Median algebra with a retraction: an example of variety closed under natural extension

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(joint work with Georges Hansoul)

The Big Picture

Claims about natural extensions

Natural extension provides

‘canonical extension’ for non lattice-based algebras ;

insight about the construction of canonical extension.

Back to the roots : canonical extension

$\mathbf{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ is a DL

Canonical extension $\mathbf{L}^\delta$ of $\mathbf{L}$ comes with topologies ι and δ :

- $\mathbf{L}^\delta$ is doubly algebraic.
- $\mathbf{L} \hookrightarrow \mathbf{L}^\delta$.
- $\mathbf{L}$ is dense in $\mathbf{L}^\delta_\iota$.
- $\mathbf{L}$ is dense and discrete in $\mathbf{L}^\delta_\delta$.

Canonical extension comes with a tool to extend maps

Problem. Given $f\colon \mathbf{L} \to \mathbf{E}$, define $f^\delta\colon \mathbf{L}^\delta \to \mathbf{E}^\delta$.

Solution.

- $\mathbf{L}$ is made of the isolated points of $\mathbf{L}^\delta$,
- $\mathbf{L}$ is dense in $\mathbf{L}^\delta_\delta$,
- $f^\delta := \liminf_\delta f$ and $f^\pi := \mathrm{limsup}_\delta f$.

Leads to canonical extension of ordered algebras :

Jónsson-Tarski (1951), Gehrke and Jónsson (1994), Dunn, Gehrke and Palmigiano (2005), Gehrke and Harding (2011), Gehrke and Vosmaer (2011), Davey and Priestley (2011)...

Why canonical extension ?

It provides completeness results for modal logics with respects to classes of Kripke frames :

Jónsson, B. (1994). On the canonicity of Sahlqvist identities. *Studia Logica*, 53(4), 473–491.

Gehrke, M., Nagahashi, H., and Venema, Y. (2005). A Sahlqvist theorem for distributive modal logic. *Ann. Pure Appl. Logic*, 131(1-3), 65–102.

Hansoul, G., and Teheux, B. (2013). Extending Łukasiewicz logics with a modality : algebraic approach to relational semantics. *Studia Logica*, 101(3), 505–545.

Is it possible to generalize canonical extension to non lattice-based algebras ?

Problem 1. Define the natural extension $\mathbf{A}^\delta$ of $\mathbf{A}$:

Davey, B. A., Gouveia, M., Haviar, M., and Priestley, H. (2011). Natural extensions and profinite completions of algebras. *Algebra Universalis*, 66, 205–241.

Problem 2. Extend $f\colon \mathbf{A} \to \mathbf{B}$ to $f^\delta\colon \mathbf{A}^\delta \to \mathbf{B}^\delta$.

We give a partial solution.

Natural extension of algebras

The framework of natural extension

$\mathbf{A}^\delta$ can be defined if $\mathbf{A}$ belongs to some

$$\mathbb{ISP}(\mathcal{M})$$

where $\mathcal{M}$ is class of finite algebras of the same type.

$\mathbf{A}^\delta$ can more easily be computed if $\mathbb{ISP}(\mathcal{M})$ is dualisable (in the sense of natural dualities).

We adopt the setting of natural dualities

$\mathbf{M} \equiv$ a finite algebra

A discrete alter-ego topological structure $\underset{\sim}{M}$

$\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$

Algebra	**Topology**
$\mathbf{M}$	$\underset{\sim}{M}$
$\mathcal{A} = \mathbb{ISP}(\mathbf{M})$	$\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{M})$
$\mathbf{A}$	$\mathbf{A}^* = \mathcal{A}(\mathbf{A}, \mathbf{M}) \leq_c \underset{\sim}{M}^{\mathbf{A}}$
$\underset{\sim}{X}_* = \mathcal{X}(\underset{\sim}{X}, \underset{\sim}{M}) \leq \mathbf{M}^X$	$\underset{\sim}{X}$

Definition. $\underset{\sim}{M}$ *yields a natural duality* for $\mathbb{ISP}(\mathbf{M})$ if

$$(\mathbf{A}^*)_* \simeq \mathbf{A}, \qquad \mathbf{A} \in \mathbb{ISP}(\mathbf{M}).$$

Natural extension of an algebra can be constructed from its dual

Priestley duality is a natural duality : $\mathbf{L} \simeq (\mathbf{L}^*)_*$

Proposition (Gehrke and Jónsson)

If $\mathbf{L} \in \mathrm{DL}$ then $\mathbf{L}^\delta$ is the algebra of order-preserving maps from $\mathbf{L}^*$ to $\underset{\sim}{2}$.

Assume that $\underset{\sim}{M}$ yields a duality for $\mathbb{ISP}(\mathbf{M})$.

Proposition (Davey and al.)

If $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, then $\mathbf{A}^\delta$ is the algebra of structure-preserving maps from $\mathbf{A}^*$ to $\underset{\sim}{M}$.

Natural extension of median algebras

The variety of median algebras is an old friend...

The expression

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

defines an operation $\mathrm{m}_{\leq}(x, y, z)$ on a distributive lattice $(L, \leq)$.

Definition. (Avann, 1948)

median algebra $\mathbf{A} = (A, \mathrm{m}) \iff$ subalgebra of some $(L, \mathrm{m}_{\leq})$

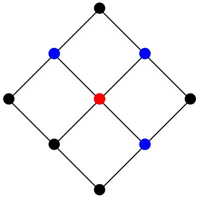
Example. Set $\mathbf{2} := \langle \{0, 1\}, \mathrm{m} \rangle$ where m is the majority function.

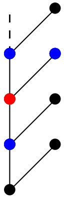
Theorem. The variety $\mathcal{A}_m$ of median algebras is $\mathbb{ISP}(\mathbf{2})$.

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

Examples.

$$\mathrm{m}(\bullet, \bullet, \bullet) = \bullet$$





Median graphs

Some metric spaces

…in which semilattice orders can be defined

For every $a \in \mathbf{A}$, the relation $\leq_a$ defined on $\mathbf{A}$ by

$$b \leq_a c \quad \text{if} \quad \mathrm{m}(a, b, c) = b.$$

is a $\wedge$-semilattice order on A with $b \wedge_a c = \mathrm{m}(a, b, c)$.

Semillatices obtained in this way are the *median semilattices*.

Proposition. In a median semilattice, principal ideals are distributive lattices.

Grau (1947), Birkhoff and Kiss (1947), Sholander (1952, 1954), Isbell (1980), Bandelt and Hedlíková (1983)…

There is a natural duality for median algebras

$$\mathcal{A}_m = \mathbb{ISP}(\mathbf{2})$$

$$\underset{\sim}{2} := \langle \{0, 1\}, \leq, \cdot^{\bullet}, 0, 1, \iota \rangle.$$

Theorem (Isbell (1980), Werner (1981)). The structure $\underset{\sim}{2}$ yields a logarithmic duality for $\mathcal{A}_m$.

$\mathbf{A}^\delta$ is the algebra of structure-preserving maps $x\colon \mathbf{A}^* \to \underset{\sim}{2}$.

Natural extension completes everything it can complete

Theorem. Let $a \in \mathbf{A}$.

- $\langle \mathbf{A}^\delta, \leq_a \rangle$ a bounded-complete extension of $\langle \mathbf{A}, \leq_a \rangle$.
- If $\mathbf{I}$ is a distributive lattice in $\mathbf{A}$ then $\mathrm{cl}_{\mathbf{A}_\iota^\delta}(\mathbf{I}) = \mathbf{I}^\delta$

Natural extension of maps

$\mathbf{A}$ can be defined topologically in $\mathbf{A}^\delta$

$\mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M}) \equiv$ set of morphisms defined on a closed substructure of $\mathbf{A}^*$.

Definition.

$$O_f := \{x \in \mathcal{X}(\mathbf{A}^*, \underset{\sim}{M}) \mid x \supseteq f\}, \qquad f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})$$

$$\Delta := \{O_f \mid f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})\}$$

Working assumption. $\underset{\sim}{M}$ yields a full logarithmic duality for $\mathbb{ISP}(\mathbf{M})$ and $\underset{\sim}{M}$ is injective in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{M})$.

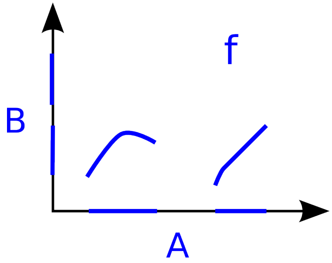
Proposition.

- Δ is a basis of topology δ
- $\mathbf{A}$ is dense and discrete in $\mathbf{A}^\delta$.
- In the settings of DL, we get the known topology.

We canonically extends maps to multi-maps

Input :

$$f : \mathbf{A} \to \mathbf{B}$$



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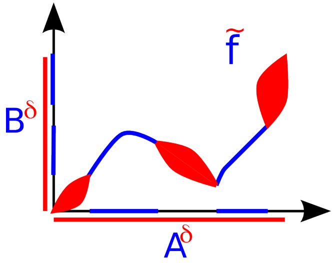


Output :

$$f^+ : \mathbf{A}^\delta \to \Gamma(\mathbf{B}_\iota^\delta)$$

The multi-extension of $f : \mathbf{A} \to \mathbf{B}$

Intermediate step : Consider

$$\bar{f} : \mathbf{A} \to \Gamma(\mathbf{B}_\iota^\delta) : a \mapsto \{f(a)\}.$$

Recall that $\mathbf{A}$ is dense in $\mathbf{A}_\delta^\delta$ and $\Gamma(\mathbf{B}_\iota^\delta)$ is a complete lattice.

Definition. The *multi-extension f^+ of f* is defined by

$$f^+ : \mathbf{A}_\delta^\delta \to \Gamma(\mathbf{B}_\iota^\delta) : x \mapsto \mathrm{limsup}_\delta \bar{f}(x),$$

In other words,

$$f^+(x) = \bigcap \{\mathrm{cl}_{\mathbf{B}_\iota^\delta}(f(A \cap V)) \mid V \in \delta_x\},$$

$$f^+(x)\big|_F = \bigcap \{f(A \cap V)\big|_F \mid V \in \delta_x\}, \qquad F \Subset \mathbf{B}^\delta$$

The multi-extension is a continuous map

Definition.

We say that f is *smooth* if $\# f^+(x) = 1$ for all $x \in \mathbf{A}^\delta$.

Let $\sigma{\downarrow}$ be the *co-Scott* topology on $\Gamma(\mathbf{B}^\delta_\iota)$.

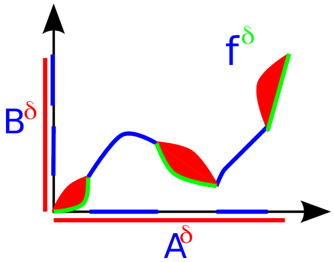
Proposition.

- f^+ is the smallest $(\delta, \sigma{\downarrow})$-continuous extension from $\mathbf{A}^\delta_\delta$ to $\Gamma(\mathbf{B}^\delta_\iota)$.
- f is smooth if and only if it admits an (δ, ι)-continuous extension $f^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ satisfying $f^\delta(x) \in f^+(x)$.

This construction sheds light on canonical extension

Proposition. If $f\colon \mathbf{A} \to \mathbf{B}$ is a map between DLs with lower extension f^δ and upper extension f^π, then for any $x \in \mathbf{L}^\delta$

$$f^\delta(x) = \bigwedge f^+(x),$$
$$f^\pi(x) = \bigvee f^+(x).$$



Natural extension of median algebras with a retraction.

Natural extension of expansions of median algebras

General framework.

Let

$$\mathbf{A} = \langle A, m, r, a \rangle$$

where $\langle A, \mathrm{m} \rangle \in \mathcal{A}_m$, $a \in \mathbf{A}$ and $r \colon \mathbf{A} \to \mathbf{A}$

Set

$$r^\delta(x) \ = \ \wedge_a \, r^+(x), \qquad x \in \langle A, m \rangle^\delta.$$

$$\mathbf{A}^\delta := \langle \langle A, m \rangle^\delta, r^\delta, a \rangle$$

Natural properties

Definition. A property P of algebras in $\mathcal{A}$ is *natural* if

$$\mathbf{A} \models P \implies \mathbf{A}^\delta \models P, \qquad \mathbf{A} \in \mathcal{A}$$

Example. The property 'being a median algebra of a Boolean algebra' is natural.

Median algebras with a retraction

Definition. An idempotent homomorphism $r\colon \mathbf{A} \to \mathbf{A}$ such that $u(A)$ is convex is called a *retraction*.

Proposition. A map $r\colon \mathbf{A} \to \mathbf{A}$ is a retraction if and only if

$$r(\mathrm{m}(x, y, z)) = \mathrm{m}(x, r(y), r(z)), \qquad x, y, z \in \mathbf{A}.$$

Definition. An algebra $\langle A, \mathrm{m}, r, a\rangle$ is a *pointed retract algebra* if r is a retraction of the median algebra $\langle A, \mathrm{m}\rangle$ and $a \in A$.

The variety of pointed retract algebras is natural

Theorem. If **A** is a pointed retract algebra then $\mathbf{A}^\delta$ is a pointed retract algebra.

Sketch of the proof.

Proves equalities of the type

$$(r \circ \mathrm{m})^\delta = r^\delta \circ \mathrm{m}$$

using continuity properties of the extensions. □

The variety of pointed algebras with operator is natural.

Definition. An algebra $\mathbf{A} = \langle A, m, f, a \rangle$ is a *pointed median algebra with operator* if $\langle A, \mathrm{m} \rangle$ is a median algebra, $a \in A$ and

$$f(\mathrm{m}(a, x, y)) = \mathrm{m}(a, f(x), f(y)), \qquad x, y \in A.$$

Theorem Let $\langle A, \mathrm{m}, f, a \rangle$ be a pointed median algebra with operator.

- f is smooth.
- $\mathbf{A}^\delta$ is a pointed median algebra with operator.

Sketch of the proof.

f can be dualized as a relation R on $\mathbf{A}^*$ and f^δ can be explicitly computed with R. □

Questions/Problems.

- Interesting instances of natural extensions of maps (in non-ordered based algebras).
- Successful applications of the whole theory.
- Find canonical (continuous) way to pick-up some element in $f^+(x)$.
- Intrinsic definition of δ in the non-dualizable setting.