Enumerating chordless circuits in directed graphs

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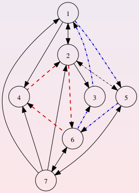
Example of chordless circuits



Two chordless circuits :

1. ① → ⑤ → ⑥ → ③ → ①
2. ② → ⑥ → ④ → ②

The absence of chordless circuits of odd length guarantees the existence of kernels – independent outranking choices – in an outranking digraph.

Seminal and similar recent work

- Finding or enumerating the elementary circuits : Tiernan (1970), Weinblatt (1972), Tarjan (1973)
- Detection of holes, i.e. chordless cycles in (non oriented) graphs ; Nikolopoulos (2007)
- The strong perfect graph theorem : A graph is perfect if, and only if, it contains no holes or antiholes (holes in the complement graph) on an odd number of vertices ; Chudnovsky, Robertson, Seymour ,Thomas (2006).

Notation

- G is a directed graph (digraph) with no multiple arcs.
- $G(V)$ and $G(A)$ are respectively the vertices and the arcs set of G.
- The number of its vertices is called the order and the number of its arcs is called the size of the digraph.
- A (directed) path $P_k^{\rightarrow}$ in G of length $k > 0$ from v_0 to v_k is a list of vertices $[v_0, v_1, \cdots, v_k]$ such that $(v_i, v_{i+1}) \in G(A)$ for $i = 0, ..., k-1$.

Notation – continue

- A path is called simple if none of its vertices occur more than once.
- A directed path $P_k^{\rightarrow}$ is called chordless if neither $(v_i, v_j) \in A$ nor $(v_j, v_i) \in A$ for any two non-consecutive vertices v_i, v_j $(i, j = 0 \ldots k)$ in the path.
- A chordless circuit $C_k^{\rightarrow}$ is a closed directed path $P_k^{\rightarrow}$ without chords, i.e. $v_k = v_0$ and no $(v_i, v_j) \in A$ such that $i - j \not\equiv k - 1 \mod k$.

Notation – continue

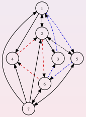
- We say that a path $[v_0, v_1, \cdots, v_k]$ is adjacent to a path $[w_0, w_1, \cdots, w_k]$ if $v_k = w_0$.
- We call a pre-chordless-circuit (of length k) a list of vertices $pC_k^{\rightarrow} := [v_0, v_1, \cdots, v_{k-2}, v_{k-1}]$ from V with $k \geqslant 3$ when both partial sublists $[v_0, v_1, \cdots, v_{k-2}]$, as well as $[v_1, \cdots, v_{k-2}, v_{k-1}]$, are chordless paths of length $k - 2$.
- The (dominated) strict neighbourhood $N(v_1)$ of a vertex $v_1 \in V$ is the set of all vertices $v_2 \in G$ such that $(v_1, v_2) \in A$ and $(v_2, v_1) \notin A$.

Detecting a chordless circuit

Lemma (1)

A digraph G contains a chordless circuit $C_k^{\rightarrow}$ of length $k \geqslant 3$, starting from a vertex v_0 if, and only if, there exists a pre-chordless-circuit $pC_{k-1}^{\rightarrow} = [v_0, v_1, \cdots, v_{k-1}]$ starting from v_0 which is followed by an adjacent chordless path of length 1 from v_{k-1} back to v_0.

Example (prechordless circuit condition)



① → ⑤ → ⑥ is a $P_2^{\rightarrow}$, and

⑤ → ⑥ → ③ is a $P_2^{\rightarrow}$, and

③ → ① is a $P_1^{\rightarrow}$.

Associated pre-chordless-circuits digraph

We consider now the auxiliary chordless line digraph L with *vertices* set $L(V)$ gathering all chordless paths of length 1 and all possible pre-chordless-circuits $[v_0, \ldots, v_{k-1}]$ of length $k - 1 \geqslant 2$ in G, with *edges* set $L(E)$ defined for $k \geqslant 2$ as follows :

$$
\begin{aligned}
L(V) &:= \{[v_i, v_j] \; : \; (v_i, v_j) \in G(A) \wedge (v_j, v_i) \notin G(A)\} \\
&\quad \cup \{pC_k^{\rightarrow} \; : \; pC_k^{\rightarrow} \text{ is a pre-chordless-circuit in } G\} \\
L(E) &:= \{([v_0, \ldots, v_{k-1}], [v_0, \ldots, v_{k-1}, v_k]) \in L(V)^2 \\
&\quad \text{s. t. } [v_{k-1}, v_k] \in L(V)\}
\end{aligned}
$$

Running a DFS on the pre-chordless-circuits digraph

Lemma (2)

Let G be a digraph and let L be the associated pre-chordless-circuits digraph defined before.

1. *G contains a chordless circuit $C_k^{\rightarrow}$ if, and only if, the DFS algorithm, when running on L, finds a sequence of increasing pre-chordless-circuits $pC_i^{\rightarrow}$ for $i = 2 \dots k-1$ that eventually meet the conditions of Lemma (1).*
2. *Running the complete DFS algorithm on L will in turn deliver all chordless circuits to be found in a given digraph G.*

The Chordless-Circuit-Enumeration (CCE) algorithm

```
Input : a digraph G ; Output : a list of chordless circuits.
def enumerateChordlessCircuits(In : G ) :
    chordlessCircuits ← [] :
    visitedLEdges ← {} :
    toBeVisited ← a copy of V :
    while toBeVisited ≠ {} :
        v ← toBeVisited.pop()
        P ← [v]
        vCC ← []
        if chordlessCircuit(P,v) :
            chordlessCircuits ← chordlessCircuits + vCC
    return chordlessCircuits
```

The CCE algorithm –continue

```
Input : a path P = [··· , v_{k−1}], and a target vertex v_k.
Output : a Boolean variable detectedChordlessCircuit
def chordlessCircuit( In : P, v_k ; Out : detectedChordlessCircuit
    v_{k−1} ← last vertex of P
    visitedLEdges ← add {v_{k−1}, v_k}
    if v_k ∈ N(v_{k−1}) :
        detectedChordlessCircuit ← True
        print 'Chordless circuit's certificate : ', P
        vCC ← append P
```

The CCE algorithm –continue

```
    else
        detectedChordlessCircuit ← False
        N ← a local copy of N(v_{k−1})
        while N not empty :
            v ← pop a neighbour of v_{k−1} from N
            if {v_{k−1}, v} ∉ visitedLEdges :
                NoChord ← True
                P_current ← a copy of the current P
                for x ∈ P_current − {v_{k−1}} :
                    if x = v_k :
                        if (x, v) ∈ A :
                            NoChord ← False
                    else
                        if (x, v) ∈ A ∨ (v, x) ∈ A :
                            NoChord ← False
                if NoChord :
                    P_current ← append v
        if chordlessCircuit(P_current, v_k) :
        detectedChordlessCircuit ← True
    return detectedChordlessCircuit
```

Complexity of the CCE algorithm

- The CCE algorithm yields a time complexity in $O(|L(V)|)$;
- A lower bound in $\Omega(4^{\sqrt{n}})$ for digraphs of order n has been provided on round grid graphs by Pierre Kelsen (UL);
- Space complexity is roughly in $O(n(n+m))$ for digraphs of order n containing m chordless circuits.

Chordless circuits in random digraphs

Chordless circuit enumeration in samples of 1 000 random digraphs

order	time		total	mean	frequency (in %) per circuit length						
(n)	(sec.)	(stdev)	freq.	length	3	4	5	6	7	8	...
10	0.0005	0.0001	4	3.00	100						
20	0.0037	0.0007	43	3.19	80	16	5				
30	0.0148	0.0022	174	3.29	73	25	2				
40	0.0466	0.0060	473	3.39	66	31	4				
50	0.1148	0.0127	1035	3.47	59	35	6				
60	0.2575	0.0252	1996	3.55	53	39	8				
70	0.5038	0.0447	3506	3.62	48	41	10	1			
80	0.9393	0.0761	5796	3.69	44	43	12	1			
90	1.6204	0.1241	9054	3.75	41	45	13	1			
100	2.6509	0.1917	13526	3.81	37	46	15	1			
110	4.1251	0.2786	19596	3.87	34	47	17	2			
...	...	...	...	...	...						
150	22.787	1.2848	68079	4.06	25	47	24	3	...		

Average execution statistics for the CCE algorithm obtained on a Dell PE 2950, 2 Quad-Core Xeon X5355 2.66GHz, 32 Gb.

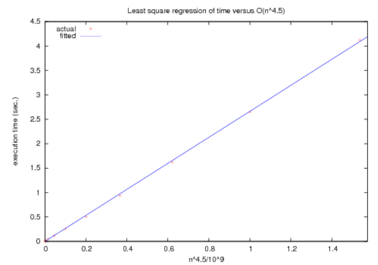
Varying the arc probability with constant order 50

Chordless circuits in samples of 1 000 random digraphs of order 50

arc	time		total	mean	frequency (in %) per circuit length							
prob.	sec.	stdev	freq.	length	3	4	5	6	7	8	9	...
0.8	0.015	0.001	181	3.01	99	1						
0.7	0.031	0.003	410	3.07	97	3						
0.6	0.061	0.006	691	3.21	80	19	1					
0.5	0.115	0.013	1035	3.47	59	35	6					
0.4	0.234	0.028	1469	3.90	36	42	19	3				
0.3	0.489	0.077	1981	4.58	17	32	31	15	4			
0.25	0.671	0.117	2194	5.07	11	23	30	23	10	3		
0.2	0.888	0.177	2215	5.70	6	15	24	26	18	8	2	
0.15	0.946	0.240	1816	6.53	4	9	16	21	21	16	9	...
0.1	0.535	0.218	728	7.40	3	6	10	14	17	17	14	...
0.05	0.025	0.022	22	6.19	9	14	14	9	9	9	9	...

Average execution statistics for the CCE algorithm

Apparent time complexity in $n^{4.5}$ for random digraphs



Chordless circuits in tournaments

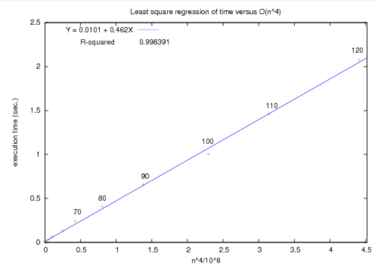
- A tournament, i.e. a complete asymmetric digraph, may only contain chordless circuits of length 3 ;
- The maximal number of such chordless 3-circuits (oriented cyclic triples) in any tournament of order n is $O(n^3)$ (Kendall and Babington Smith, 1940) ;
- Expected number of chordless 3-circuits in a random tournament of order n and arc probablity 0.5 is $\frac{1}{4}\binom{n}{3}$ (Moon, 1968).

Average execution statistics for the CCE algorithm

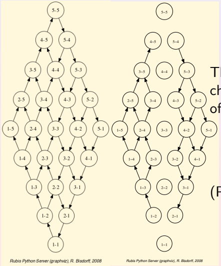
Circuits in samples of 1 000 random tournaments

order (n)	expected #	frequency (#)	(stdev)	run time (sec.)	(stdev)
10	30	30	5	0.0008	0.0001
20	285	286	15	0.0057	0.0005
30	1015	1015	27	0.0019	0.0011
40	2470	2470	43	0.0490	0.0027
50	4900	4897	60	0.1018	0.0056
60	8555	8553	77	0.2530	0.0445
70	13865	13864	106	0.4272	0.0767
80	20540	20544	128	0.8038	0.1357
90	29370	29365	153	1.3800	0.1981
100	40425	40430	172	2.2979	0.2630
110	53955	53949	199	3.1396	0.4101
120	70210	70207	226	4.4086	0.6939

Apparent time complexity in n^4



Round grid graphs



The maximum number of chordless circuits in a digraph of order n is at least equal to

$$\frac{4^{2\sqrt{n}-2}}{(\sqrt{n}-2)^4}$$

(Pierre Kelsen, 2010).

Concluding remarks

- We provide a chordless circuits enumeration (CCE) algorithm with a time complexity that is proportional to the number of pre-chordless-circuits contained in the digraph.
- Lower bound time complexity of CCE is at least exponential in $\sqrt{n}$.
- The space complexity of CCE for a digraph of order n containing m chordless circuits is $O(n(n+m))$.
- Average time complexity of CCE is apparently polynomial – $O(n^{4.5})$ – for random digraphs with arc probability 0.5.
- Average time complexity of CCE for random tournaments is apparently $O(n^4)$.