ON 1-DIMENSIONAL SHEAVES ON PROJECTIVE PLANE

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ABSTRACT. Let M be the Simpson moduli space of semistable sheaves on the projective plane with fixed linear Hilbert polynomial $P(m) = dm + c$. A generic sheaf in M is a vector bundle on its Fitting support, which is a planar projective curve of degree d. The sheaves that are not vector bundles on their support constitute a closed subvariety M' in M.

We study the geometry of M' in the case of Hilbert polynomials $dm - 1$, $d > 3$, and demonstrate that M' is a singular variety of codimension 2 in M.

We speculate on how the question we study is related to recompactifying of the Simpson moduli spaces by vector bundles.

CONTENTS

0. INTRODUCTION

Let $\Bbbk$ be an algebraically closed field of characteristic zero, say $\mathbb{C}$. Let V be a vector space over $\Bbbk$ of dimension 3, and let $\mathbb{P}_2 = \mathbb{P}V$ be the corresponding projective plane.

Definition 0.1. *A 1-dimensional sheaf $\mathcal{F}$ on $\mathbb{P}_2$ is a pure coherent with* $\dim \operatorname{Supp} \mathcal{F} = 1$*, i. e., $C = \operatorname{Supp} \mathcal{F} \subseteq \mathbb{P}_2$ is a curve.*

Since purity implies torsion-freeness on support and since torsion free sheaves on smooth curves are locally free, a generic 1-dimensional sheaf is a vector bundle on an algebraic curve.

One can see (cf. [14]) that 1-dimensional sheaves on $\mathbb{P}_2$ are in one-to-one correspondence with the pairs (E, f), where $E = \oplus \mathcal{O}_{\mathbb{P}_2}(a_i)$ is a direct sum of line bundles on $\mathbb{P}_2$ and $E \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{f} E$ is an injective morphism of sheaves.

For a sheaf $\mathcal{F}$, let $P(m) = P_{\mathcal{F}}(m)$ be its Hilbert polynomial. Its degree equals the dimension of the support of $\mathcal{F}$, so 1-dimensional sheaves have linear Hilbert polynomials $dm+c$, $d, m \in \mathbb{Z}$.

Let $M = M_{dm+c}(\mathbb{P}_2)$ be the Simpson moduli space of semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm + c$.

Recall that $\mathcal{F}$ is called semistable resp. stable if it is pure and for every proper subsheaf $\mathcal{E} \subseteq \mathcal{F}$ with $P_{\mathcal{E}}(m) = d'm + c'$ it holds $c'/d' \leqslant c/d$ resp. $c'/d' < c/d$.

Properties of M.

- M is projective, irreducible, locally factorial, $\dim M = d^2+1$ (Le Potier [7]). If $\gcd(d, c) = 1$, M is a fine moduli space, there are only stable sheaves, M is smooth (Le Potier [7]).

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- Canonical singularities for $\gcd(d,c) \neq 1$ (Woolf [13]).
- For small d one has
  - the Betti numbers (Choi, Chung, Maican [1], [2], [3]);
  - description of M in terms of locally closed strata (Drézet, Maican [4], [10], [11]).
- Isomophisms:
  - an obvious one $M_{dm+c} \cong M_{dm+c+d}$, $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2}(1)$;
  - a non-obvious one $M_{dm+c} \cong M_{dm-c}$, $\mathcal{F} \mapsto \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}_2})$ (Maican [9])
- $M_{dm+c} \cong M_{d'm+c'}$ iff $d = d'$ and $c = \pm c' \mod d$ (Woolf [13]).

Definition 0.2. *A 1-dimensional sheaf $\mathcal{F}$ is called singular if it is locally free on its support.*

Let $M' \subseteq M$ be the closed subvariety of singular sheaves. If M' is non-empty, then $M \setminus M'$ is a space of vector bundles on support and M is its compactification. Since $\operatorname{codim}_M M' > 1$ in general, this compactification is not maximal.

Questions.

- Study M'.
- Find a maximal compactification with a geometric meaning.
- Find a maximal compactification by vector bundles (on support).

We restrict ourselves to the case $\gcd(d,c) = 1$, i. e., to the case of the moduli spacesof isomorphism classes.

1. First examples

Trivial examples.

- A sheaf $\mathcal{F}$ belongs to $M_{m+1}(\mathbb{P}_2)$ if and only $\mathcal{F} \cong \mathcal{O}_L$ for a line $L \subseteq \mathbb{P}_2$.
- A sheaf $\mathcal{F}$ belongs belongs to $M_{m+1}(\mathbb{P}_2)$ if and only $\mathcal{F} \cong \mathcal{O}_C$ for a conic $C \subseteq \mathbb{P}_2$.

In these cases $M' =$.

A non-trivial example. A sheaf $\mathcal{F}$ belongs to $M = M_{3m-1}(\mathbb{P}_2)$ if and only if it is isomorphic to the ideal sheaf of a point p on a cubic planar curve $C \subseteq \mathbb{P}_2$, i. e., there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_C \to \mathcal{O}_{\{p\}} \to 0.$$

Then M is isomorphic to the universal cubic curve

$$\{(p, C) \mid p \in C, \quad C \text{ is a cubic curve in } \mathbb{P}_2\}.$$

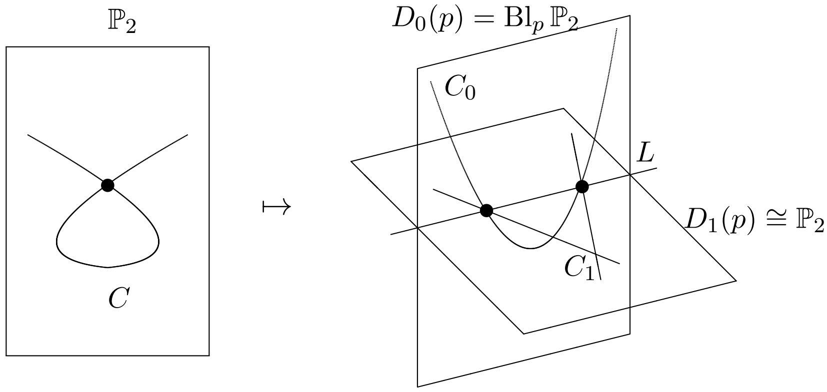
$\mathcal{F}$ is singular iff $\mathcal{F}_p \not\cong \mathcal{O}_{C,p}$ iff $p \in \operatorname{Sing} C$, hence M' is the universal singular locus

$$\{(p, C) \mid p \in \operatorname{Sing} C, \quad C \text{ is a cubic curve in } \mathbb{P}_2\},$$

which is a smooth subvariety of codimension 2 in M.

A construction that interprets $\operatorname{Bl}_{M'} M$ as a compactification of $M \setminus M'$ by vector bundles on curves in reducible surfaces $D(p)$ was given in [6]. A singular sheaf $\widetilde{\mathcal{F}}$ given by $p \in C$ is substituted by sheaves on curves $C_0 \cup C_1 \subseteq D(p)$.

The surface $D(p)$ consists of two irreducible components $D_0(p) \cup D_1(p)$, where $D_0(p) = \operatorname{Bl}_p \mathbb{P}_2$ is the blow up of $\mathbb{P}_2$ at p and $D_1(p)$ is a projective plane attached to $D_0(p)$ along the exceptional line.



This construction is very explicit and uses heavily the properties of M', which motivated us to study the subvarieties of singular sheaves in the Simpson moduli spaces for Hilbert polynomials $dm + c$, $d \geqslant 4$.

2. Moduli spaces $M_{dm-1}(\mathbb{P}_2)$

We consider the moduli spaces $M = M_{dm-1}(\mathbb{P}_2)$, $d \geqslant 4$. Their description in terms of locally closed strata, each of which is described as a quotient, is given in [4], [10], [11]. For an arbitrary d one has a good understanding [8] of the open Brill-Noether locus

$$M_0 = \{\mathcal{F} \in M \mid h^0(\mathcal{F}) = 0\}.$$

$\mathcal{F} \in M_0$ if and only if $\mathcal{F}$ has a locally free resolution

$$0 \to \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \to \mathcal{F} \to 0, \quad \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (d-2)\mathcal{O}_{\mathbb{P}_2}(-2), \quad \mathcal{E}_0 = (d-1)\mathcal{O}_{\mathbb{P}_2}(-1)$$

with $A = \left(\begin{smallmatrix} Q \\ \Phi \end{smallmatrix}\right)$ such that Q is a $1 \times (d-1)$ of quadratic forms and Φ is a stable Kronecker module, i. e., a $(d-2) \times (d-1)$ matrix of linear forms that is not equivalent under the action of $\mathrm{GL}_{d-2}(\Bbbk) \times \mathrm{GL}_{d-1}(\Bbbk)$ to a matrix with a zero block of size $j \times (d-1-j)$, $j = 1, \ldots, d-2$.

This describes M_0 as a quotient

$$\{A = \left(\begin{smallmatrix} Q \\ \Phi \end{smallmatrix}\right) \text{ as above}\}/\operatorname{Aut}(\mathcal{E}_1) \times \operatorname{Aut}(\mathcal{E}_0)$$

with the induced map to the quotient space of the stable Kronecker modules Φ

$$M_0 \to N, \quad N = N(3; d-2, d-1) = \{\Phi\}/\mathrm{GL}_{d-2}(\Bbbk) \times \mathrm{GL}_{d-1}(\Bbbk).$$

Taking all matrices A as above (not necessarily injective), one gets a projective quotient $\mathbb{B} = \{A\}^{ss}/\operatorname{Aut}(\mathcal{E}_1) \times \operatorname{Aut}(\mathcal{E}_0)$, with a map $\mathbb{B} \to N$, which is a projective bundle associated to a vector bundle of rank $3d$ over N (cf. [8]).

Consider an open subvariety $N_0 \subseteq N$ corresponding to Φ with coprime maximal minors. Then N_0 is isomorphic to an open subvariety $H_0 \subseteq \mathbb{P}_2^{[l]}$, $l = (d-1)(d-2)/2$, in the Hilbert scheme of l points on $\mathbb{P}_2$ that do not lie on a curve of degree $d-3$. The class of $[\Phi]$ is sent to the zero scheme of its maximal minors.

Put $M_{00} = \mathbb{B}|_{N_0}$, then $\operatorname{codim}_M M \setminus M_0 \geqslant 2$ as shown in [14].

$\mathcal{F} \in M_{00}$ if and only if $\mathcal{F}$ is a twisted ideal sheaf of $Z \in H_0$ on $C = \operatorname{Supp} \mathcal{F}$:

$$0 \to \mathcal{F} \to O_C(d-3) \to \mathcal{O}_Z \to 0.$$

The fibre over $[\Phi] \in N_0$ can be interpreted as curves of degree d through $Z \in H_0$ that corresponds to $[\Phi]$ under the isomorphism $N_0 \cong H_0$. Thus M_{00} can be seen as an open subvariety of the Hilbert flag scheme $H(l, d)$ of l points on a curve of degree d.

3. Singular sheaves in M_{00}

The results in this section are obtained in [5] together with Alain Leytem, a PhD student of Martin Schlichenmaier at the University of Luxembourg.

Let us mention some necessary conditions for $\mathcal{F} \in M_{00}$ to be singular.

- $C = \operatorname{Supp} \mathcal{F}$ must be singular as torsion free sheaves on a smooth curves are locally free.

- As $\mathcal{F}$ is a twisted ideal sheaf of $Z \subseteq C$, $\mathcal{F}$ can only be singular at points from Z, thus $Z \cap \operatorname{Sing} C \neq \emptyset$ if $\mathcal{F}$ is singular.

Claim. *If Z consists of l different points, then $\mathcal{F}$ is singular if and only if $Z \cap \operatorname{Sing} C \neq \emptyset$.*

Indeed, $\mathcal{F}$ is singular if and only if there exists $p \in Z$ with $\mathcal{F}_p \not\cong \mathcal{O}_{C,p}$, which, in turn, holds if and only if there is a point $p \in Z \cap \operatorname{Sing} C$.

Let now $Z = \bigsqcup Z_i$, where Z_i is a (fat) point at $p_i \in \mathbb{P}_2$. Assume that for a given i, Z_i is a curvilinear point that in some local coordinates x, y at p_i is given as $Z(x - h(y), y^n)$. Let (C_i, p_i) be the germ of the smooth curve $C_i = Z(x - h(y))$.

Claim. *$\mathcal{F}$ is non-singular at $p_i \in \operatorname{Sing} C$ if and only if $(C, p_i) \cap (C_i, p_i) = (Z_i, p_i)$ (intersection of germs of curves).*

Proof. Straightforward: write $C = Z(\det \left(\begin{smallmatrix} x-h(y) & y^n \\ u(y) & v(x,y) \end{smallmatrix}\right))$ and study when the ideal of $Z_i \subseteq C$ is 1-generated. □

If Z_i is a double point on a line L, then $\mathcal{F}$ is non-singular at $p_i \in \operatorname{Sing} C$ iff the tangent cone of C at p_i consists of two lines different from L. This shows that the sheaves singular at double points are limits of the sheaves singular at simple points.

Now fix a basis $(x_0, x_1, x_2) \in V^*$, assume $p_1 = \langle 1, 0, 0 \rangle = Z(x_1, x_2)$, and assume that Z contains at most 1 fat point and this fat point can only be a double point. Then the requirement for $\mathcal{F}$ to be singular at p_1 imposes 2 linear independent conditions on C (independent from the conditions imposed by the condition $Z \subseteq C$):

- vanishing of the coefficients of the monomials $x_0^{d-1}x_1$, $x_0^{d-1}x_2$ in the equation of C if $Z_1 = p_1$ is a simple point;
- vanishing of the coefficients of the monomials $x_0^{d-1}x_1$, $x_0^{d-2}x_2^2$ in the equation of C if $Z_1 = Z(x_1, x_2^2)$ is a double point.

The imposed conditions are independent because Z does not lie on a curve of degree $d-3$.

We conclude that for $M'_{00} = M_{00} \cap M' \subseteq M_{00} \to N_0 \cong H_0$, the fibre over the locus H_c of the configurations (l different points) is a union of l linear subspaces of codimension 2 in $\mathbb{P}_{3d-1}$ (the fibre of $M_{00} \to N_0$). The fibre over

$$H_1 = \{Z \mid \text{with exactly 1 fat point}\}$$

is a union of $l-1$ linear subspaces of codimension 2.

Conclusion. Fibres of M'_{00} over $N_c \sqcup N_1 \cong H_c \sqcup H_1$ are singular of codimension 2. Therefore, M'_{00} is singular of codimension 2 in M_{00}. Since $\operatorname{codim}_M M \setminus M_{00} \geqslant 2$ (Yuan, [14]), we obtain the following theorem.

Theorem. *M' is singular of codimension 2 in M.*

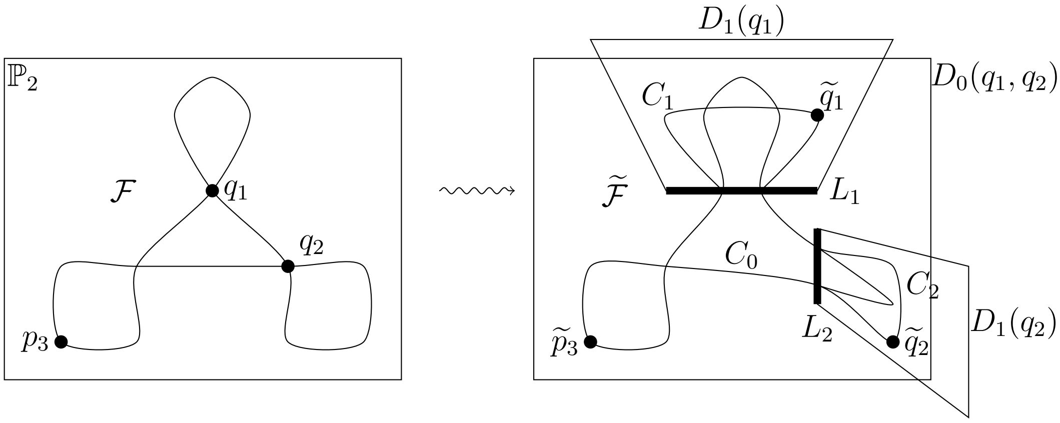
The restriction $M'_{00}|_{N_c}$ is a family of arrangements of linear subspaces of codimension 2 in $\mathbb{P}_{3d-1}$.

4. Speculations on recompactifying the Simpson moduli spaces

Aim. Interpret the blow-up

$$\operatorname{Bl}_{M'} M \to M$$

as a process that substitutes a singular sheaf $\mathcal{F} \in M_{00}|_{H_c}$ given by $Z \subseteq C$ with non-empty $Z \cap \operatorname{Sing} C = \{q_1, \ldots, q_r\}$ by sheaves $\widetilde{\mathcal{F}}$ on a curve $C' = C_0 \cup C_1 \cup \cdots \cup C_r$ in a reduced surface $D(q_1, \ldots, q_r)$ obtained by blowing up the points $q_1, \ldots, q_r$ and attaching to the exceptional lines $L_1, \ldots, L_r$ surfaces $D_1(q_i) \cong \mathbb{P}_2$ such that C_0 is the proper transform of C in $D_0(q_1, \ldots, q_r) = \operatorname{Bl}_{\{q_1, \ldots, q_r\}} \mathbb{P}_2$ and $C_i \subseteq D_1(q_i)$. $\widetilde{\mathcal{F}}$ is a twisted ideal sheaf of l points $\{\widetilde{q}_1, \ldots, \widetilde{q}_r, \widetilde{p}_{r+1}, \ldots, \widetilde{p}_l\}$ in C' with $\widetilde{q}_i \in C_i \subseteq D_1(q_i)$ and the points $\widetilde{p}_{r+1}, \ldots, \widetilde{p}_l$ being preimages of $p_{r+1}, \ldots, p_l$ in $D_0(q_1, \ldots, q_r)$. The sheaf $\widetilde{\mathcal{F}}$ is locally free on C' or "less singular".



Iterating this (to be) construction we want to get a recompatification of the Simpson moduli spaces by vector bundles (on 1-dimensional support).

Remark. It should be mentioned that the construction indicated here resembles the construction from [12, Theorem 4.3]. I was happy to learn this from the talk of Szilárd Szabó given at this conference.

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