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Robust PI controller design for integrator plus dead-time process with stochastic uncertainties using operational matrix

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Abstract–To increase the precision and reliability of process control, random uncertainty factors affecting the control system must be accounted for. We propose a novel approach based on the operational matrix technique for robust PI controller design for dead-time processes with stochastic uncertainties in both process parameters and inputs. The use of the operational matrix drastically reduces computational time in controller design and statistical analysis with a desired accuracy over that of the traditional Monte-Carlo method. Examples with deterministic and stochastic inputs were considered to demonstrate the validity of the proposed method. The computational effectiveness of the proposed method was shown by comparison with the Monte-Carlo method. The proposed approach was mainly derived based on the integrator plus dead-time process, but can be easily extended to other types of more complex stochastic systems with dead-time, such as a first-order plus dead-time or a second-order plus dead-time system.

Key words: Dead-time Process, Operational Matrix, Robust PI Controller Design, Statistical Analysis, Stochastic Process, Walsh Functions

INTRODUCTION

In most engineering applications, one aims to solve physical problems by converting them into a deterministic mathematical model. This is a rough approximation of reality, as many physical input parameters describing the problem are fixed through this conversion. In reality, however, these parameters exhibit randomness with definite influences over behavior of the solution. Accordingly, it becomes increasingly important to quantify uncertainties associated with model predictions.

A representative and most popular traditional statistical approach for uncertainty quantification is the Monte-Carlo (MC) method [1-3]. With the brute force MC implementation, one first generates an ensemble of random realizations with each parameter drawn from its uncertainty distribution. Deterministic solvers are then applied to each member to obtain an ensemble of results. The ensemble of results is then post-processed to obtain the relevant statistical properties of the results, such as mean and standard deviation, as well as the probability density function. Since estimation of the mean converges with the inverse square root of the number of runs, the MC approach is often computationally too expensive [4].

Polynomial chaos [4,5] is another frequently used technique for quantifying uncertainties. However, the random inputs of many systems involve random processes approximated by truncated Karhunen-Loeve (KL) expansions, and the input's dimensionality depends on the correlation lengths of these processes. For input processes with low correlation lengths, the number of dimensions required for accurate representation can be extremely large [4].

Operational matrix is one technique that can be used to solve problems such as calculus of variation, differential equation, optimal control etc. [5-9]. In [5], an operational matrix with the Neumann series was used for quantification uncertainty in system models without time delay. This technique, also known as the spectral method, is based on a finite-dimensional approximation of the mathematical model of a system using orthogonal expansions. The main characteristic of this technique is reduction of a system of differential equations into algebraic equations, thus greatly simplifying the problem. This method gives algebraic relationships between the first- and second-order stochastic moments of a system's input and output, hence bypassing the KL expansions that can require large dimensions for accurate results.

Many processes can be described as an integrator plus dead-time process [10], such as the linearized Nomad 200 in [11]. We used the operational matrix for robust PID controller design to account for the influence of the random changes in the parameter of the integrator plus dead-time control system on the statistical characteristics of its output when the disturbance is either deterministic or stochastic. Other types of more complex stochastic systems with dead-time, such as a first order plus dead-time process, were examined.

MOMENT OF ARBITRARY ORDER FOR RANDOM VARIABLES

1. Moment of Normal Random Variables

For normal random variables, the moments of arbitrary order can be expressed through the cumulants [12]. The relation between the moment α_r and the cumulant χ_r is given in reference [13] as follows:

$$\sum_{r=0}^{\infty}\alpha_r\frac{\lambda^r}{r!}=\prod_{r=1}^{\infty}e^{\left(\frac{\chi_r\lambda^r}{r!}\right)} \tag{1}$$

where

$$e^{\left(\frac{\chi_r\lambda^r}{r!}\right)}=\sum_{k=0}^{\infty}\left(\frac{\chi_r\lambda^r}{r!}\right)^k\frac{1}{k!} \tag{2}$$

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Expansion of the exponential in the right hand side of Eq. (1) into series Eq. (2) gives:

$$\prod_{r=1}^{\infty} e^{\left(\frac{\chi_r\lambda^r}{r!}\right)} \prod_{r=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\chi_r\lambda^r}{r!}\right)^k\frac{1}{k!}=1+\frac{\chi_1}{1!}\lambda+\frac{\chi_1^2+\chi_2}{2!}\lambda^2 +\frac{\chi_1^3+3\chi_1\chi_2+\chi_3}{3!}\lambda^3+\frac{\chi_1^4+4\chi_1\chi_3+3\chi_2^2+6\chi_2\chi_1^2+\chi_4}{4!}\lambda^4+\dots \quad (3)$$

Rearrangement of the left hand side of Eq. (1) into a polynomial of λ gives:

$$\sum_{r=0}^{\infty}\alpha_r\frac{\lambda^r}{r!}=\alpha_0+\alpha_1\lambda+\frac{\alpha_2}{2!}\lambda^3+\frac{\alpha_3}{3!}\lambda^3+\frac{\alpha_4}{4!}+\dots \quad (4)$$

Comparing terms Eq. (3) and Eq. (4) with the same order of λ, the relations between the moments and cumulants are obtained.

For a normal random variable, the first order cumulant is expectation and the second-order cumulant is equal to the second central moment, the variance. Furthermore, for a normal random variable, all the cumulants higher than the second order are zero.

Thus, the moments of arbitrary orders for the Gaussian random variable can be expressed in terms of the cumulants:

$$\begin{aligned}&\alpha_0=1\\&\alpha_1=m\\&\alpha_2=D+m^2\\&\alpha_3=3mD+m^3\\&\alpha_4=3D^2+6Dm^2+m^4\\&\dots\end{aligned} \quad (5)$$

where m is the mathematical expectation and D is the variance.

2. Moment of Uniform Random Variables

For certain types of distributions, the moment expansions of the arbitrary order are more straightforward. In particular, for the uniform distributed random variable x on the interval [a, b] with the probability density

$$f_x(x)=\frac{1}{b-a} \quad (6)$$

The moment of arbitrary order random variable x is:

$$\alpha_k^x=\frac{1}{b-a}\int_a^b x^k dx=\frac{b^{k+1}-a^{k+1}}{(k+1)(b-a)} \quad (7)$$

In the spectral models of stochastic systems, the random factors a_i are transformed into the form:

$$a_i=\bar{a}_i+a_i^r \quad (8)$$

where $\bar{a}$=M[a_i] is the mean of a_i and a_i^r is the random central component.

Thus, if a random variable a_i is in the interval [g_i^L, g_i^R], the central random variable a_i^r will be in the range:

$$V_i=[g_i^L-M[a_i],\ g_i^R-M[a_i]] \quad (9)$$

The term V_i can also be defined as:

$$V_i=[-r_i\ \ r_i] \quad (10)$$

where r_i is half the length of the interval; g_i^L and g_i^R are the left and the right borders, respectively.

Then, the k[th] order moment of a_i^r will be determined on the basis of Eq. (7) as follows:

$$\begin{aligned}&\alpha_1^{a_i^r}=\frac{1}{2r_i}\int_{-r_i}^{r_i}x\,dx=\frac{r_i^2-r_i^2}{2r_i}=0\\&\alpha_2^{a_i^r}=\frac{1}{2r_i}\int_{-r_i}^{r_i}x^2dx=\frac{r_i^3+r_i^3}{3*2r_i}=\frac{r_i^2}{3}\\&\vdots\end{aligned} \quad (11)$$

Note that all odd moments are zero since the segment is symmetrical about 0. Thus, the general formula, which determines the k[th] moments of central uniformly distributed random variables, has the form:

$$\alpha_k^{a_i^r}=\begin{cases}0 & \text{k-odd}\\ \dfrac{r_i^k}{(k+1)} & \text{k-even}\end{cases} \quad (12)$$

STATISTICAL ANALYSIS FOR INTEGRATOR PLUS DEAD-TIME PROCESS USING OPERATIONAL MATRIX

1. Stochastic Operational Matrix of Integrator Plus Dead-time Process

Consider an integrator plus dead-time (IPDT) system with random gain in Fig. 1:

$$\frac{K}{s}e^{(-Ls)} \quad (13)$$

Now, introduce the output, setpoint, and disturbance signals in the form of a Fourier series

$$\begin{cases}D(t)\cong D_l(t)=\sum_{k=1}^{l}c_k^D\varphi_k(t)=\Phi^T(t)C^d\\ R(t)\cong R_l(t)=\sum_{k=1}^{l}c_k^R\varphi_k(t)=\Phi^T(t)C^r\\ Y(t)\cong Y_l(t)=\sum_{k=1}^{l}c_k^Y\varphi_k(t)=\Phi^T(t)C^Y\end{cases} \quad (14)$$

where $\Phi^T(t)=[\varphi_1, \dots, \varphi_l]$ is a set of orthonormal basis (superscript T denotes operation transpose).

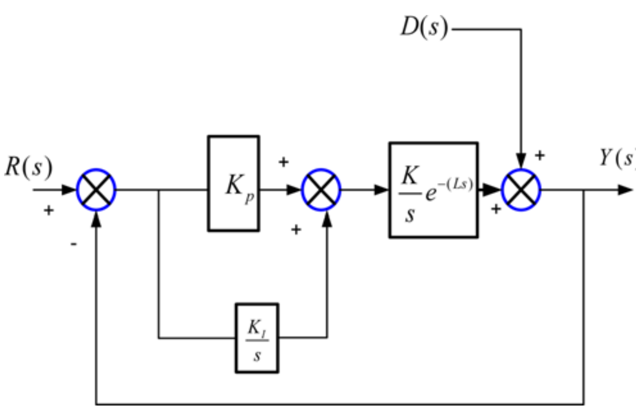

Fig. 1. Control system block diagram of the IPDT system.

C^d is the vector coefficient of the Fourier expansion or spectral characteristic of the disturbance.

C^r is the vector coefficient of the Fourier expansion or spectral characteristic of the setpoint.

C^y is the vector coefficient or spectral characteristic of the output signal.

Using the Pade approximation for the delay term, an operational matrix for the delay term is obtained as A_{pa}. Using block matrix algebra for the operational matrix [3], the operational matrix for the open loop system is thus achieved:

$$A_l = A_c A_{pa} A_i K \quad (15)$$

where A_c is the operational matrix of the proportional-integral (PI) controller :

$$A_c = K_p I + K_i A_i \quad (16)$$

I is the unity matrix and A_i the operational matrix of the integrator.

Let us denote

$$A_l = A_c A_{pa} A_i \quad (17)$$

The relation between the spectral characteristics of the disturbance and output is given by:

$$C^y = (I + A_1 K)^{-1} C^d \quad (18)$$

Analogous to the regulatory problem, the relation between the spectral characteristics of the setpoint and output is given by:

$$C^y = K A_1 (I + K A_1)^{-1} C^r \quad (19)$$

Denoting $A_0 = (I + A_1 \overline{K})^{-1}$ and using a Neumann series gives:

$$(I + A_1 K)^{-1} = (I + A_1 \overline{K} + K_r A_1)^{-1} = A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} (K_r A_1 A_0)^{\nu} \quad (20)$$

For the servo problem, Eq. (19) can thus be rewritten as:

$$C^y = \sum_{\nu=0}^{\infty} (\overline{K} + K_r)(A_1) A_0 (-1)^{\nu} (K_r A_1 A_0)^{\nu} C^r = A^{ser} C^r \quad (21.a)$$

For the regulatory problem, Eq. (18) can be rewritten as:

$$C^y = \sum_{\nu=0}^{\infty} A_0 (-1)^{\nu} (K_r A_1 A_0)^{\nu} C^b = A^{reg} C^d \quad (21.b)$$

where $\overline{K} = M[K]$ is the mean of random gain and K_r the random central component of uniform variable K. From now on, the superscript denoting the stochastic operator of system for servo or regulator case will be dropped for convenience of notations. The matrix A in Eq. (21) is called by the stochastic operational matrix of the system.

2. Statistical Analysis Using Operational Matrix Technique

Consider the output and input signals in the form of Fourier series expansions:

$$\begin{cases} Y(t) \cong Y_l(t) = \sum_{k=1}^{l} c_k^Y \varphi_k(t) = \Phi^T(t) C^Y \\ X(t) \cong X_l(t) = \sum_{k=1}^{l} c_k^X \varphi_k(t) = \Phi^T(t) C^X \end{cases} \quad (22)$$

and the spectral characteristics of the output and input are linked by: $C^Y = AC^X$.

Thus, an equation for the output of stochastic systems is:

$$Y_l(t) = \Phi^T(t) C^Y = \Phi^T(t) A C^X \quad (23)$$

where A is the stochastic matrix operator defined by Eq. (21.a) or Eq. (21.b).

The mean of Eq. (23) can be calculated as:

$$m_Y^l(t) = M[Y_l(t)] = M[\Phi^T(t) C^{m_Y}] = \Phi^T(t) M[C^Y] = \Phi^T(t) M[AC^X] \quad (24)$$

From the statistical independence of matrix A and column vector of coefficient expansion of input C^X

$$m_Y^l(t) = \Phi^T(t) C^{m_Y} = \Phi^T(t) M[A] M[C^X] = \Phi^T(t) \overline{A} C^{m_X} \quad (25)$$

where $M[A] = \overline{A}$.

Thus, the spectral characteristic of the mathematical expectations of the output and input signals of the stochastic system are related by:

$$C^{m_Y} = M[A] C^{m_X} = \overline{A} C^{m_X} \quad (26)$$

Accordingly, the spectral characteristic of the mathematical expectation of the output signal is defined as a linear transformation of the spectral characteristic of the mathematical expectation input.

Deterministic matrix operator $\overline{A}$ is the expectation of random stochastic matrix operator A. To determine the deterministic matrix operator $\overline{A}$ for the regulatory problem, the expectation of stochastic matrix operator A is calculated:

$$\overline{A} = M[A] = M\left\{ A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} (K_r A_1 A_0)^{\nu} \right\} = A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} M\{(K_r)^{\nu}\} (A_1 A_0)^{\nu} \quad (27)$$

or for the servo problem:

$$\overline{A} = M[A] = M\left\{ \sum_{\nu=0}^{\infty} (\overline{K} + K_r) A_1 A_0 (-1)^{\nu} (K_r A_1 A_0)^{\nu} \right\} = \sum_{\nu=0}^{\infty} A_1 A_0 (-1)^{\nu} [\overline{K} M\{(K_r)^{\nu}\} + M\{(K_r)^{\nu+1}\}] (A_1 A_0)^{\nu} \quad (28)$$

The stochastic moments of arbitrary-order $M\{(K_r\}^{\nu}\}$ in Eqs. (27) and (28) are calculated for each ν using the method mentioned in section II.

Eq. (25) shows how the random parameters given in $\overline{A}$ affect the expectation of the output. The mathematical expectation of the output system, as determined by Eqs. (24), (25), (26), (27), and (28), can be calculated with a desired accuracy that depends on the expectation of stochastic matrix operator, which in turn is determined by ν, the number of terms for approximation in Eqs. (27) and (28).

The correlation function of the output stochastic system and its second central moment are next defined. By introducing the signal of the system in the form of Eq. (22), the equation to define the second moment of output can be written as:

$$\theta_{YY}^l(t_1, t_2) = M[Y_l(t_1) Y_l(t_2)] = M[\Phi^T(t_1) C^Y (C^Y)^T \Phi(t_2)] = \Phi^T(t_1) M[C^Y (C^Y)^T] \Phi(t_2) = \Phi^T(t_1) M[AC^X (C^X)^T A^T] \Phi(t_2) \quad (29)$$

Thus, Eq. (29) can take the form of:

$$\theta_{YY}^l(t_1, t_2) = \Phi^T(t_1) M[AC^{\theta_{XX}} A^T] \Phi(t_2) \quad (30)$$

where $C^{\theta_{XX}}$ is the square matrix of the spectral characteristic of the second moment of the input of the system, which is determined using Eq. (22):

$$\theta^l_{XX}(t_1, t_2)=M[X_l(t_1)X_l(t_2)]=\Phi^T(t_1)M[C^X(C^X)^T]\Phi^T(t_2)$$
$$=\Phi^T(t_1)M[C^{\theta_{XX}}]\Phi^T(t_2) \quad (31)$$

The covariance function or the second central moment of the output system is defined as:

$$\kappa^l_{YY}(t_1, t_2)=M\{[Y_l(t_1)-m^l_Y(t_1)][Y_l(t_2)-m^l_Y(t_2)]\}$$
$$=M[Y_l(t_1)Y_l(t_2)]-m^l_Y(t_1)m^l_Y(t_2)=\theta^l_{YY}(t_1, t_2)-m^l_Y(t_1)m^l_Y(t_2) \quad (32)$$

where the first order moment $m^l_Y(t_1)$ is determined by Eq. (25) and the second moment by Eq. (32).

The covariance function of the input signal is similarly associated with the second-order moment:

$$\kappa^l_{XX}(t_1, t_2)=\theta^l_{XX}(t_1, t_2)-m^l_X(t_1)m^l_X(t_2) \quad (33)$$

where $m^l_X(t_2)$ is the mathematical expectation of the input signal.

Furthermore, the covariance function of the input signal can be expanded in terms of the orthonormal basis:

$$\kappa^l_{XX}(t_1, t_2)=\Phi^T(t_1)C^{\kappa_{XX}}\Phi(t_2)=\Phi^T(t_1)C^{\theta_{XX}}\Phi(t_2)-\Phi^T(t_1)C^{m_X}(C^{m_X})^T\Phi(t_2) \quad (34)$$

Thus, the spectral characteristic of the moments of input signal are related by:

$$C^{\kappa_{XX}}=C^{\theta_{XX}}-C^{m_X}(C^{m_X})^T \quad (35)$$

Eq. (29) can thus be rewritten as follows:

$$\theta^l_{YY}(t_1, t_2)=\Phi^T(t_1)M\{A[C^{\kappa_{XX}}+C^{m_X}(C^{m_X})^T]A^T\}\Phi(t_2) \quad (36)$$

Taking into account Eq. (32) and Eq. (36), the following equation is obtained for the covariance function of the output stochastic system:

$$\kappa^l_{YY}(t_1, t_2)=\Phi^T(t_1)C^{R_{YY}}\Phi(t_2)=\Phi^T(t_1)M\{A[C^{\kappa_{XX}}+C^{m_X}(C^{m_X})^T]A^T\}\Phi(t_2)$$
$$-\Phi^T(t_1)C^{m_Y}(C^{m_Y})^T\Phi(t_2) \quad (37)$$

or

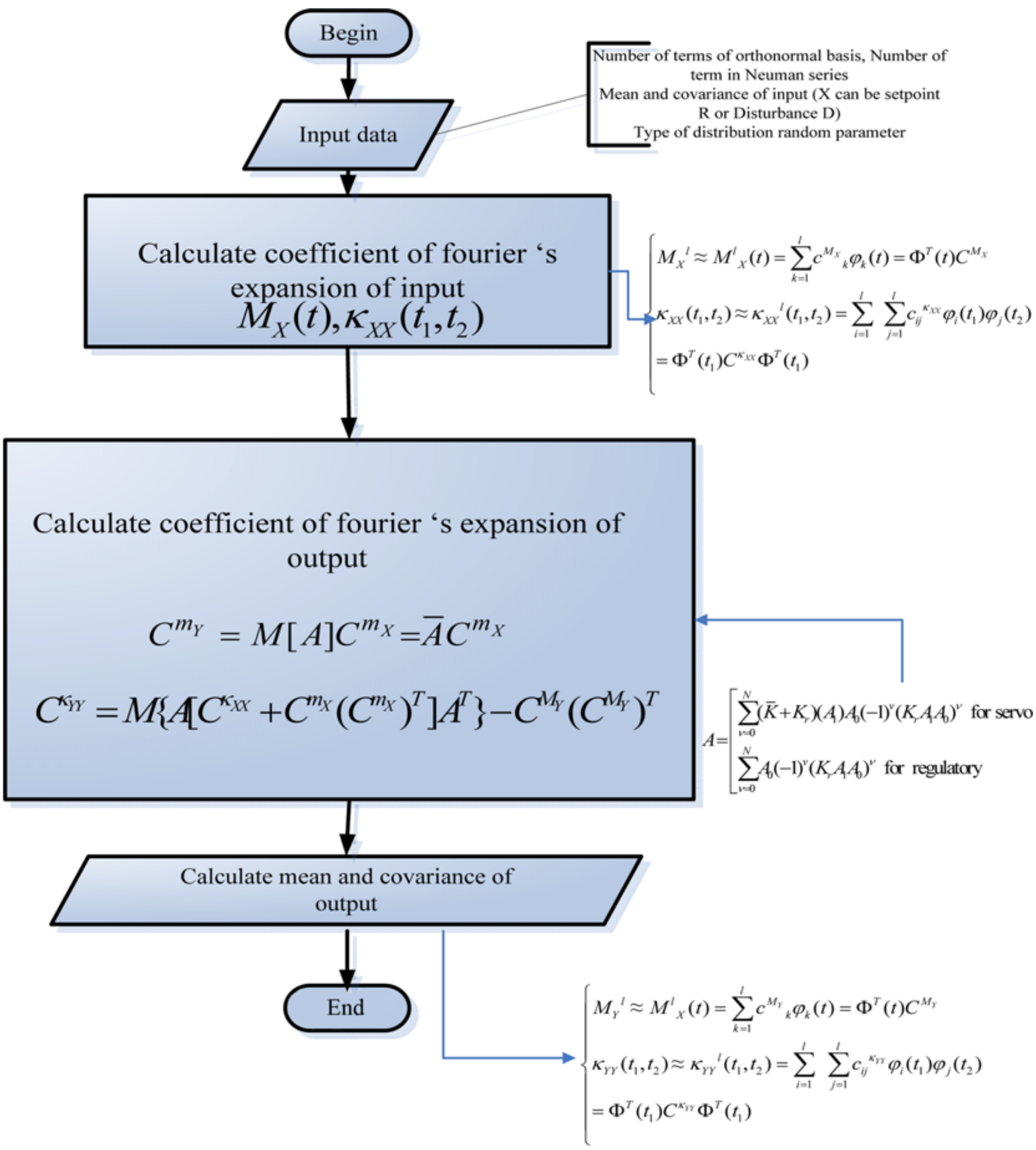

Fig. 2. Proposed numerical algorithm for calculate the mean and the variance of dead-time system output with random input and parameter.

$$C^{\kappa_{YY}}=C^{\theta_{YY}}-C^{m_Y}(C^{m_Y})^T=M\{A[C^{\kappa_{XX}}C^{m_X}(C^{m_X})^T]A^T\}-C^{m_Y}(C^{m_Y})^T \quad (38)$$

where A is the stochastic operational matrix defined by Eq. (21).

It is assumed that all eigenvalues of the random matrix variable $K_r A_1 A_0 = A_r$ are inside the unit circle or $|\lambda_j|_{A_r}<1$ for the convergences of the series by Eq. (20) [13]. If this condition is not satisfied, one can apply the precondition technique in [14] or stochastic collocation approach [4,5,15] for estimation moment of random matrix in Eqs. (26) and (38).

Eq. (37) gives the relation between the spectral characteristics of the covariance function of the output and input signal, and the mathematical expectations of the output and input signal. To summarize, the proposed algorithm is presented briefly in Fig. 2.

Remark: Eqs. (25-28), (37) and (38) give the semi-analytical relationship between spectral characteristics of the first- and second-order moments for system random input and output. Therefore, the computational demand for repetitive simulations of the system as in the traditional MC method is unnecessary.

ILLUSTRATIVE EXAMPLES

1. Examples 1.a-1.h: Integrator Plus Dead Time with Non-white Noise Forcing

Consider a robust controller design problem for an IPDT process, K/s $e^{(-s)}$, where is a random variable. Several simulation examples with different types of random gain and input were used to validate the correctness of the method. Third-order Pade approximation was used in all cases. The stochastic signals used in the simulation are all Gaussian random process. All simulation parameters are described in Table 1. Statistical characteristics of closed loop systems for both regulatory and servo problems are shown in Figs. 3 and 4, which delineate the consistency between the operational matrix and MC method. A computer, with AMD Phenom II X3 2.81 GHz 2GB RAM, was used for the test with simulation times shown in Table 2. From Table 2 and Figs. 3 and 4, it is clear the operational matrix method can reduce computing time drastically, while it gives the statistical characteristics quite identical to the Monte-Carlo method. Some guidelines for the number of samples for the Monte-Carlo method can be found in [1] and [3]. Calculations were made using the library SML [3].

Fig. 4 also indicates the differences between the variance of the output for the different types of random gain and random input. As seen in the figure, the variance is increased when the disturbance is

Table 2. Computation time

Example	Computation time (sec)	
	Monte-carlo	Operational matrix
1.a	46	0.3
1.b	57	0.3
1.c	900	0.3
1.d	2880	0.3
1.e	169	3
1.f	302	3
1.g	1740	3
1.h	2160	3
3	2040	3

Table 1. Simulation parameters

Case	Simulation parameters		
	Monte-carlo		Operational matrix
	N.o.S K	N.o.S D	N.o.W
1.a) $K_p=1$; $K_i=0$; $K\in U[0.5, 1.5$; $R=0$; $D=1(t)$	2000	-	128
1.b) $K_p=1$; $K_i=0$; $K\in N(1, 0.1)$; $R=0$; $D=1(t)$	2000	-	128
1.c) $K_p=1$; $K_i=0$; $K\in U[0.5, 1.5]$; $R(t)=0$; $M_D(t)=1(t)$; $\kappa_{DD}(t_1, t_2)=0.01e^{-5\|t_1-t_2\|}$	70	2000	128
1.d) $K_p=1$; $K_i=0$; $K\in N(1, 0.01)$; $R(t)=0$; $M_D(t)=1(t)$; $\kappa_{DD}(t_1, t_2)=0.01e^{-5\|t_1-t_2\|}$	200	2000	128
1.e) $K_p=1$; $K_i=0$; $K\in U[0.5, 1.5]$ $R(t)=1$; $D(t)=0$	2000	-	128
1.f) $K_p=1$; $K_i=0$; $K\in N(1, 0.1)$ $R(t)=1$; $D(t)=0$	3000	-	128
1.g) $K_p=0.609$; $K_i=0.002$; $K\in U[0.5, 1.5]$; $D(t)=0$; $M_R(t)=1(t)$; $\kappa_{RR}(t_1, t_2)=0.01e^{-5\|t_1-t_2\|}$	200	2000	128
1.h) $K_p=0.609$; $K_i=0.002$; $K\in N(1, 0.1)$; $D(t)=0$; $M_R(t)=1(t)$; $\kappa_{RR}(t_1, t_2)=0.01e^{-5\|t_1-t_2\|}$	200	2000	128
3) $K_p=1$; $K_i=0$; $K\in U[0.5, 1]$; $R(t)=0$; $M_D(t)=1(t)$; $\kappa_{DD}(t_1, t_2)=0.01e^{-5\|t_1-t_2\|}$	200	2000	128

N.o.S K, N.o.S D, and N.o.W denote the number of samples for K, the number of samples for D(t) or R(t), and the number of Walsh functions, respectively

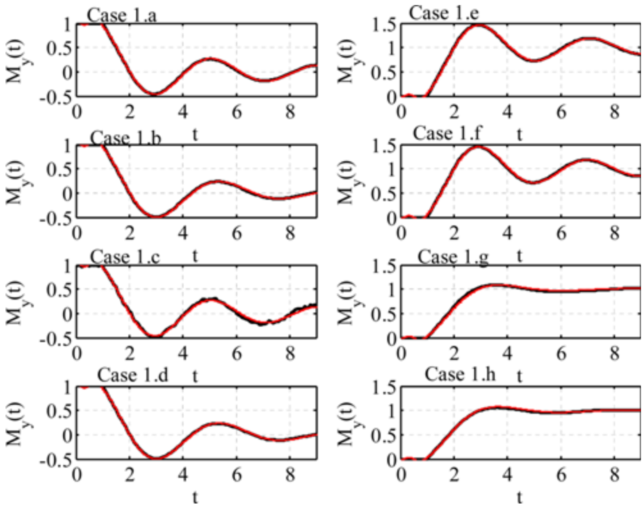

Fig. 3. Means of system output for Example 1.a-1.h.

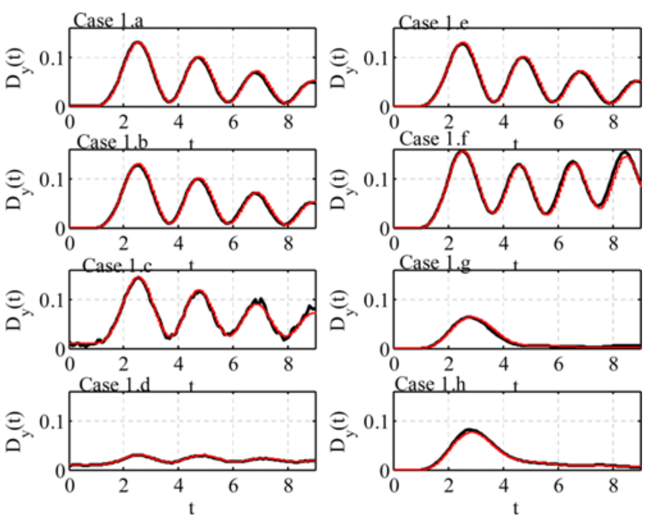

Fig. 4. Variances of system output for Example 1.a-1.h.

stochastic. By using the operational matrix method, one can quickly derive a dynamic bound for an integrator plus dead-time system with random parameters and random input, which can be in turn used for the design of a controller for a robust transient response. In this paper, in case 1.g and 1.h, PI controllers were designed by minimization of $\int M[e(t)]dt$ to mitigate the effect of both parameter and additive input uncertainties, whereas most papers for PI controller design have restricted to only parametric uncertainties. The improved action by the optimally tuned PI controller on stochastic gain and set-point variation can be seen from the variance of output in Fig. 4.

2. Example 2: IPDT with White Noise Forcing

In this example, we will study the effect of order of Pade approximation in the proposed method.

Consider an IPDT process $1/s\ e^{(-Ls)}$ under the feedback configuration (servo problem) as in Fig. 1 with a unit proportional controller. Reference input is an ideal white Gaussian noise with zero mean and covariance $\kappa_{RR}(t_1, t_2)=\delta(t_1-t_2)$. The exact (analytical) steady state variance of the system output is given by [16,17].

$$D_{y_{ss}} = \frac{1}{2}\frac{\cos(L)}{1-\sin(L)} \tag{39}$$

Since this system does not have random gain, the matrix operator of the open-loop system is deterministic and given by

$$A_l=A_iA_p \tag{40}$$

where A_p is the operational matrix of delay part using Pade approximation. For example, if a third-order approximation is used, the matrix A_p is given by:

$$A_p=(IL^3+12A_iL^2+60A_i^2L+120A_i^3)^{-1}(-IL^3+12A_iL^2-60A_i^2L+120A_i^3) \tag{41}$$

where I and A_i are the identity matrix and operational matrix of integration, respectively. Details about the operational matrix of integration with different orthogonal functions can be found in [5,6] and references therein.

The closed loop operator is then given by:

$$A=(I+A_l)^{-1}A_l \tag{42}$$

Fig. 5 compares the analytical variances of system outputs under random white noise forcing and those by the proposed method with different order of Pade approximation versus the time delay L. The plots show that the low order Pade approximations provide a satisfactory approximation unless the time delay is somewhat significant.

3. Example 3: First-order Plus Dead-time System (FOPDT)

The algorithm proposed in section 3 can be easily extended for more general dead-time processes. This section demonstrates how the proposed method can be extended for the analysis of FOPDT systems under random gain and random output disturbance. The mean and covariance of random output disturbance are given in Table 1.

Consider an FOPDT system:

$$G=\frac{K}{s+1}e^{(-s)} \tag{43}$$

where K is a random variable with parameters given in Table 1. This FOPDT system is in the closed-loop feedback configuration with a PI controller.

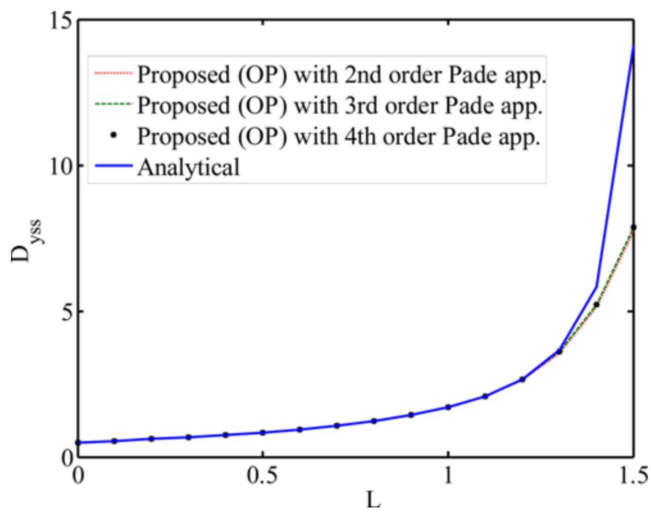

Fig. 5. Steady state variances of system as a function of time delay for Example 2.

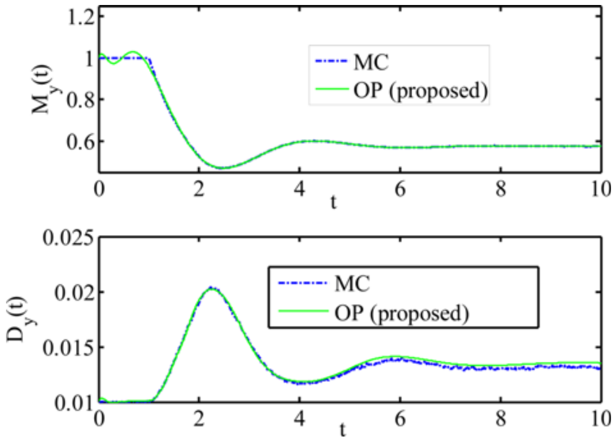

Fig. 6. Means and variances of system output for Example 3.

The operational matrix for a PI controller is:

$$A_c=K_pI+K_iA_i \tag{44}$$

The open-loop operational matrix is given by:

$$A_l=A_cA_p(I+A_i)^{-1}A_iK \tag{45}$$

where A_p is an operational matrix for time delay using Pade approximation as in the above example.

By splitting the random gain into $K=\overline{K}+K_r$ as in section 2, the stochastic operational matrix for the closed-loop system is:

$$A=(I+A_lK)^{-1}=A_0\sum_{\nu=0}^{N}(-1)^{\nu}(K_rA_lA_0)^{\nu} \tag{46}$$

where $A_0=(I+A_l\overline{K})^{-1}$.

By substituting the stochastic operator A in Eq. (46) into Eqs. (25-28), (37) and (38), the semi-analytical relationship between the spectral characteristics of the first- and second-order moments can be obtained for random disturbance and output.

The mean and variance of the system output by the proposed and MC methods are shown in Fig. 6. Computational time for obtaining these statistical characteristics is given in Table 2. From Fig. 6 and Table 2, it is clear that the proposed method can reduce computing time drastically while it gives statistical characteristics quite identical to the MC method.

CONCLUSIONS

Robust PI controller design for an SISO plus dead-time system with random gain and random input was proposed using the operational matrix technique. The use of operational matrix explicitly gives a semi-analytical relationship for the spectral characteristics between the first- and second-order moments of system random input and output, thus bypassing the computationally demanding repetitive simulations of the system with samples as in the MC method. The use of the operational matrix in illustrative examples drastically reduced computational time with a desired accuracy over that of the traditional MC method. However, the method is restricted only to relatively small time delay due to inherent limitation of Pade approximation.

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NOMENCLATURE

$K\in N(\mu, \sigma^2)$: K is a Gaussian random variable with a mean μ and variance σ^2

$K\in U[a, b]$: K is a uniform random variable in the interval [a, b]

κ_{XX} : covariance function of stochastic Gaussian input

REFERENCES

1. W. L. Dunn and J. K. Shultis, *Exploring monte carlo method*, Elsevier, San Diego (2011).
2. J. S Liu, *Monte carlo strategies in scientific computing*, Springer-Verlag (2001).
3. K. A. Puvkov and N. D. Egupov, *Classical and modern theory of control system (2nd Ed.)*, Vol. 2, MGTU Press, Moscow (2003) [in Russian].
4. D. Xiu, *Numerical method for stochastic computation: Spectral approach*, Princeton University Press (2011).
5. K. A. Puvkov, N. D Egupov, A. M. Makarenkov and A. I. Trofimov, *Theory and numerical methods for studying stochastic systems*, Moscow, Fizmatlits (2003) [in Russian].
6. C. F Chen, Y. T. T Tsay and T. T. Wu, *J. Franklin Inst.*, **303**, 267 (1977).
7. J. S. Gu and W. S. Jiang, *Int. J. Syst. Sci.*, **27**, 623 (1996).
8. F. Khellat and A. Yousefi, *J. Franklin Inst.*, **343**, 181 (2006).
9. W. Marsalek, *J. Franklin Inst.*, **320**(3), 105 (1985).
10. I. L. Chien and P. S. Fruehauf, *Chem. Eng. Prog.*, **86**, 33 (1990).
11. J. E. N. Rico, I. Alcala, J. G. Ortega and E. F. Camacho, *Control Engineering Practice*, **9**, 1209 (2001).
12. L. G. Evlanov, *System with random parameter*, Nauka (1976) [in Russian].
13. V. I. Smirnov, *Course of higher mathematics,* Moscow, Nauka (1967) [in Russian].
14. E. V. Seregina, M. A. Makarenkov and M. A. Stepovich, *Journal of Surface Investigation. X-ray, Synchrotron and Neutron Techniques*, **3**, 468 (2009).
15. J. Foo, X. Wan and G. E. Karniadakis, *J. Comput. Phys.*, **227**, 9572 (2008).
16. E. Jarlebring, J. Vanbiervliet and W. Michiels, *IEEE Transactions on Automatic Control*, **56**(4), 814 (2011).
17. S. Guilouzic, I. L. Heureux and A. Longtin, *Phys. Rev. E*, **59**(4), 3970 (1999).