$,

$$\widehat{\mu}_n(4) = \frac{1}{n^2\mathcal{N}_n}\sum_{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n}\left(\lambda_1^4+\lambda_2^4-6\lambda_1^2\lambda_2^2\right),$$

as well as from elementary symmetry considerations.

Lemma 4.1. *For every $n \in S$ we have*

$$\frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n} \lambda_\ell^4 = \frac{3+\widehat{\mu}_n(4)}{8},$$

where $\ell = 1, 2$, and moreover

$$\frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n} \lambda_1^2\lambda_2^2 = \frac{1-\widehat{\mu}_n(4)}{8}.$$

Let us state now the above mentioned CLT result. Let us define, for $n \in S$,

$$W(n) := \begin{pmatrix} W_1(n) \\ W_2(n) \\ W_3(n) \\ W_4(n) \end{pmatrix} := \frac{1}{n\sqrt{\mathcal{N}_n/2}} \sum_{\substack{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n \\ \lambda_2>0}} \left(|a_\lambda|^2 - 1\right) \begin{pmatrix} n \\ \lambda_1^2 \\ \lambda_2^2 \\ \lambda_1\lambda_2 \end{pmatrix}. \tag{4.48}$$

Exploiting the representation (3.31) in the case $q = 2$, one can show the following.

Lemma 4.2. *We have, for diverging subsequences $\{n_j\} \subseteq S$ such that $\mathcal{N}_{n_j} \to +\infty$ and $\widehat{\mu}_{n_j}(4)$ converges,*

$$\mathcal{L}_{n_j}[4] = \sqrt{\frac{E_{n_j}}{512\,\mathcal{N}_{n_j}^2}} \Big(1 + W_1(n_j)^2 - 2W_2(n_j)^2 - 2W_3(n_j)^2 - 4W_4(n_j)^2 + o_{\mathbb{P}}(1)\Big). \tag{4.49}$$

The proof of Lemma 4.2 will be given in §5.

Lemma 4.3. *Assume that the subsequence $\{n_j\} \subseteq S$ is such that $\mathcal{N}_{n_j} \to +\infty$ and $\widehat{\mu}_{n_j}(4) \to \eta \in [-1, 1]$. Then, as $n_j \to \infty$, the following CLT holds:*

$$W(n_j) \xrightarrow{\mathrm{d}} Z(\eta) = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}, \tag{4.50}$$

where $Z(\eta)$ is a centered Gaussian vector with covariance

$$\Sigma = \Sigma(\eta) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3+\eta}{8} & \frac{1-\eta}{8} & 0 \\ \frac{1}{2} & \frac{1-\eta}{8} & \frac{3+\eta}{8} & 0 \\ 0 & 0 & 0 & \frac{1-\eta}{8} \end{pmatrix}. \tag{4.51}$$

The eigenvalues of Σ are $0, \frac{3}{2}, \frac{1-\eta}{8}, \frac{1+\eta}{4}$ and hence, in particular, Σ is singular.

Proof. Each component of the vector $W(n_j)$ is an element of the second Wiener chaos associated with $\mathbf{A}$ (see §2.1). As a consequence, by e.g. [N-P, Theorem 6.2.3], to prove the desired result it is sufficient to establish the following relations (as $n_j \to \infty$): (a) the covariance matrix of $W(n_j)$ converges to Σ, and (b) for every $k = 1, 2, 3, 4$, $W_k(n_j)$ converges in distribution to a one-dimensional centered Gaussian random variable. Part (a) follows by a direct computation based on Lemma 4.1, as well as on the fact that the random variables in the set

$$\left\{|a_\lambda|^2 - 1 : \lambda \in \Lambda_{n_j},\, \lambda_2 > 0\right\}$$

are centered, independent, identically distributed and unit variance. To prove part (b), write $\Lambda^+_{n_j} := \{\lambda \in \Lambda_{n_j}, \lambda_2 > 0\}$ and observe that, for every k and every n_j, the random variable $W_k(n_j)$ is of the form

$$W_k(n_j) = \sum_{\lambda\in\Lambda^+_{n_j}} c_k(n_j,\lambda) \times (|a_\lambda|^2 - 1)$$

where $\{c_k(n_j,\lambda)\}$ is a collection of positive deterministic coefficients such that

$$\max_{\lambda\in\Lambda^+_{n_j}} c_k(n_j,\lambda) \to 0,$$

as $n_j \to \infty$. An application of the Lindeberg criterion, e.g. in the quantitative form stated in [N-P, Proposition 11.1.3], yields that $W_k(n_j)$ converges in distribution to a Gaussian random variable. Since it is easy to verify the claimed eigenvalues of Σ in (4.51) via an explicit computation, this concludes the proof of Lemma 4.2.

□

4.2 Proof of Proposition 2.2: asymptotic behaviour of $\mathcal{L}_n[4]$

Proof of Proposition 2.2 assuming Lemma 4.2. Let $\{n_j\} \subseteq S$ be such that $\mathcal{N}_{n_j} \to \infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0,1]$. For each subsequence $\{n'_j\} \subseteq \{n_j\}$ there exists a subsubsequence $\{n''_j\} \subseteq \{n'_j\}$ such that it holds either (i) $\widehat{\mu}_{n''_j}(4) \to \eta$ or (ii) $\widehat{\mu}_{n''_j}(4) \to -\eta$. Set

$$v(n''_j) := \sqrt{\frac{E_{n''_j}}{512\,\mathcal{N}^2_{n''_j}}}, \quad j \geq 1.$$

Then, as $n''_j \to \infty$

$$Q(n''_j) := \frac{\mathcal{L}_{n''_j}[4]}{v(n''_j)} \xrightarrow{\mathrm{d}} 1 + Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2, \tag{4.52}$$

by Lemma 4.3 and Lemma 4.2; here $Z = Z(\eta) \in \mathbb{R}^4$ is as in (4.50), i.e. a centred Gaussian 4-variate vector with covariance matrix as in (4.51).
Actually, the multidimensional CLT stated in (4.50) implies that

$$(W_1(n_j)^2, W_2(n_j)^2, W_3(n_j)^2, W_4(n_j)^2) \xrightarrow{\mathrm{d}} (Z_1^2, Z_2^2, Z_3^2, Z_4^2).$$

A simple computation of Gaussian moments now yields

$$\begin{aligned}&\mathrm{Var}\left(Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2\right)\\ &= 2 + 8\left(\frac{3+\eta}{8}\right)^2 + 8\left(\frac{3+\eta}{8}\right)^2 + 32\left(\frac{1-\eta}{8}\right)^2 - 2 - 2 + 4\left(\frac{1-\eta}{4}\right)^2 = \eta^2 + 1,\end{aligned}$$

entailing in particular that $\mathrm{Var}\left(Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2\right)$ is the same in both cases (i) – (ii). We can rewrite (4.52) as, for $n''_j \to +\infty$,

$$\frac{\mathcal{L}_{n''_j}[4]}{\sqrt{1+\eta^2}\, v(n''_j)} \xrightarrow{\mathrm{d}} \frac{1}{\sqrt{1+\eta^2}}(1 + Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2).$$

A direct computation (obtained e.g. by diagonalising the covariance matrix Σ appearing in (4.51)) reveals that, in both cases (i)–(ii), the random variable

$$\frac{1}{\sqrt{1+\eta^2}}\left(1+Z_1^2-2Z_2^2-2Z_3^2-4Z_4^2\right)$$

has the same law as $\mathcal{M}_\eta$, as defined in (1.11). Therefore in particular we have

$$\frac{\mathcal{L}_{n_j}[4]}{\sqrt{1+\eta^2}\,v(n_j)} \xrightarrow{\mathrm{d}} \frac{1}{\sqrt{1+\eta^2}}(1+Z_1^2-2Z_2^2-2Z_3^2-4Z_4^2). \tag{4.53}$$

To conclude, it is enough to note that from (4.53), since we are in a fixed Wiener chaos (see [N-P] e.g.) we can take the limit of the variance on the left-hand side as the variance of the limit on the right-hand side, thus obtaining (2.20). □

4.3 Proof of Proposition 2.3: $\mathcal{L}_{n_j}[4]$ dominates $\mathcal{L}_{n_j}$

Now we are able to prove one of the main findings in this paper, i.e. that the fourth-chaotic projection and the total nodal length have the same asymptotic behavior.

Proof. Let us first prove (2.22). Note that (1.9) and Proposition 2.2 immediately give (2.22) i.e., as $\mathcal{N}_{n_j} \to +\infty$,

$$\mathrm{Var}(\mathcal{L}_{n_j}) \sim \mathrm{Var}(\mathcal{L}_{n_j}[4]).$$

Now let us show that (2.22) is equivalent to (2.21).

Since different chaotic projections are orthogonal in L^2, from part (b) of Proposition 3.2 we have

$$\mathrm{Var}(\mathcal{L}_{n_j}) = \mathrm{Var}(\mathcal{L}_{n_j}[4]) + \sum_{q=3}^{+\infty} \mathrm{Var}(\mathcal{L}_{n_j}(2q)) \tag{4.54}$$

or, equivalently,

$$\mathrm{Var}(\mathcal{L}_{n_j} - \mathcal{L}_{n_j}[4]) = \sum_{q=3}^{+\infty} \mathrm{Var}(\mathcal{L}_{n_j}(2q)). \tag{4.55}$$

(2.22) and (4.54) entail that

$$\sum_{q=3}^{+\infty} \mathrm{Var}(\mathcal{L}_{n_j}(2q)) = o\left(\mathrm{Var}(\mathcal{L}_{n_j}[4])\right), \tag{4.56}$$

hence using (2.20) in (4.56) we get (2.21).

Conversely, if (2.21) holds, keeping in mind (2.20) from (4.55) we have (4.56). Let us substitute (4.56) in (4.54): we immediately have (2.22).

□

5 Proof of Lemma 4.2: explicit formula for $\mathcal{L}_{n_j}[4]$

Consider the following representation of $\mathcal{L}_{n_j}[4]$, that is a particular case $q = 2$ of (3.31):

$$\begin{aligned}
\mathcal{L}_{n_j}[4] \quad = \quad & \sqrt{4\pi^2}\sqrt{\frac{n}{2}}\Big(\frac{\alpha_{0,0}\beta_4}{4!}\int_{\mathbb{T}} H_4(T_n(x))\,dx \\
& +\frac{\alpha_{0,4}\beta_0}{4!}\int_{\mathbb{T}} H_4(\partial_2\widetilde{T}_n(x))\,dx + \frac{\alpha_{4,0}\beta_0}{4!}\int_{\mathbb{T}} H_4(\partial_1\widetilde{T}_n(x))\,dx + \\
& +\frac{\alpha_{0,2}\beta_2}{2!2!}\int_{\mathbb{T}} H_2(T_n(x))H_2(\partial_2\widetilde{T}_n(x))\,dx \\
& +\frac{\alpha_{2,0}\beta_2}{2!2!}\int_{\mathbb{T}} H_2(T_n(x))H_2(\partial_1\widetilde{T}_n(x))\,dx + \\
& +\frac{\alpha_{2,2}\beta_0}{2!2!}\int_{\mathbb{T}} H_2(\partial_1\widetilde{T}_n(x))H_2(\partial_2\widetilde{T}_n(x))\,dx\Big),
\end{aligned} \tag{5.57}$$

where the coefficients $\alpha_{\cdot,\cdot}$ and $\beta_{\cdot}$ are defined according to equation (3.30) and equation (3.29), respectively.

5.1 Auxiliary results

The next four lemmas yield a useful representations for the six summands appearing on the right-hand side of (5.57). In what follows, $n \in S$ and, moreover, to simplify the discussion we will sometimes use the shorthand

$$\sum_{\lambda} = \sum_{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n}, \qquad \sum_{\lambda,\lambda'} = \sum_{\lambda,\lambda'\in\Lambda_n} \qquad \text{and} \qquad \sum_{\lambda:\lambda_2>0} = \sum_{\substack{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n \\ \lambda_2>0}},$$

in such a way that the exact value of the integer n will always be clear from the context. Also, the symbol $\{n_j\}$ will always denote a subsequence of integers contained in S such that $\mathcal{N}_{n_j} \to \infty$ and $\widehat{\mu}_{n_j}(4) \to \eta \in [-1, 1]$, as $n_j \to \infty$. As before, we write '$\xrightarrow{\mathbb{P}}$' to denote convergence in probability, and we use the symbol $o_{\mathbb{P}}(1)$ to denote a sequence of random variables converging to zero in probability, as $\mathcal{N}_n \to \infty$.

Following [K-K-W], we will abundantly use the fine structure of the *length-4 spectral correlation set*:

$$S_n(4) := \{(\lambda, \lambda', \lambda'', \lambda''') \in (\Lambda_n)^4 : \lambda + \cdots + \lambda''' = 0\}. \tag{5.58}$$

Lemma 5.1 ([K-K-W], p. 31). *Let $S_n(4)$ be the length-4 spectral correlation set defined in (5.58). Then $S_n(4)$ is the disjoint union*

$$S_n(4) = A_n(4) \cup B_n(4),$$

where $A_n(4)$ is all the 3 permutations of

$$\widetilde{A}_n(4) = \{(\lambda, \lambda', -\lambda, -\lambda') : \lambda, \lambda' \in \Lambda_n, \lambda \neq \lambda'\},$$

and $B_n(4)$ is all the 3 permutations of

$$\widetilde{B}_n(4) = \{(\lambda, \lambda, -\lambda, -\lambda) : \lambda \in \Lambda_n\}.$$

In particular, using the inclusion-exclusion principle,

$$|S_n(4)| = 3\mathcal{N}_n(\mathcal{N}_n - 1).$$

Lemma 5.2. *One has the following representation:*

$$\int_{\mathbb{T}} H_4(T_n(x))\,dx = \frac{6}{\mathcal{N}_n}\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda:\lambda_2>0}(|a_\lambda|^2-1)+o_{\mathbb{P}}(1)\right)^2 - \frac{3}{\mathcal{N}_n^2}\sum_{\lambda}|a_\lambda|^4. \tag{5.59}$$

Also, as $n_j\to\infty$,

$$\frac{3}{\mathcal{N}_{n_j}}\sum_{\lambda}|a_\lambda|^4 \xrightarrow{\mathbb{P}} 6. \tag{5.60}$$

Proof. Using the explicit expression $H_4(x)=x^4-6x^2+3$, we deduce that

$$\begin{aligned}\int_{\mathbb{T}} H_4(T_n(x))\,dx &= \int_{\mathbb{T}}\left(T_n(x)^4-6T_n(x)^2+3\right)\,dx\\ &= \frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\dots,\lambda'''\in\Lambda_n} a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''}\int_{\mathbb{T}}\exp(2\pi i\langle\lambda-\lambda'+\lambda''-\lambda''',x\rangle)\,dx+\\ &- 6\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n}a_\lambda\overline{a}_{\lambda'}\int_{\mathbb{T}}\exp(2\pi i\langle\lambda-\lambda',x\rangle)\,dx+3\\ &= \frac{1}{\mathcal{N}_n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''}-6\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}|a_\lambda|^2+3, \end{aligned} \tag{5.61}$$

where the summation with the subscript $\lambda-\lambda'+\lambda''-\lambda'''=0$ is over $(\lambda,-\lambda',\lambda'',-\lambda''')\in S_n(4)$. By the fine structure of $S_n(4)$ described in Lemma 5.1, the right-hand side of (5.61) simplifies to

$$\begin{aligned}\int_{\mathbb{T}} H_4(T_n(x))\,dx &= 3\frac{1}{\mathcal{N}_n^2}\Big(\sum_{\lambda,\lambda'\in\Lambda_n}|a_\lambda|^2|a_{\lambda'}|^2-\sum_{\lambda}|a_\lambda|^4\Big)-6\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}|a_\lambda|^2+3\\ &= 3\frac{1}{\mathcal{N}_n}\Big(\frac{1}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2-1)\Big)^2-3\frac{1}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n}|a_\lambda|^4\\ &= \frac{6}{\mathcal{N}_n}\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda:\lambda_2>0}(|a_\lambda|^2-1)+o_{\mathbb{P}}(1)\right)^2-\frac{3}{\mathcal{N}_n^2}\sum_{\lambda}|a_\lambda|^4,\end{aligned}$$

where $o_{\mathbb{P}}(1)=0$ if $n^{1/2}$ is not an integer, otherwise

$$o_{\mathbb{P}}(1)=(\mathcal{N}_{n_j}/2)^{-1/2}(|a_{(n^{1/2},0)}|^2-1),$$

thus yielding (5.59) immediately. The limit (5.60) follows from a standard application of the law of large numbers to the sum,

$$\frac{3}{\mathcal{N}_{n_j}}\sum_{\lambda}|a_\lambda|^4=\frac{3}{\mathcal{N}_{n_j}/2}\sum_{\lambda:\lambda_2>0}|a_\lambda|^4+o_{\mathbb{P}}(1),$$

as well all the variables a_λ are i.i.d with

$$\mathbb{E}\left[|a_\lambda|^4\right]=2.$$

□

Lemma 5.3. *For $\ell = 1, 2$,*

$$\int_{\mathbb{T}} H_4(\partial_\ell \widetilde{T}_n(x))\, dx = \frac{24}{\mathcal{N}_n}\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}\left(\frac{\lambda_\ell^2}{n}\left(|a_\lambda|^2-1\right)\right)+o_{\mathbb{P}}(1)\right)^2 - \left(\frac{2}{n}\right)^2\frac{3}{\mathcal{N}_n^2}\sum_\lambda \lambda_\ell^4|a_\lambda|^4.$$

Moreover, as $n_j \to \infty$,

$$\left(\frac{2}{n_j}\right)^2\frac{3}{\mathcal{N}_{n_j}}\sum_\lambda \lambda_\ell^4|a_\lambda|^4 \xrightarrow{\mathbb{P}} 3(3+\eta).$$

Proof. The proof is similar to that of Lemma 5.2. We have that

$$\begin{aligned}
&\int_{\mathbb{T}} H_4(\partial_\ell \widetilde{T}_n(x))\, dx = \int_{\mathbb{T}} (\partial_\ell \widetilde{T}_n(x)^4 - 6\partial_\ell \widetilde{T}_n(x)^2 + 3)\, dx \\
&= \frac{1}{\mathcal{N}_n^2}\frac{4}{n^2}\sum_{\lambda,\dots,\lambda'''\in\Lambda_n}\lambda_\ell\lambda'_\ell\lambda''_\ell\lambda'''_\ell a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''}\int_{\mathbb{T}}\exp(2\pi i\langle\lambda-\lambda'+\lambda''-\lambda''',x\rangle)\,dx + \\
&-6\frac{1}{\mathcal{N}_n}\frac{2}{n}\sum_{\lambda,\lambda'}\lambda_\ell\lambda'_\ell a_\lambda\overline{a}_{\lambda'}\int_{\mathbb{T}}\exp(2\pi i\langle\lambda-\lambda',x\rangle)\,dx + 3 \\
&= \frac{1}{\mathcal{N}_n^2}\frac{4}{n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}\lambda_\ell\lambda'_\ell\lambda''_\ell\lambda'''_\ell a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''} - 6\frac{1}{\mathcal{N}_n}\frac{2}{n}\sum_{\lambda\in\Lambda_n}\lambda_\ell^2|a_\lambda|^2+3 \\
&= \frac{3}{\mathcal{N}_n^2}\frac{4}{n^2}\left(\sum_{\lambda,\lambda'}\lambda_\ell^2(\lambda'_\ell)^2|a_\lambda|^2|a_{\lambda'}|^2 - \sum_\lambda\lambda_\ell^4|a_\lambda|^4\right) - 6\frac{1}{\mathcal{N}_n}\frac{2}{n}\sum_{\lambda\in\Lambda_n}\lambda_\ell^2|a_\lambda|^2+3 \\
&= \frac{24}{\mathcal{N}_n}\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}\left(\frac{\lambda_\ell^2}{n}\left(|a_\lambda|^2-1\right)\right)+o_{\mathbb{P}}(1)\right]^2 - \left(\frac{2}{n}\right)^2\frac{3}{\mathcal{N}_n^2}\sum_\lambda\lambda_\ell^4|a_\lambda|^4. \qquad (5.62)
\end{aligned}$$

To conclude the proof, we first observe that the last term in the rhs of (5.62) may be written as

$$\begin{aligned}
&\left(\frac{2}{n_j}\right)^2\frac{3}{\mathcal{N}_{n_j}}\sum_\lambda\lambda_\ell^4|a_\lambda|^4 \\
&= o_{\mathbb{P}}(1) + \underbrace{\left(\frac{2}{n_j}\right)^2\frac{3}{\mathcal{N}_{n_j}/2}\sum_{\lambda:\lambda_2>0}\lambda_\ell^4(|a_\lambda|^4-2)}_{=:K_1(n_j)} + \underbrace{\frac{24}{n_j^2\mathcal{N}_{n_j}}\sum_\lambda\lambda_\ell^4}_{=:K_2(n_j)}. \qquad (5.63)
\end{aligned}$$

Now for the last term in the rhs of (5.63) we have from Lemma 4.1

$$K_2(n_j) = 3(3+\widehat{\mu}_n(4)),$$

so that the conclusion follows from the fact that $\widehat{\mu}_n(4)\to\eta$, as well as from the fact that, since the random variables $\{|a_\lambda|^4-2 : \lambda\in\Lambda_{n_j},\ \lambda_2>0\}$ in $K_1(n_j)$ are i.i.d., square-integrable and centered and $\lambda_\ell^4/n^2\le 1$, $\mathbb{E}K_1(n_j)^2 = O(\mathcal{N}_{n_j}^{-1})\to 0$.

□

Lemma 5.4. *One has that*

$$\int_{\mathbb{T}} H_2(T_n(x))\Big(H_2(\partial_1\widetilde{T}_n(x)) + H_2(\partial_2\widetilde{T}_n(x))\Big)\,dx \tag{5.64}$$
$$= \frac{4}{\mathcal{N}_n}\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}\left(|a_\lambda|^2-1\right) + o_{\mathbb{P}}(1)\right)^2 - \frac{2}{\mathcal{N}_n^2}\sum_{\lambda}|a_{\lambda'}|^4.$$

Proof. For $\ell = 1, 2$,

$$\int_{\mathbb{T}} H_2(T_n(x))H_2(\partial_\ell\widetilde{T}_n(x))\,dx = \int_{\mathbb{T}}(T_n(x)^2-1)(\partial_\ell\widetilde{T}_n(x)^2-1)\,dx$$
$$= \int_{\mathbb{T}}\left(\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda\overline{a}_{\lambda'}e_\lambda(x)e_{-\lambda'}(x)-1\right)\left(\frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_{\lambda'',\lambda'''}\lambda''_\ell\lambda'''_\ell a_{\lambda''}\overline{a}_{\lambda'''}e_{\lambda''}(x)e_{-\lambda'''}(x)-1\right)dx$$
$$= \frac{2}{n}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}\lambda''_\ell\lambda'''_\ell a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''} - \frac{1}{\mathcal{N}_n}\sum_\lambda|a_\lambda|^2 - \frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda\lambda_\ell^2|a_\lambda|^2+1. \tag{5.65}$$

An application of the inclusion-exclusion principle yields that the first summand in the rhs of (5.65) equals

$$\frac{2}{n}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}\lambda''_\ell\lambda'''_\ell a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''} \tag{5.66}$$
$$= \frac{2}{n}\frac{1}{\mathcal{N}_n^2}\left(\sum_{\lambda,\lambda'}\lambda_j^2|a_\lambda|^2|a_{\lambda'}|^2 + 2\sum_{\lambda,\lambda'}\lambda_j\lambda'_j|a_\lambda|^2|a_{\lambda'}|^2 - 2\sum_\lambda\lambda_j^2|a_\lambda|^4 + \sum_\lambda\lambda_j^2|a_\lambda|^4\right).$$

Using the relation $a_{-\lambda} = \overline{a}_\lambda$, we also infer that

$$\sum_{\lambda,\lambda'}\lambda_j\lambda'_j|a_\lambda|^2|a_{\lambda'}|^2 = \left(\sum_\lambda\lambda_j|a_\lambda|^2\right)^2 = 0. \tag{5.67}$$

Substituting (5.67) into (5.66) and then (5.66) into (5.65) we rewrite (5.65) as

$$\int_{\mathbb{T}} H_2(T_n(x))H_2(\partial_\ell\widetilde{T}_n(x))\,dx$$
$$= \frac{2}{n}\frac{1}{\mathcal{N}_n^2}\left(\sum_{\lambda,\lambda'}\lambda_j^2|a_\lambda|^2|a_{\lambda'}|^2 - \sum_\lambda\lambda_j^2|a_\lambda|^4\right) - \frac{1}{\mathcal{N}_n}\sum_\lambda|a_\lambda|^2 \tag{5.68}$$
$$-\frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda\lambda_\ell^2|a_\lambda|^2+1.$$

Summing the terms corresponding to ∂_1 and ∂_2 up, i.e. (5.68) for $\ell = 1$ and $\ell = 2$, we deduce that the lhs of (5.64) equals

$$
\begin{aligned}
&\int_{\mathbb{T}} H_2(T_n(x))\Big(H_2(\partial_1\widetilde{T}_n(x)) + H_2(\partial_2\widetilde{T}_n(x))\Big)\,dx \\
&= \frac{2}{n}\frac{1}{\mathcal{N}_n^2}\left(\sum_{\lambda,\lambda'}\left(\lambda_1^2+\lambda_2^2\right)|a_\lambda|^2|a_{\lambda'}|^2 - \sum_{\lambda}\left(\lambda_1^2+\lambda_2^2\right)|a_{\lambda'}|^4\right) \\
&-\frac{2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2 - \frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda\left(\lambda_1^2+\lambda_2^2\right)|a_\lambda|^2 + 2 \\
&= \frac{2}{n}\frac{1}{\mathcal{N}_n^2}\left(\sum_{\lambda,\lambda'} n|a_\lambda|^2|a_{\lambda'}|^2 - \sum_\lambda n|a_{\lambda'}|^4\right) - \frac{2}{\mathcal{N}_n}\sum_\lambda|a_\lambda|^2 - \frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda n|a_\lambda|^2 + 2 \\
&= 2\frac{1}{\mathcal{N}_n^2}\left(\sum_{\lambda,\lambda'}|a_\lambda|^2|a_{\lambda'}|^2 - \sum_\lambda|a_{\lambda'}|^4\right) - \frac{2}{\mathcal{N}_n}\sum_\lambda|a_\lambda|^2 - \frac{2}{\mathcal{N}_n}\sum_\lambda|a_\lambda|^2 + 2 \\
&= \frac{2}{\mathcal{N}_n}\left(\frac{\sqrt{2}}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}\left(|a_\lambda|^2-1\right) + o_{\mathbb{P}}(1)\right)^2 - \frac{2}{\mathcal{N}_n^2}\sum_\lambda|a_{\lambda'}|^4,
\end{aligned}
$$

which equals to the rhs of (5.64). □

Our last lemma allows one to deal with the most challenging term appearing in (5.57).

Lemma 5.5. *We have that*

$$
\begin{aligned}
&\int H_2(\partial_1\widetilde{T}_n)H_2(\partial_2\widetilde{T}_n)\,dx \\
&= -4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_2^2(|a_\lambda|^2-1)\right]^2 - 4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_1^2(|a_\lambda|^2-1) + o_{\mathbb{P}}(1)\right]^2 \\
&+4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}(|a_\lambda|^2-1) + o_{\mathbb{P}}(1)\right]^2 \\
&+16\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_1\lambda_2\left(|a_\lambda|^2-1\right) + o_{\mathbb{P}}(1)\right]^2 - \frac{12}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_\lambda\lambda_1^2\lambda_2^2|a_\lambda|^4,
\end{aligned}
$$

and the following convergence takes place as $n_j\to\infty$:

$$
\frac{12}{n_j^2}\frac{1}{\mathcal{N}_{n_j}^2}\sum_\lambda\lambda_1^2\lambda_2^2|a_\lambda|^4 \xrightarrow{\mathbb{P}} 3(1-\eta). \tag{5.69}
$$

Proof. One has that

$$
\begin{aligned}
\int H_2(\partial_1\widetilde{T}_n)H_2(\partial_2\widetilde{T}_n)dx &= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}\lambda_1\lambda_1'\lambda_2''\lambda_2'''a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''} \\
&-\frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda\lambda_1^2|a_\lambda|^2 - \frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_\lambda\lambda_2^2|a_\lambda|^2 + 1.
\end{aligned}
\tag{5.70}
$$

First of all, we note that for the first two terms in (5.70)

$$\mathbb{E}\left[\frac{2}{n}\frac{1}{\mathcal{N}_n}\sum_{\lambda}(\lambda_1^2+\lambda_2^2)|a_\lambda|^2\right]=\mathbb{E}\left[\frac{2}{\mathcal{N}_n}\sum_{\lambda}|a_\lambda|^2\right]=2.$$

Let us now focus on (5.70). Using the structure of $S_4(n)$ in Lemma 5.1, we obtain

$$\begin{aligned}
&\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda-\lambda'+\lambda''-\lambda'''=0}\lambda_1\lambda_1'\lambda_2''\lambda_2'''a_\lambda\overline{a}_{\lambda'}a_{\lambda''}\overline{a}_{\lambda'''} \qquad (5.71)\\
&=\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\left[\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2+2\sum_{\lambda,\lambda'}\lambda_1\lambda_2\lambda_1'\lambda_2'|a_\lambda|^2|a_{\lambda'}|^2-3\sum_{\lambda}\lambda_1^2\lambda_2^2|a_\lambda|^4\right].
\end{aligned}$$

Let us now denote

$$\begin{aligned}
A &:= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2,\\
B &:= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}2\sum_{\lambda,\lambda'}\lambda_1\lambda_2\lambda_1'\lambda_2'|a_\lambda|^2|a_{\lambda'}|^2,\\
C &:= -3\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda}\lambda_1^2\lambda_2^2|a_\lambda|^4, \qquad (5.72)\\
D &:= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\left\{-N_n\frac{n}{2}\sum_{\lambda}|a_\lambda|^2+\frac{\mathcal{N}_n^2n^2}{4}\right\}, \qquad (5.73)
\end{aligned}$$

so that (5.70) with (5.71) read

$$\int H_2(\partial_1\widetilde{T}_n)H_2(\partial_2\widetilde{T}_n)\,dx=A+B+C+D. \qquad (5.74)$$

We have that

$$\begin{aligned}
A &= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2\\
&= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2+\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2\right\}\\
&= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{\sum_{\lambda,\lambda'}(n-\lambda_2^2)(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2+\sum_{\lambda,\lambda'}\lambda_1^2(n-(\lambda_1')^2)|a_\lambda|^2|a_{\lambda'}|^2\right\},
\end{aligned}$$

which we may rewrite as

$$
\begin{aligned}
A \quad &= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{-\sum_{\lambda,\lambda'}\lambda_2^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2-\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_1')^2|a_\lambda|^2|a_{\lambda'}|^2\right\}\\
&+\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{n\sum_{\lambda,\lambda'}(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2+n\sum_{\lambda,\lambda'}\lambda_1^2|a_\lambda|^2|a_{\lambda'}|^2\right\}\\
&= \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{-\sum_{\lambda,\lambda'}\lambda_2^2(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2-\sum_{\lambda,\lambda'}\lambda_1^2(\lambda_1')^2|a_\lambda|^2|a_{\lambda'}|^2\right\}\\
&+\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left\{n\sum_{\lambda,\lambda'}(\lambda_2')^2|a_\lambda|^2|a_{\lambda'}|^2+n\sum_{\lambda,\lambda'}\lambda_1^2|a_\lambda|^2|a_{\lambda'}|^2\right\}. \qquad (5.75)
\end{aligned}
$$

From (5.73) and (5.75), we get

$$
\begin{aligned}
A+D \quad &= \quad -\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left[\sum_{\lambda}\lambda_2^2(|a_\lambda|^2-1)\right]^2-\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}\frac{1}{2}\left[\sum_{\lambda}\lambda_1^2(|a_\lambda|^2-1)\right]^2\\
&\quad +\frac{4}{\mathcal{N}_n^2}\frac{1}{2}\left[\sum_{\lambda}(|a_\lambda|^2-1)\right]^2\\
&= \quad -4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_2^2(|a_\lambda|^2-1)+o_{\mathbb{P}}(1)\right]^2 \qquad (5.76)\\
&\quad -4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_1^2(|a_\lambda|^2-1)+o_{\mathbb{P}}(1)\right]^2\\
&\quad +4\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda,\lambda_2>0}(|a_\lambda|^2-1)+o_{\mathbb{P}}(1)\right]^2.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
B \quad &= \quad \frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}2\sum_{\lambda,\lambda'}\lambda_1\lambda_2\lambda_1'\lambda_2'|a_\lambda|^2|a_{\lambda'}|^2=\frac{4}{n^2}\frac{1}{\mathcal{N}_n^2}2\left[\sum_{\lambda}\lambda_1\lambda_2\left(|a_\lambda|^2-1\right)\right]^2\\
&= \quad 16\left[\frac{1}{\sqrt{\mathcal{N}_n/2}}\frac{1}{n}\sum_{\lambda,\lambda_2>0}\lambda_1\lambda_2\left(|a_\lambda|^2-1\right)+o_{\mathbb{P}}(1)\right]^2. \qquad (5.77)
\end{aligned}
$$

The first statement of Lemma 5.5 then follows upon substituting (5.76), (5.77) and (5.72) into (5.74).

Now to prove (5.69) it suffices to write

$$\begin{aligned}
&\frac{12}{{n_j}^2}\frac{1}{\mathcal{N}_{n_j}^2}\sum_{\lambda}\lambda_1^2\lambda_2^2|a_\lambda|^4\\
&=\frac{12}{{n_j}^2}\frac{1}{\mathcal{N}_{n_j}^2}\sum_{\lambda}\lambda_1^2\lambda_2^2(|a_\lambda|^4-2)+\frac{12}{{n_j}^2}\frac{2}{\mathcal{N}_{n_j}^2}\sum_{\lambda}\lambda_1^2\lambda_2^2\\
&=o_{\mathbb{P}}(1)+\frac{12}{{n_j}^2}\frac{2}{\mathcal{N}_{n_j}^2}\sum_{\lambda:\lambda_2>0}\lambda_1^2\lambda_2^2(|a_\lambda|^4-2)+\frac{24}{{n_j}^2}\frac{1}{\mathcal{N}_{n_j}^2}\sum_{\lambda}\lambda_1^2\lambda_2^2,
\end{aligned}\tag{5.78}$$

and then Lemma 4.1 and an argument similar to the one that concluded the proof of Lemma 5.3 allow to prove the result.

□

5.2 Proof of Lemma 4.2: technical computations

Substituting the results of Lemmas 5.2-5.5 into (5.57) we obtain, as $n_j\to+\infty$,

$$\begin{aligned}
\mathcal{L}_{n_j}[4]=\sqrt{\frac{E_{n_j}}{2\mathcal{N}_{n_j}^2}}\Bigg(&\frac{\alpha_{0,0}\beta_4}{4!}6\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\sum_{\lambda:\lambda_2>0}(|a_\lambda|^2-1)\Big)^2\\
&+\frac{\alpha_{0,4}\beta_0}{4!}\cdot 24\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\sum_{\lambda,\lambda_2>0}\Big(\frac{\lambda_1^2}{n_j}(|a_\lambda|^2-1)\Big)\Big)^2\\
&+\frac{\alpha_{4,0}\beta_0}{4!}\cdot 24\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\sum_{\lambda,\lambda_2>0}\Big(\frac{\lambda_2^2}{n_j}(|a_\lambda|^2-1)\Big)\Big)^2\\
+\frac{\alpha_{0,2}\beta_2}{2!2!}\cdot 4\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}&\sum_{\lambda,\lambda_2>0}(|a_\lambda|^2-1)\Big)^2+\frac{\alpha_{2,2}\beta_0}{2!2!}\Bigg(-4\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\frac{1}{n_j}\sum_{\lambda,\lambda_2\ge 0}\lambda_2^2(|a_\lambda|^2-1)\Big)^2\\
&-4\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\frac{1}{n_j}\sum_{\lambda,\lambda_2>0}\lambda_1^2(|a_\lambda|^2-1)\Big)^2+4\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\sum_{\lambda,\lambda_2>0}(|a_\lambda|^2-1)\Big)^2\\
&+16\Big(\frac{1}{\sqrt{\mathcal{N}_{n_j}/2}}\frac{1}{n_j}\sum_{\lambda,\lambda_2>0}\lambda_1\lambda_2\Big(|a_\lambda|^2-1\Big)\Big)^2\Bigg)+\widetilde{R}(n_j)\Bigg),
\end{aligned}\tag{5.79}$$

where $\widetilde{R}(n_j)$ is a sequence of random variables converging in probability to some constant $\in\mathbb{R}$. Computing the coefficients $\alpha_{\cdot,\cdot}$ (see (3.30)) and $\beta_\cdot$ (see (3.29)), from (5.79) we obtain that, as $n_j\to+\infty$,

$$\mathcal{L}_{n_j}[4]=\sqrt{\frac{E_{n_j}}{512\,\mathcal{N}_{n_j}^2}}\Big(W_1(n_j)^2-2W_2(n_j)^2-2W_3(n_j)^2-4W_4(n_j)^2+R(n_j)\Big),$$

where $W_k(n_j)$, $k=1,2,3,4$ are as in (4.48) and $R(n_j)$ is a sequence of random variables converging in probability to 1. The proof is now complete.

□

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