Enumerating chordless circuits in directed graphs

Raymond Bisdorff
Université du Luxembourg
FSTC/ILAS
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Example of chordless circuits

Two chordless circuits:
1. \(1 \rightarrow 5 \rightarrow 6 \rightarrow 3 \rightarrow 1\)
2. \(2 \rightarrow 6 \rightarrow 4 \rightarrow 2\)

The absence of chordless circuits of odd length guarantees the existence of kernels – independent outranking choices – in an outranking digraph.

Seminal and similar recent work

• Finding or enumerating the elementary circuits: Tiernan (1970), Weinblatt (1972), Tarjan (1973)
• Detection of holes, i.e. chordless cycles in (non oriented) graphs; Nikolopoulos (2007)
• The strong perfect graph theorem: A graph is perfect if, and only if, it contains no holes or antiholes (holes in the complement graph) on an odd number of vertices; Chudnovsky, Robertson, Seymour, Thomas (2006).

Notation

• \(G\) is a directed graph (digraph) with no multiple arcs.
• \(G(V)\) and \(G(A)\) are respectively the vertices and the arcs set of \(G\).
• The number of its vertices is called the order and the number of its arcs is called the size of the digraph.
• A (directed) path \(P_k\) in \(G\) of length \(k > 0\) from \(v_0\) to \(v_k\) is a list of vertices \([v_0, v_1, \ldots, v_k]\) such that \((v_i, v_{i+1}) \in G(A)\) for \(i = 0, \ldots, k - 1\).
A path is called **simple** if none of its vertices occur more than once.

A directed path $P^\rightarrow_k$ is called **chordless** if neither $(v_i, v_j) \in A$ nor $(v_j, v_i) \in A$ for any two non-consecutive vertices $v_i$, $v_j$ ($i, j = 0...k$) in the path.

A **chordless circuit** $C^\rightarrow_k$ is a closed directed path $P^\rightarrow_k$ without chords, i.e. $v_k = v_0$ and no $(v_i, v_j) \in A$ such that $i - j \not\equiv k - 1 \mod k$.

- We say that a path $[v_0, v_1, \cdots, v_k]$ is **adjacent** to a path $[w_0, w_1, \cdots, w_k]$ if $v_k = w_0$.
- We call a **pre-chordless-circuit** (of length $k$) a list of vertices $pC^\rightarrow_k := [v_0, v_1, \cdots, v_{k-2}, v_{k-1}]$ from $V$ with $k \geq 3$ when both partial sublists $[v_0, v_1, \cdots, v_{k-2}]$, as well as $[v_1, \cdots, v_{k-2}, v_{k-1}]$, are chordless paths of length $k - 2$.
- The (dominated) **strict neighbourhood** $N(v_1)$ of a vertex $v_1 \in V$ is the set of all vertices $v_2 \in G$ such that $(v_1, v_2) \in A$ and $(v_2, v_1) \not\in A$.

**Detecting a chordless circuit**

**Example (prechordless circuit condition)**

![Graph](image)

**Lemma (1)**

A digraph $G$ contains a chordless circuit $C^\rightarrow_k$ of length $k \geq 3$, starting from a vertex $v_0$ if, and only if, there exists a pre-chordless-circuit $pC^\rightarrow_{k-1} = [v_0, v_1, \cdots, v_{k-1}]$ starting from $v_0$ which is followed by an adjacent chordless path of length 1 from $v_{k-1}$ back to $v_0$.

**Associated pre-chordless-circuits digraph**

We consider now the auxiliary chordless line digraph $L$ with vertices set $L(V)$ gathering all chordless paths of length 1 and all possible pre-chordless-circuits $[v_0, \ldots, v_{k-1}]$ of length $k - 1 \geq 2$ in $G$, with edges set $L(E)$ defined for $k \geq 2$ as follows:

$L(V) := \{[v_i, v_j] : (v_i, v_j) \in G(A) \land (v_j, v_i) \not\in G(A)\}$

$\cup \{pC^\rightarrow_k : pC^\rightarrow_k$ is a pre-chordless-circuit in $G\}$

$L(E) := \{([v_0, \ldots, v_{k-1}], [v_0, \ldots, v_{k-1}, v_k]) \in L(V)^2 \}$

s. t. $[v_{k-1}, v_k] \in L(V)$
Running a DFS on the pre-chordless-circuits digraph

Lemma (2)

Let $G$ be a digraph and let $L$ be the associated pre-chordless-circuits digraph defined before.

1. $G$ contains a chordless circuit $C_k$ if, and only if, the DFS algorithm, when running on $L$, finds a sequence of increasing pre-chordless-circuits $pC_i$ for $i = 2...k - 1$ that eventually meet the conditions of Lemma (1).

2. Running the complete DFS algorithm on $L$ will in turn deliver all chordless circuits to be found in a given digraph $G$.

The 

**Chordless-Circuit-Enumeration (CCE)** algorithm

1: Input: a digraph $G$; Output: a list of chordless circuits.
2: def enumerateChordlessCircuits(In : $G$):
3:     chordlessCircuits ← []:
4:     visitedLEdges ← {}:
5:     toBeVisited ← a copy of $V$:
6:     while toBeVisited ≠ {}:
7:         v ← toBeVisited.pop()
8:         P ← [v]
9:         vCC ← []
10:        if chordlessCircuit($P$, $v$):
11:           chordlessCircuits ← chordlessCircuits + vCC
12:           return chordlessCircuits
13:     Input: a path $P = [···, v_{k-1}]$, and a target vertex $v_k$.
14:     Output: a Boolean variable detectedChordlessCircuit.
15:     def chordlessCircuit(In : $P$, $v_k$; Out : detectedChordlessCircuit
16:         v_{k-1} ← last vertex of $P$
17:         visitedLEdges ← add $\{v_{k-1}, v_k\}$
18:         if $v_k \in N(v_{k-1})$:
19:             detectedChordlessCircuit ← True
20:             print 'Chordless circuit's certificate : ', $P$
21:             vCC ← append $P$
22: else detectedChordlessCircuit ← False
23:     $N$ ← a local copy of $N(v_{k-1})$
24:     while $N$ not empty:
25:         v ← pop a neighbour of $v_{k-1}$ from $N$
26:         if $\{v_{k-1}, v\} \notin \text{visitedLEdges}$:
27:             NoChord ← True
28:         $P_{current}$ ← a copy of the current $P$
29:         for $x \in P_{current} \setminus \{v_{k-1}\}$:
30:             if $x = v_k$:
31:                 if $(x, v) \in A$:
32:                     NoChord ← False
33:                 else
34:                     if $(x, v) \in A \lor (v, x) \in A$:
35:                         NoChord ← False
36:                 else NoChord ← True
37:             if NoChord:
38:                 $P_{current}$ ← append $v$
39:             if chordlessCircuit($P_{current}$, $v_k$):
40:                 detectedChordlessCircuit ← True
41:     return detectedChordlessCircuit
The CCE algorithm yields a time complexity in $O(|L(V)|)$;

A lower bound in $\Omega(4^{\sqrt{n}})$ for digraphs of order $n$ has been provided on round grid graphs by Pierre Kelsen (UL);

Space complexity is roughly in $O(n(n + m))$ for digraphs of order $n$ containing $m$ chordless circuits.

### Varying the arc probability with constant order 50

<table>
<thead>
<tr>
<th>arc prob.</th>
<th>time (sec.)</th>
<th>(stdev)</th>
<th>total freq.</th>
<th>mean length</th>
<th>frequency (in %) per circuit length</th>
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<td>3 4 5 6 7 8 9 ...</td>
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</table>

Average execution statistics for the CCE algorithm

Apparent time complexity in $n^{4.5}$ for random digraphs

![Graph showing apparent time complexity vs $O(n^{4.5})$](image-url)
Chordless circuits in tournaments

- A tournament, i.e. a complete asymmetric digraph, may only contain chordless circuits of length 3;
- The maximal number of such chordless 3-circuits (oriented cyclic triples) in any tournament of order \( n \) is \( O(n^3) \) (Kendall and Babington Smith, 1940);
- Expected number of chordless 3-circuits in a random tournament of order \( n \) and arc probability 0.5 is \( \frac{1}{4}(\frac{n}{3}) \) (Moon, 1968).

<table>
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<tr>
<th>order ((n))</th>
<th>expected # frequency ((#)) (stdev)</th>
<th>run time ((\text{sec.})) (stdev)</th>
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The maximum number of chordless circuits in a digraph of order \( n \) is at least equal to

\[
\frac{4^2\sqrt{n} - 2}{(\sqrt{n} - 2)^4}
\]

(Pierre Kelsen, 2010).
Concluding remarks

- We provide a chordless circuits enumeration (CCE) algorithm with a time complexity that is proportional to the number of pre-chordless-circuits contained in the digraph.
- Lower bound time complexity of CCE is at least exponential in $\sqrt{n}$.
- The space complexity of CCE for a digraph of order $n$ containing $m$ chordless circuits is $O(n(n + m))$.
- Average time complexity of CCE is apparently polynomial – $O(n^{4.5})$ – for random digraphs with arc probability 0.5.
- Average time complexity of CCE for random tournaments is apparently $O(n^4)$.