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Deterministic analysis of distributed order systems using operational matrix

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ABSTRACT

Recently, distributed order systems as a generalized concept of fractional order have been a major focus in science and engineering areas, and have rapidly extended application across a wide range of disciplines. However, only a few numerical methods are available for analyzing the distributed order systems. This paper proposes a novel numerical scheme to analyze the behavior of single input single output linear systems in the time domain with a single distributed order differentiator/integrator by using operational matrix technique. The proposed method reduces different analysis problems to a system of algebraic equations by using block pulse functions, which makes it easy to handle an arbitrary input. Numerical examples were used to illustrate the accuracy and computational efficiency of the proposed method. The proposed method was found to be an efficient tool for analyzing linear distributed order systems.

1. Introduction

Fractional/distributed order calculus has wide applicability across a wide range of disciplines, such as physics, biology, chemistry, mechanical, physiology, and control engineering [1–6]. Caputo first proposed the distributed order (DO) equation, which is a generalized concept fractional order, in 1969 [7] and solved it in 1995 [8]. The general solution of linear DO systems is discussed elsewhere [9]. Later, the DO concept was used to examine the diffusion equation [10] and other real complex physical phenomena [11–14]. To the authors' knowledge, only a few numerical methods have been reported for the time domain analysis of the DO systems [15–18]. The numerical methods for the analysis of the DO operator are still immature and require further development. This has motivated the development of a computational scheme for the analysis basics of a linear DO system. The operational matrix (OP) method has attracted considerable attention for the analysis of a range of dynamic systems [19–22]. The main characteristic of this technique is that different analysis problems can be reduced to a system of algebraic equations using different orthogonal functions, which simplifies the problem. On the other hand, to the best of the author's knowledge, there are no reports on the analysis of DO systems using the operational matrix technique.

This paper proposes a numerical scheme based on the operational matrix technique for the deterministic analysis of linear DO systems. Section 2 briefly introduces the DO system and the operational matrix technique. Section 3 provides examples to demonstrate the use of the proposed method. The results of the proposed method are also compared with those of other existing numerical and analytical methods.

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2. Preliminary of fractional and distributed order systems

First, the concepts of the fractional order derivative and integral are described briefly. The concept of the DO operator is then introduced as a generalization of the fractional order one. A DO term can be approximated as a weighted sum of fractional terms. This procedure is known as a discretized procedure for the DO integral. Finally, due to the discretized procedure, the OP technique can be used to solve the discretized DO differential equation, which is actually a multi-term fractional order equation.

2.1. Governing equation for system dynamics with fractional order dynamics

Fractional calculus considers the generalization of an integration and differentiation operator to non-integer order [23,24]. Among the many formulations of the generalized fractional order derivative, the Riemann–Liouville definition is used most often:

$$ {}_{RL}D_t{}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau, \tag{1}$$

where $\Gamma(x)$ denotes the gamma function, and m is a positive integer satisfying $m-1 \le \alpha < m$. The subscript RL is for the Riemann–Liouville definition.

The Riemann–Liouville (RL) fractional order integral of a function $f(t)$ is defined as:

$$ {}_{RL}I_t{}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{2}$$

Another popular definition of the fractional order derivative is the Caputo (C) definition:

$$ {}_{C}D_t{}^{\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{3}$$

The subscript C denotes the Caputo definition.

The Laplace transform for a fractional order derivative under zero initial conditions can be defined as follows:

$$ L\{D_t{}^{\alpha} f(t)\} = s^{\alpha} F(s) \tag{4}$$

Note that under a zero initial condition, the two Riemann–Liouville and Caputo definitions are equivalent. Hence, the zero initial conditions are assumed and the subscripts for the two definitions are dropped.

Therefore, a fractional order single input single output (SISO) system can be described using a fractional order differential equation,

$$ a_0 D_t{}^{\alpha_0} y(t) + a_1 D_t{}^{\alpha_1} y(t) + \cdots + a_n D_t{}^{\alpha_n} y(t) = b_0 D_t{}^{\beta_0} u(t) + b_1 D_0{}^{\beta_1} u(t) + \cdots + b_m D_t{}^{\beta_m} u(t), \tag{5}$$

or a transfer function,

$$ G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \cdots + a_0 s^{\alpha_0}}, \tag{6}$$

where $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^m$ are arbitrary real positive numbers, n and m are the numbers of fractional order terms in the left hand side (LHS) and right hand side (RHS) of Eq. (5), respectively, and $u(t)$ and $y(t)$ are the input and output of the system, respectively.

2.2. Distributed order (DO) systems

The DO derivative of a function $f(t)$ with respect to $\rho(\alpha)$ is defined as follows [13–17]:

$$ D_t{}^{\rho(\alpha)} f(t) = \int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t{}^{\alpha} f(t) d\alpha, \tag{7}$$

where $\rho(\alpha)$ denotes the distribution function of order α.

Therefore, the general form of the distributed order differential equation can be expressed as

$$ \sum_{i=0}^{n} a_i D_t{}^{\rho_i(\alpha)} y(t) = \sum_{j=0}^{m} b_j D_t{}^{\rho_j(\alpha)} u(t), \tag{8}$$

where $\{a_i\}_{i=0}^n$and$\{b_j\}_{j=0}^m$ are the real constant parameters of the DO equation; $\{\rho_i(\alpha)\}_{i=0}^n$ and$\{\rho_j(\alpha)\}_{j=0}^m$ are the distribution functions corresponding to $\{a_i\}_{i=0}^n$and$\{b_j\}_{j=0}^m$, respectively; n and m are numbers of the DO derivative terms in LHS and RHS of the DO differential equation.

For time domain analysis of the DO, the distributed order integral in Eq. (8) is discretized using the quadrature formula as follows [16]:

$$\int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t^{\alpha} f(t) d\alpha \approx \sum_{l=1}^{Q} \rho(\alpha_l) \left(D_t^{\alpha_l} f(t) \right) \upsilon_l, \tag{9}$$

where α_l, υ_l are the node and weights from the quadrature formula, respectively; Q is the number of nodes in the quadrature. In other words, a single DO term is approximated as a weighted sum of Q fractional order derivative terms; and the DO equation can be approximated as a multi-term fractional order equation, and rearranged as Eq. (5).

2.3. Operational matrices of the block pulse function for the analysis of DO systems

Block pulse functions (BPFs) are a complete set of orthogonal functions that are defined over the time interval, $[0,\tau]$:

$$\psi_i(t) = \begin{cases} 1 & \frac{i-1}{N}\tau \le t \le \frac{i}{N}\tau \\ 0 & \text{elsewhere} \end{cases}, \tag{10}$$

where N is the number of block pulse functions.

Therefore, any function that can be absolutely integrated over the time interval, $[0,\tau]$, can be expanded to a series based on the BPFs:

$$f(t) = \boldsymbol{\psi}_N{}^T(t) C_f = \sum_{i=1}^{N} c_{f_i} \psi_i(t), \tag{11}$$

where $\boldsymbol{\psi}_N{}^T(t) = [\psi_1(t), \ldots, \psi_N(t)]$ is a vector component of the block pulse basis with respect to independent variable t (time). From here, the subscript $\boldsymbol{N}$ of $\boldsymbol{\psi}_N{}^T(t)$ is discarded for the convenience of notation. $C_f = [c_{f_1}, \ldots, c_{f_N}]^T$ is a vector of the coefficients expansion of $f(t)$ using the BPFs.

The expansion coefficients (or spectral characteristics), c_{f_i}, can be evaluated as follows:

$$c_{f_i} = \frac{N}{\tau} \int_{[(i-1)/N]\tau}^{(i/N)\tau} f(t) \psi_i(t) dt, \quad i = 1, \ldots, N \tag{12}$$

For a function, $f(t)$, with the vector of coefficients expansion, C_f, a fractional order integral defined in Eq. (2) can be expressed in terms of the BPFs using the generalized operational matrix of the fractional order integral [19]:

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \approx \boldsymbol{\psi}(t)^T A_{\alpha} C_f, \tag{13}$$

where the generalized operational matrix integration of BPFs, A_{α}, is

$$A_{\alpha} = P_{\alpha}{}^T = \left(\frac{\tau}{N}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_N \\ 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \cdots & \cdots & \zeta_1 \end{pmatrix}^T, \tag{14}$$

The elements of the generalized operational matrix integration can be given by the following:

$$\zeta_1 = 1; \ \ \zeta_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1} \ \text{ for } p = 2, 3 \ldots, N \tag{15}$$

B_{α}, the generalized operational matrix of a fractional order derivative of order α, is defined as:

$$B_{\alpha} A_{\alpha} = I, \tag{16}$$

where I is the identity matrix.

For a function $f(t)$ with the vector of coefficients expansion, C_f, the generalized operational matrix of the derivative can be used to approximate Eq. (3) as follows:

$$D_t^{\alpha} f(t) \approx \boldsymbol{\psi}(t)^T B_{\alpha} C_f \tag{17}$$

Algorithm 1
Procedure for obtaining a solution of the deterministic DO equation.

a) Calculate the vector of the coefficients of expansions, $C_U = [c_{u_1}, \ldots, c_{u_N}]^T$, of the input in terms of the block pulse functions as in Eq. (12)
b) Rewrite the DO differential equation in Eq. (8) with a suitable quadrature, in terms of the operational matrix, as expressed in Eq. (19). In this step, different quadrature rules can be used for the discretization of various DO systems, such as Gauss–Legendre, Gauss–Kronrod–Patterson, etc.
c) The expansion coefficients of the output is obtained as $C_Y = A_G C_U$
d) The output is $Y(t) = (C_Y)^T \boldsymbol{\psi}(t)$

By substituting Eq. (17) into Eq. (9), discretization of the DO derivative can be expressed as:

$$D_t^{\rho(\alpha)} f(t) = \int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t^{\alpha} f(t) d\alpha \approx \sum_{l=1}^{Q} \rho(\alpha_l)(D_t^{\alpha_l} f(t))\upsilon_l$$
$$= \sum_{l=1}^{Q} \upsilon_l \rho(\alpha_l)(\boldsymbol{\psi}(t)^T B_{\alpha_l} C_f) = \boldsymbol{\psi}(t)^T \sum_{l=1}^{Q} \upsilon_l \rho(\alpha_l)(B_{\alpha_l} C_f) \tag{18}$$

By substituting Eq. (18) into Eq. (8), the DO differential equation given by Eq. (8) can be rewritten in terms of the operational matrix fractional order derivative as follows:

$$\boldsymbol{\psi}(t)^T \left[\sum_{i=0}^{n} a_i \sum_{l=1}^{Q} \upsilon_l \rho_i(\alpha_l) B_{\alpha_l}\right] C_Y = \boldsymbol{\psi}(t)^T \left[\sum_{j=0}^{m} b_j \sum_{l=1}^{Q} \upsilon_l \rho_j(\alpha_l) B_{\alpha_l}\right] C_U \tag{19}$$

Let an operational matrix, A_G, for the system in Eq. (19) be defined as:

$$A_G = \left[\sum_{i=0}^{n} a_i \sum_{l=1}^{Q} \upsilon_l \rho_i(\alpha_l) B_{\alpha_l}\right]^{-1} \left[\sum_{j=0}^{m} b_j \sum_{l=1}^{Q} \upsilon_l \rho_j(\alpha_l) B_{\alpha_l}\right]. \tag{20}$$

Using Eq. (20), the input and output of the DO system are related by the following equation:

$$C_Y = A_G C_U; \quad Y(t) = \boldsymbol{\psi}^T(t) C_Y; \quad U(t) = \boldsymbol{\psi}(t)^T C_U. \tag{21}$$

Algorithm 1 summarizes the proposed method for analyzing the deterministic DO SISO system.

3. Examples

3.1. Example 1

Consider a deterministic distributed order integrator,

$$G(s) = \frac{Y(s)}{U(s)} = \int_{0.5}^{0.8} s^{-\alpha} d\alpha, \tag{22}$$

The analytical inverse Laplace transform of Eq. (22), which is a system impulse response, can be expressed as follows [15]:

$$h(t) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-xt}}{(\log(x))^2 + \pi^2} [x^{-0.5}(\sin(0.5\pi)\log(x) + \pi\cos(0.5\pi)) - x^{-0.8}(\sin(0.8\pi)\log(x) + \pi\cos(0.8\pi))] dx \tag{23}$$

where log() is a natural logarithm function.

For the proposed method, the DO is discretized with a three point Gauss–Legendre quadrature rule obtained from the OPQ suite (a MATLAB suite of programs for generating orthogonal polynomials and related quadrature) [25], and the operational matrix for this distributed integrator can be calculated by

$$A_G = \sum_{i=1}^{3} A_{\alpha_i} \upsilon_i, \tag{24}$$

where α_i, υ_i are the nodes and weights of the quadrature, respectively.

Because the input here is a Dirac delta function, a rectangular pulse is used to approximate the singular Dirac delta functions in the proposed method. Therefore, the system output can be calculated as follows:

$$C_y = A_G C_u;$$
$$y(t) = \boldsymbol{\psi}^T(t) C_y, \tag{25}$$

where $y(t)$ is the system output; C_u and C_y are the spectral characteristics of the input and output, respectively.

Fig. 1 presents the impulse responses for this system by various methods: the exact solution obtained using Eq. (23), the proposed method, the numerical inverse Laplace transform (NILT) method [15,26], and the improved Talbot (IT) method [27]. The

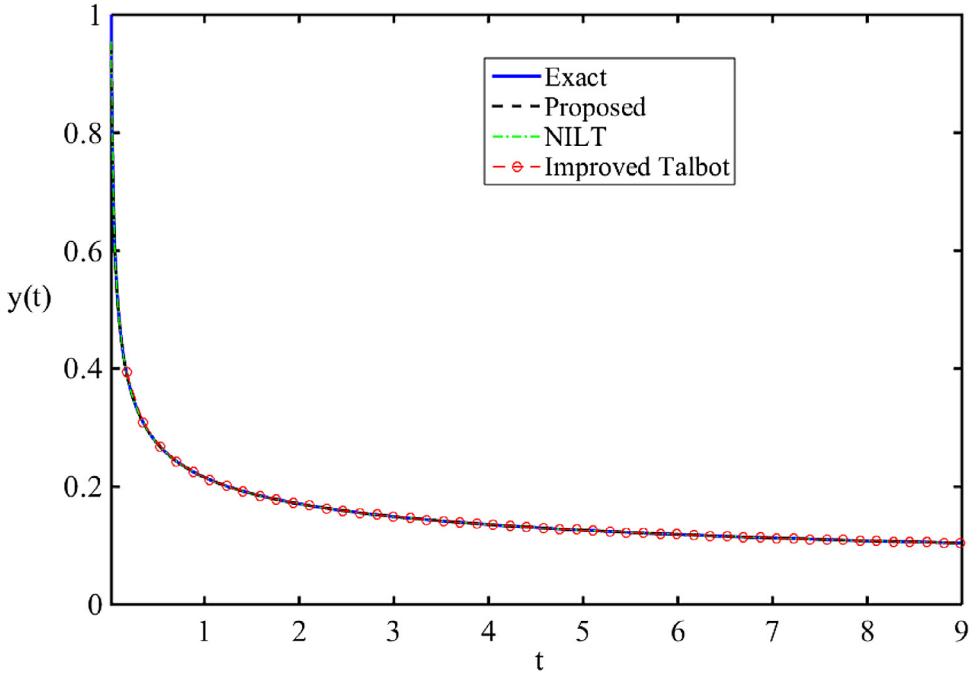

Fig. 1. Impulse responses of distributed order integrator in Example 1.

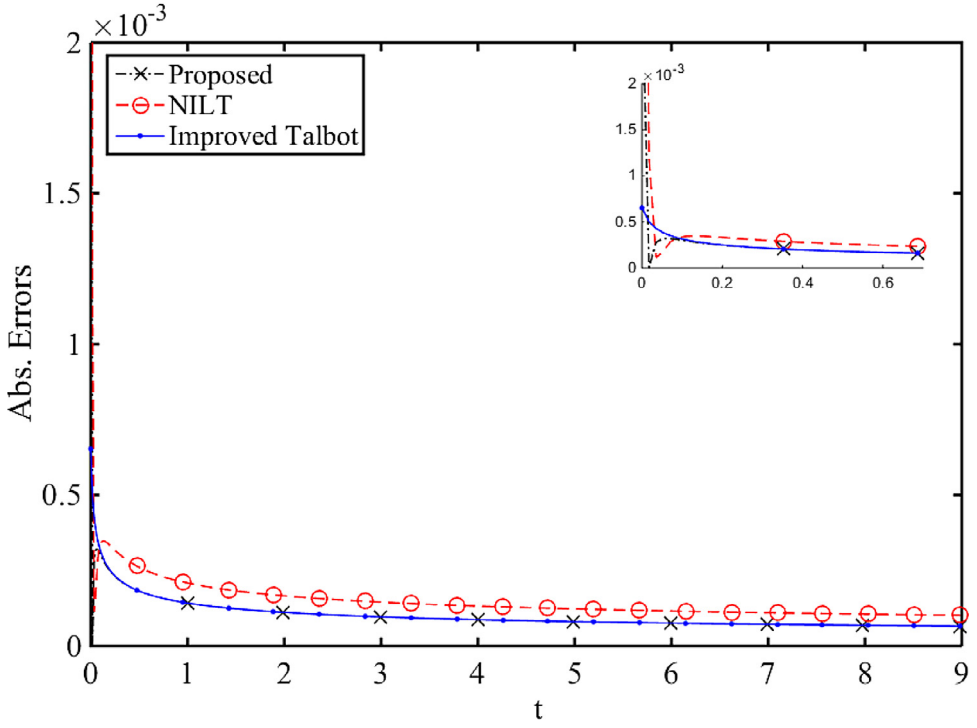

Fig. 2. Errors of the proposed, NILT, and IT methods for Example 1: small figure is the zoom to the time interval [0,0.7] s.

NILT method is based on the application of a fast Fourier transformation with a quotient difference algorithm. In the proposed method, the number of block pulse functions was chosen in such a way that the computational time of the two methods is similar. In overall, the three methods closely approximated the exact solution. The performances of the three methods were compared precisely in terms of the absolute error in Fig. 2. As expressed in Fig. 2, the proposed method showed better accuracy than the NILT method. In addition, the proposed method resulted in an algebraic equation that is more convenient to manipulate. When

the system includes several simple components, block algebra (similar to block algebra for transfer function) can be applied to obtain the system operational matrix from the operational matrix of the components [21].

Remarks. The main characteristic of the proposed technique is that different analysis problems can be reduced to a system of algebraic equations using block pulse functions, which makes it easy to handle an arbitrary input. On the other hand, the knowledge of Laplace transform of the input is required when using either the NILT or IT method for solving a linear DO differential equation. Therefore, the proposed method takes a clear advantage specially when the Laplace transform of the input is not available and/or a DO differential equation with thousand sample of random process should be solved such as Monte Carlo simulation.

3.2. Example 2

Consider the initial value problem for a distributed order relaxation equation taken from [17]:

$$\begin{aligned} &{}_C D_t^{\rho(\alpha)} y(t) + 0.1 y(t) = 0 \\ &y(0) = 1 \\ &\rho(\alpha) = 6\alpha(1-\alpha),\ 0 \le \alpha \le 1 \end{aligned} \tag{26}$$

where the Caputo fractional derivative is used in the distributed orders of the fractional derivative.

Similar to the matrix approach reported in [17], in order to use the operational matrix approach, Eq. (26) needs to be converted to an equivalent problem with a zero initial condition, as follows:

$$\begin{aligned} &y(t) = x(t) + 1 \\ &D_t^{\rho(\alpha)} x(t) + 0.1 x(t) = -0.1 \\ &x(0) = 0; \end{aligned} \tag{27}$$

where $x(t)$ is an additional variable for converting the problem to a zero initial condition. Note that the Caputo fractional derivative satisfies the relevant property of being zero when applied to a constant, which allows Eq. (26) to be converted to Eq. (27). For Eq. (27), the initial condition is zero, the subscript for the Caputo definition of fractional order derivative is dropped because the RL definition and Caputo definition are equivalent for zero initial condition.

The solution of this equivalent problem can be approximated in term of the OP of the fractional order derivative B_α as follows:

$$C_x = (A_G + 0.1I)^{-1} C_{-0.1}; \quad A_G = \sum_{i=1}^{3} B_{\alpha_i} \upsilon_i; \quad x(t) = \boldsymbol{\psi}^T(t) C_x, \tag{28}$$

where $C_{-0.1}$ is the spectral characteristics of the step function with magnitude -0.1.

Fig. 3 shows the results of the computations using the proposed method and the matrix approach method [17]. Similar to Example 1, a three point Gauss–Legendre quadrature was used to discretize the DO term in the proposed method. The values of the solution using the proposed method showed good agreement with the method reported elsewhere [17].

The exact solution of this relaxation equation in the Laplace domain was as follows [13]:

$$Y(s) = \frac{B(s)/s}{\lambda + B(s)}, \tag{29}$$

where $\lambda = 0.1$, $B(s) = \frac{\log(s)(6s+6) - 12s + 12}{\log(s)^3}$.

The solution in the time domain can be found using the NILT and IT methods by inversing Eq. (29).

Fig. 3 shows the solutions of the proposed method with the same discretization step in time 0.01, matrix approach, NILT, and IT methods. In the proposed method, the time step discretization of 0.01 for a time interval [0 10] is equivalent to the use of 1000 BPFs.

Note that the NILT and IT methods can be applied only when an explicit solution formula can be found, as in Eq. (29). It is clear the proposed and matrix approaches have advantages for a more complex DO function, $\rho(\alpha)$.

Remarks. : In the matrix approach [17], a fractional difference approximation was used to derive an approximated solution. In the proposed method, the solution was found in the form of a series of orthogonal functions.

3.3. Example 3

In this example, the equation in [18],

$$\int_{0.2}^{1.5} \Gamma(3-p) D_t^p u(t) dp = 2\frac{t^{1.8} - t^{1.5}}{\log(t)}, \tag{30}$$

with the initial condition, $u(0) = \dot{u}(0) = 0$, was solved using the proposed method. The exact solution is $u(t) = t^2$.

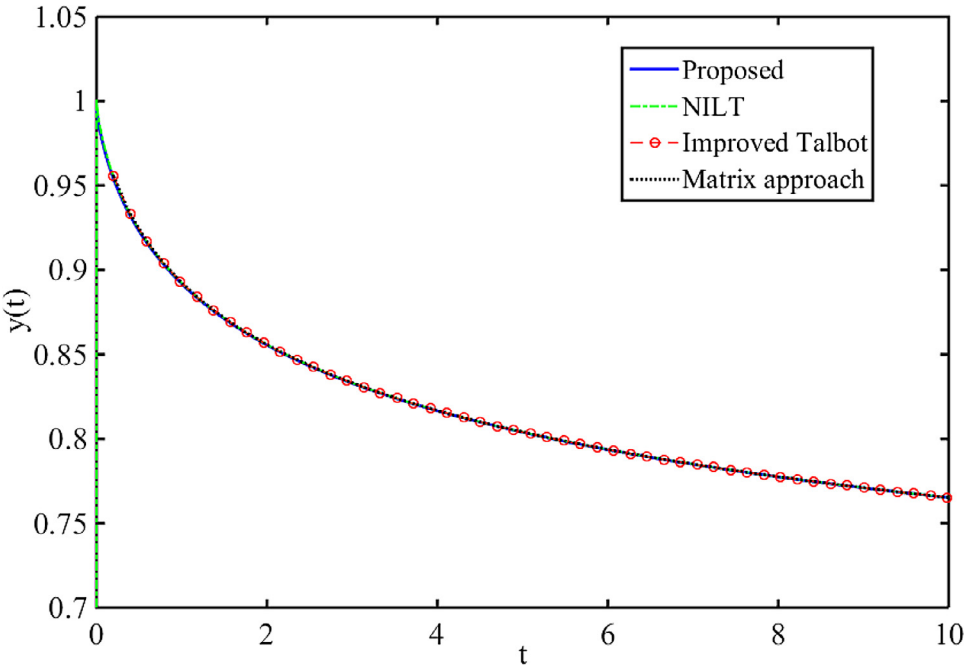

Fig. 3. Solutions of distributed order relaxation equation with $\rho(\alpha) = 6\alpha(1-\alpha)$ in Example 2.

Using the Gauss–Legendre quadrature and operational matrix integration, Eq. (30) can be approximated in terms of OP as follows:

$$\left[\Gamma(3-p_1)I\upsilon_1 + \sum_{i=2}^{Q}\Gamma(3-p_i)A_{p_1-p_i}\upsilon_i\right]C_u = A_{p_i}C_f, \tag{31}$$

where I is an identity matrix, C_f is the coefficient expansion of function, $f(t) = 2\frac{t^{1.8}-t^{1.5}}{\log(t)}$, in terms of the BPFs, and $\{p_i, \upsilon_i\}_{i=1}^{Q}$ are nodes and weights from Gauss–Legendre quadrature.

Therefore, the coefficient of expansion of solution is as follows:

$$C_u = \left[\Gamma(3-p_1)I\upsilon_1 + \sum_{i=2}^{Q}\Gamma(3-p_i)A_{p_1-p_i}\upsilon_i\right]^{-1} A_{p_1}C_f \tag{32}$$

Here, it was assumed that the quadrature $\{p_i, \upsilon_i\}_{i=1}^{Q}$ is sorted, so $p_i > p_{i+1}$.

Fig. 4 presents the solutions by the proposed method and the matrix method [17] along with the exact solution. Fig. 4 also shows the absolute error of the proposed method with 2000 BPFs. The error magnitude of the proposed method was found to better than the methods reported elsewhere [17,18] with the same step size, $h = 2/2000 = 0.001$. Fig. 5 presents the absolute errors using the proposed method with different numbers of BPFs and a five point quadrature rule. Fig. 6 shows the error of the proposed method with different numbers of points of quadrature and 2000 BPFs. Fig. 7 compares the absolute errors by the proposed method ($h = 0.001$ and 5 quadrature nodes) and the matrix approach [17] ($h = 0.001$ and 100 quadrature nodes). Note that the matrix approach [17] uses the trapezoid rule with 100 quadrature nodes for discretizing the distributed order. The computational times for obtaining the results by the proposed and matrix approach methods were 0.4 s and 1.75 s, respectively.

The solution in the Laplace domain is difficult to obtain explicitly, as in the previous examples. Note that the NILT and IT methods have difficulty in tackling this example because they commonly require the knowledge of Laplace transform of the input for solving a linear DO differential equation. The proposed method can take its unique advantage in this case.

Remarks. Example 3 is a good example for checking the accuracy of different methods because the exact analytical solution is available. Note that exact solutions are available for Examples 1 and 2, but not analytically: Example 1 requires a numerical integration of a definite integral and only the Laplace transform of the solution is available for Example 2.

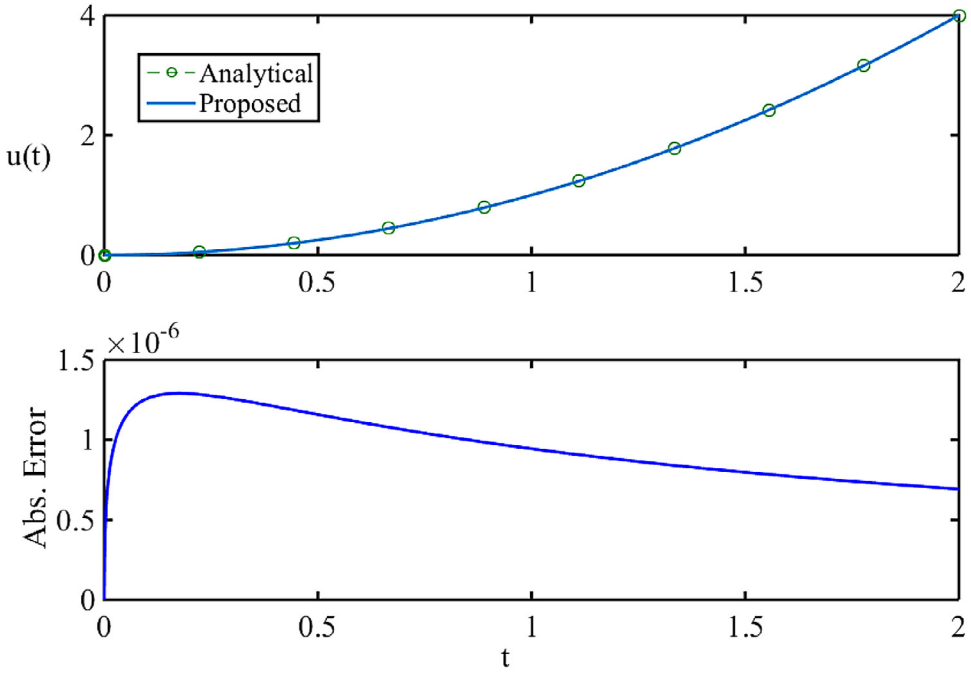

Fig. 4. Solutions and error of distributed order differential equation in Example 3.

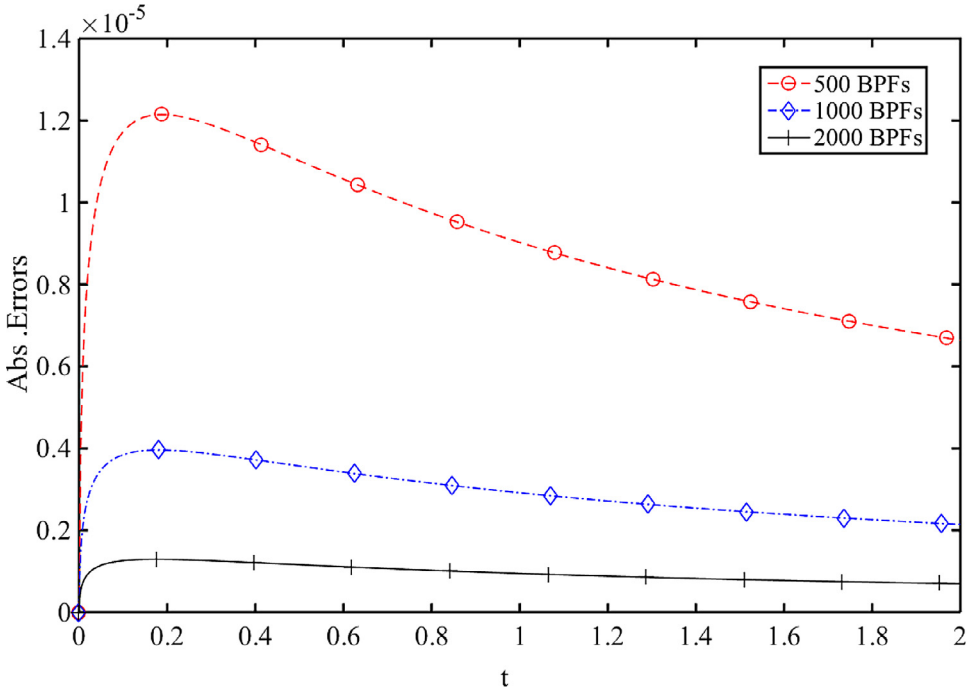

Fig. 5. Errors of the proposed method with different number of BPFs and 5 point quadrature in Example 3.

3.4. Example 4

This example considers the time domain analysis of a DO mass spring viscoelastic damper system from [17] with the transfer function of

$$G = \frac{1}{s^2 + 10\int_{0.8015}^{0.8893} s^{\alpha} d\alpha + 1} = \frac{Y(s)}{U(s)} \tag{33}$$

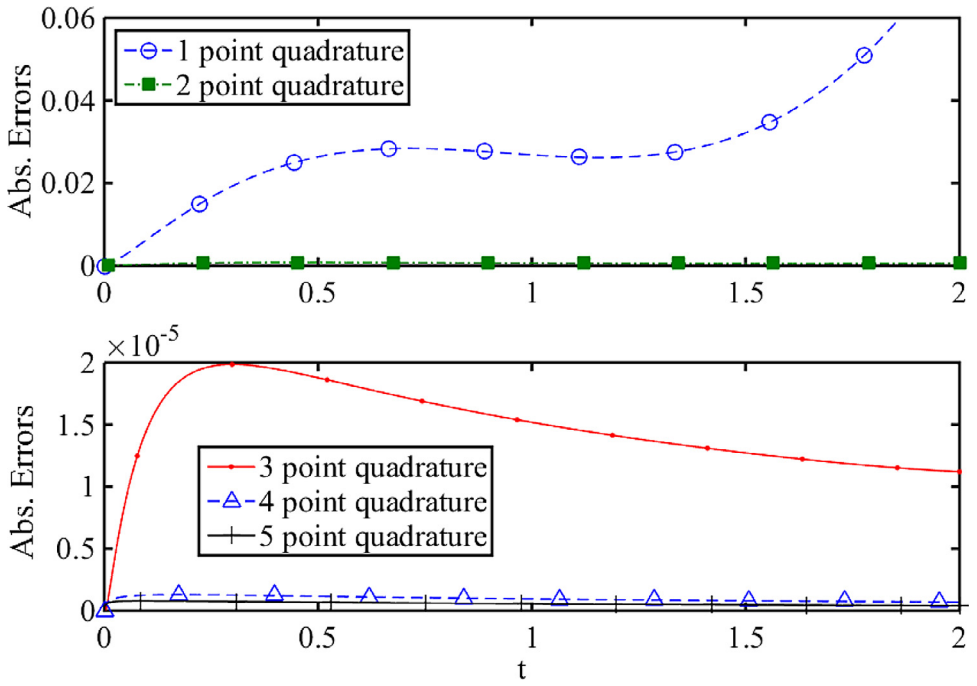

Fig. 6. Errors of the proposed method with different number quadrature points and 2000 BPFs in Example 3.

This system is a fractional mass spring viscoelastic damper system with a mass, a spring and an assembly of layers of different viscoelastic components. As the thickness of the layer approaches zero, the DO system appears. Fig. 8 shows the system outputs for a unit step by the proposed, NILT, and IT methods.

For the proposed method, the operational matrix of this system is

$$A_G = \left(I + 10 \sum_{i=1}^{Q} \upsilon_i A_{2-\alpha_i} + A_2 \right)^{-1} A_2, \tag{34}$$

where $\{\upsilon_i, \alpha_i\}_{i=1}^{Q}$ are the weights and nodes of the Legendre quadrature for the interval [0.8015, 0.8893]; I,$A_{2-\alpha_i}$ and A_2 are the identity matrix, operational matrices of fractional order integral with order of $2-\alpha_i$ and 2, respectively. As shown in Fig. 8, the three methods gave almost identical results. Note that the proposed method is more convenient because it can be used directly with a given arbitrary input. On the other hand, when using the NILT and IT methods, it is important to obtain the Laplace transform of a given arbitrary input, which may be very difficult.

3.5. Example 5

In this example, the proposed method was used to solve the equation describing the motions of the fractional distributed order oscillator [18]:

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t) + \sigma(t) = f(t) \tag{35a}$$

$$\int_0^1 a^p D_t{}^p \sigma(t) dp = \lambda \int_0^1 b^p D_t{}^p u(t) dp, \tag{35b}$$

with zero initial conditions, $u(0) = \dot{u}(0) = 0$; where u and σ are the displacement and dissipation force, respectively; λ, a and b are constant parameters; ω is the Eigen frequency of an underdamped system; $f(t)$ is the external forcing function.

For $a = b$, Eq. (35a) becomes an integer order differential equation:

$$\frac{d^2u(t)}{dt^2} + (\lambda + \omega^2) u(t) = f(t). \tag{36}$$

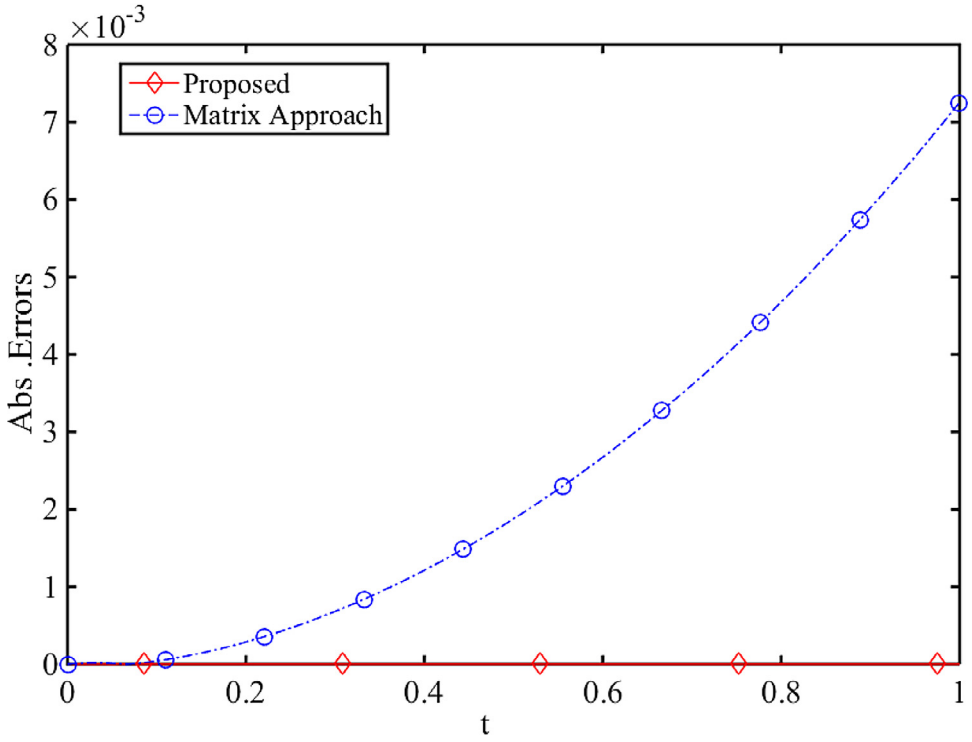

Fig. 7. Errors of the proposed ($h = 0.001$ and 5 quadrature nodes) and matrix approach ($h = 0.001$ and 100 quadrature nodes) methods for Example 3.

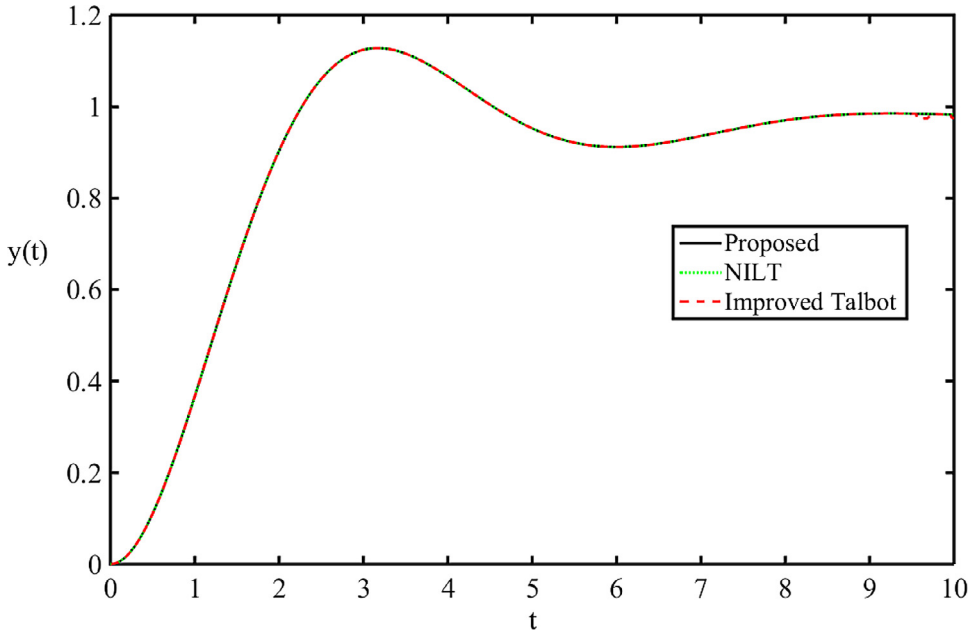

Fig. 8. Outputs by the proposed with 1024 BPFs, NILT, IT methods for Example 4.

Using the operational matrix, Eq. (35) can be rewritten as:

$$\left[\upsilon_1 a^{p_1} I + \sum_{i=2}^{Q} \upsilon_i a^{p_i} A_{p_1-p_2}\right] C_\sigma = \left[\upsilon_1 b^{p_1} I + \sum_{i=2}^{Q} \upsilon_i b^{p_i} A_{p_1-p_2}\right] C_u, \tag{37a}$$

$$(I + A_2) C_u + A_2 C_\sigma = C_f. \tag{37b}$$

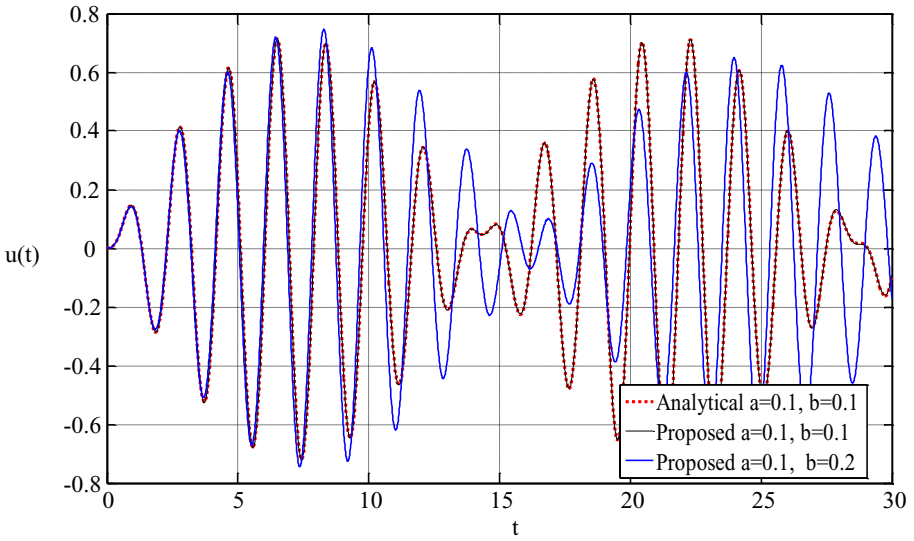

Fig. 9. Outputs by the proposed method with 3000 BPFs for Example 5.

The quadrature $\{p_i, \upsilon_i\}_{i=1}^{Q}$ was assumed to be sorted so that $p_i > p_{i+1}$.

After some algebraic manipulations, the relationship between the spectral characteristics (coefficient expansion) of the external force $f(t)$ and the displacement $u(t)$ can be expressed as:

$$C_u = \left[I + A_2 + A_2 \left[\upsilon_1 a^{p_1} I + \sum_{i=2}^{Q} \upsilon_i a^{p_i} A_{p_1 - p_2} \right]^{-1} \left[\upsilon_1 b^{p_1} I + \sum_{i=2}^{Q} \upsilon_i b^{p_i} A_{p_1 - p_2} \right] \right]^{-1} C_f. \tag{38}$$

Therefore, from this relationship, when the input $f(t)$ is given, one can easily obtain the spectral characteristics for the displacement, and hence the displacement itself.

Fig. 9 shows the displacements of the system using the proposed method with 3000 BPFs and 5 quadrature points for various a and b values when the input and the parameters are $f(t) = \sin(1.2\omega t)$, $\omega = 3$, and $\lambda = 1$.

The analytical solution from Eq. (36) is also shown in this figure for the case of $a = b$. Note that in the case of $a = b$, the displacement is calculated as expressed in Eq. (38) (as the solution of a DO equation) using the proposed method. For the case of $a = b$, the proposed method gave an identical solution to the analytical ones. For the other case, the proposed method produced similar results to the results reported elsewhere [18].

4. Conclusions

A novel operational matrix method was proposed to analyze the linear DO system. In the proposed method, the DO equation was approximated as a multi-term fractional order equation. The main advantage of the proposed method is that different analysis problems can be reduced to a system of algebraic equations using block pulse functions, which makes it easy to handle an arbitrary input. The proposed method does not require the knowledge of Laplace transform of the input for solving a linear DO differential equation, it will be very useful specially when the Laplace transform of the input is not available and/or a DO differential equation of thousand sample of random process should be solved such as Monte Carlo simulation. The algorithm is also easy to implement. Several examples were considered to illustrate the accuracy and convergence of the proposed method. The proposed method was found to be an efficient tool for studying the linear DO system. Future studies will extend this method to nonlinear cases.

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