Some unexpected facts about Lie Algebras

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Abstract : Lie algebras and their morphisms do not behave as nicely as one might expect. We will discuss some of the most important illustrations. For example, the category of Lie algebras admits direct products, but no coproducts, and the object called "direct sum" is actually not a direct sum in the categorical sense. It admits kernels and cokernels, but it is neither abelian, nor additive, and the image of a morphism is not a categorical image. Nevertheless, one can still define short exact sequences of Lie algebras. We finish by discussing the Splitting Lemma, which for Lie algebras, in general, only holds in a weaker form than in the abelian case.

0 Internal and external direct sum of vector spaces

a) Let V be a given vector space and $V_1, V_2 \subseteq V$ vector subspaces such that $V_1 \cap V_2 = \{0\}$. The internal direct sum is defined as

$$V_1 \oplus^i V_2 := \left\{ v_1 + v_2 \in V \;\middle|\; v_i \in V_i \right\} \subseteq V$$

It is another vector subspace of the given vector space V.

Example : graded vector spaces
Consider the Witt Algebra $\mathcal{W}$ and its basis elements $e_n \in \mathcal{W}$ for $n \in \mathbb{Z}$. Then

$$\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n \qquad \text{where } \mathcal{W}_n = \langle\, e_n \,\rangle$$

which means that elements in $\mathcal{W}$ write as finite linear combinations of these e_n.

b) Let V, W be 2 arbitrary vector spaces (no relations between them). We define a new vector space, called external direct sum, as

$$V \oplus^e W := \left\{ (v, w) \;\middle|\; v \in V,\, w \in W \right\}$$

It is equal to the direct product $V \times W$ (see universal properties of direct products and direct sums).

c) Nevertheless, the 2 concepts are isomorphic.
If we set $\widetilde{V} := V \times \{0\}$ and $\widetilde{W} := \{0\} \times W$, then $\widetilde{V}, \widetilde{W}$ are both vector subspaces of $V \oplus^e W$ and we have

$$V \oplus^e W = \widetilde{V} \oplus^i \widetilde{W}$$

1 The category of Lie algebras

a) A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space (of some dimension) endowed with a Lie bracket, i.e. a bilinear map $[\,\cdot\,,\,\cdot\,] : L \times L \to L$ which is skew-symmetric and satisfies the Jacobi identity.
If $\operatorname{char}\mathbb{K} = 2$, then the condition $[x, x] := 0,\ \forall\, x \in L$ must be added. Otherwise it follows from skew-symmetry.

Examples :
− abelian Lie algebras : $[x, y] := 0,\ \forall\, x, y \in L$
− If $(A, \cdot)$ is an associative algebra, then A is also a Lie algebra with respect to the commutator bracket

$$[a, b]_c := a \cdot b - b \cdot a\,, \quad \forall\, a, b \in A$$

b) A Lie algebra morphism $\phi : L_1 \to L_2$ between 2 Lie algebras L_1 and L_2 is a $\mathbb{K}$-linear map which respects the Lie bracket, i.e.

$$\phi\big([x,y]_1\big) = \big[\phi(x), \phi(y)\big]_2 , \quad \forall\, x, y \in L_1$$

The set of Lie algebra morphisms $L_1 \to L_2$ is denoted by $\mathrm{Hom}_{\mathtt{LA}}(L_1, L_2)$. It is closed under the composition $\circ$ and $\mathrm{id}_L \in \mathrm{Hom}_{\mathtt{LA}}(L, L)$ for every Lie algebra L.
Hence Lie algebras over $\mathbb{K}$ and their morphisms form a category, denoted by $\mathtt{LA}(\mathbb{K})$. This category admits $\{0\}$ as a zero-object since it is initial and terminal. In particular, there exists a zero morphism

$$0 \in \mathrm{Hom}_{\mathtt{LA}}(L_1, L_2) , \quad \forall\, L_1, L_2 \in \mathtt{LA}(\mathbb{K})$$

Moreover we have the forgetful functor $\mathtt{LA}(\mathbb{K}) \to \mathtt{Vect}_{\mathbb{K}}$ since Lie algebras are vector spaces (with some additional structure). Note however that Lie algebras do not form a subcategory of $\mathtt{Vect}_{\mathbb{K}}$ since one may put different Lie algebra structures on the same vector space.

c) Problem : $\mathrm{Hom}_{\mathtt{LA}}(L_1, L_2)$ is not an abelian group with respect to $(+, 0)$, for example

$$\mathrm{id}_L + \mathrm{id}_L \notin \mathrm{Hom}_{\mathtt{LA}}(L, L) \qquad \text{since } 2 \cdot [x, y] \neq [2x, 2y] \quad (\text{unless } L \text{ is abelian})$$

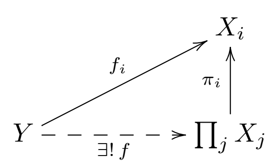
In particular, it is not a vector space neither. This is already "very bad" and actually the main reason why most of the usual "nice" properties will fail.

d) A Lie subalgebra of L is a vector subspace $L' \subseteq L$ which is closed under the Lie bracket of L, i.e. $[L', L'] \subseteq L'$. We denote $L' \leq L$.

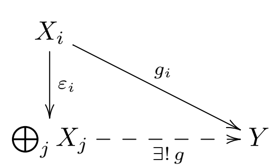
2 Direct products and direct sums

a) Let $\mathcal{C}$ be any category and consider a family of objects $X_i \in \mathcal{C}$ with $i \in I$ for some small index set I.

A direct product, if it exists, is a pair $\big(\prod_j X_j, \pi_i\big)$ where $\prod_j X_j \in \mathcal{C}$ and $\pi_i \in \mathrm{Hom}_{\mathcal{C}}\big(\prod_j X_j, X_i\big)$ are morphisms called projections such that these data satisfy the following universal property :
$\forall\, Y \in \mathcal{C}$, $\forall\, f_i \in \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$, there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{C}}\big(Y, \prod_j X_j\big)$ such that $f_i = \pi_i \circ f$, $\forall\, i \in I$.



A direct sum (also called coproduct), if it exists, is a pair $\big(\bigoplus_j X_j, \varepsilon_i\big)$ where $\bigoplus_j X_j \in \mathcal{C}$ and $\varepsilon_i \in \mathrm{Hom}_{\mathcal{C}}\big(X_i, \bigoplus_j X_j\big)$ are morphisms called coprojections such that these data satisfy the following universal property :
$\forall\, Y \in \mathcal{C}$, $\forall\, g_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$, there exists a unique morphism $g \in \mathrm{Hom}_{\mathcal{C}}\big(\bigoplus_j X_j, Y\big)$ such that $g_i = g \circ \varepsilon_i$, $\forall\, i \in I$.



Being solutions of universal problems, we know that direct products and direct sums, if they exist, are uniquely given up to canonical isomorphism.

b) $\mathcal{C}$ is called an additive category if
1) $\forall\, X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group with respect to $(+, 0)$.
2) $\forall\, X, Y, Z \in \mathcal{C}$, the composition $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is $\mathbb{Z}$-bilinear
3) there exists a zero object, denoted by 0
4) there exist finite direct products

5) there exist finite direct sums

Proposition :
If $\mathcal{C}$ satisfies the properties 1) − 3), then 4) ⇔ 5) and in this case $X \times Y \cong X \oplus Y$, $\forall X, Y \in \mathcal{C}$.
Indeed, one shows that the direct sum satisfies the universal property of the direct product, and vice-versa.

c) By 1c), we see that $\mathtt{LA}(\mathbb{K})$ is not additive.

d) For two Lie algebras L_1, L_2, we can define their so-called "direct sum" $L_1 \oplus_L L_2$ by

$$L_1 \oplus_L L_2 := L_1 \oplus^e L_2 \qquad \text{as vector spaces}$$

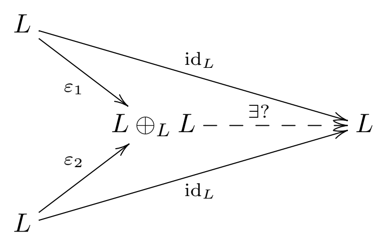
with Lie algebra structure given by $[L_1, L_2] = \{0\}$, i.e.

$$\big[(x_1, x_2), (y_1, y_2)\big] := \big([x_1, y_1]_1, [x_2, y_2]_2\big)\,, \quad \forall\, x_i, y_i \in L_i \tag{1}$$

Moreover we have the canonical injections

$$\varepsilon_1 \;:\; L_1 \longrightarrow L_1 \oplus_L L_2 \;:\; x_1 \longmapsto (x_1, 0) \qquad , \qquad \varepsilon_2 \;:\; L_2 \longrightarrow L_1 \oplus_L L_2 \;:\; x_2 \longmapsto (0, x_2)$$

which are Lie algebra morphisms by definition (1) : $\varepsilon_i \in \mathrm{Hom}_{\mathtt{LA}}\big(L_i\,,\, L_1 \oplus_L L_2\big)$.
However, this is not a categorical direct sum as defined above. As an example, let $L \in \mathtt{LA}(\mathbb{K})$ be non-abelian.



If there exists a linear map $g : L \oplus_L L \to L$ making this diagram commute, then necessarily

$$g(x_1, x_2) = g(x_1, 0) + g(0, x_2) = (g \circ \varepsilon_1)(x_1) + (g \circ \varepsilon_2)(x_2) = \mathrm{id}_L(x_1) + \mathrm{id}_L(x_2) = x_1 + x_2$$

But this is not a Lie algebra morphism since L is not abelian :

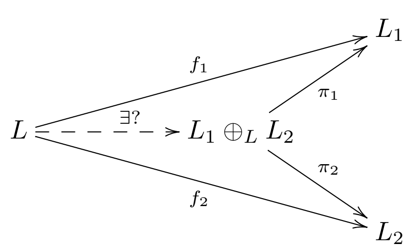
$$g\Big(\big[(x_1, x_2), (y_1, y_2)\big]\Big) = [x_1, y_1] + [x_2, y_2] \neq \big[x_1 + x_2\,,\, y_1 + y_2\big] = \big[g(x_1, x_2), g(y_1, y_2)\big]$$

<u>Conclusion</u> : "The direct sum is not a direct sum."

e) But : It is a direct product. Let $L_1, L_2, L \in \mathtt{LA}(\mathbb{K})$ and $f_i \in \mathrm{Hom}_{\mathtt{LA}}(L, L_i)$ be arbitrary. The projections

$$\pi_i \;:\; L_1 \oplus_L L_2 \longrightarrow L_i \;:\; (x_1, x_2) \longmapsto x_i$$

are Lie algebra morphisms by (1) $\Rightarrow$ $\pi_i \in \mathrm{Hom}_{\mathtt{LA}}\big(L_1 \oplus_L L_2\,,\, L_i\big)$. Now consider



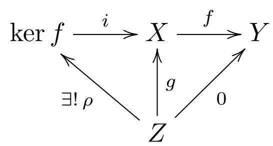
If there exists a map $f : L \to L_1 \oplus_L L_2$ making this diagram commute, then necessarily $f(x) = \big(f_1(x), f_2(x)\big)$.
And this is indeed a Lie algebra morphism since $f\big([x, y]\big) = \big[f(x), f(y)\big]$, $\forall\, x, y \in L \;\Rightarrow\; f \in \mathrm{Hom}_{\mathtt{LA}}\big(L\,,\, L_1 \oplus_L L_2\big)$.

By uniqueness of solutions of universal problems, we thus know that $L_1 \oplus_L L_2$ is the direct product $\prod_i L_i$.

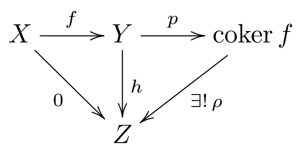
3 Kernels and cokernels

a) Let $\mathcal{C}$ be a category with a zero-object and consider $X, Y \in \mathcal{C}$ with a fixed morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

A kernel of f, if it exists, is a pair $(\ker f, i)$ where $\ker f \in \mathcal{C}$ and $i \in \mathrm{Hom}_{\mathcal{C}}(\ker f, X)$ is a morphism such that $f \circ i = 0$ and these data satisfy the following universal property :
$\forall\, Z \in \mathcal{C}$, $\forall\, g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$ such that $f \circ g = 0$, g uniquely factorizes though $\ker f$, which means that there is a unique morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(Z, \ker f)$ such that $g = i \circ \rho$.



A cokernel of f, if it exists, is a pair $(\operatorname{coker} f, p)$ where $\operatorname{coker} f \in \mathcal{C}$ and $p \in \mathrm{Hom}_{\mathcal{C}}(Y, \operatorname{coker} f)$ is a morphism such that $p \circ f = 0$ and these data satisfy the following universal property :
$\forall\, Z \in \mathcal{C}$, $\forall\, h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ such that $h \circ f = 0$, $\exists!\, \rho \in \mathrm{Hom}_{\mathcal{C}}(\operatorname{coker} f, Z)$ such that $h = \rho \circ p$.



Again, kernels and cokernels, if they exist, are uniquely given up to canonical isomorphism.

Example :
If $\mathcal{C} = \mathtt{Vect}_{\mathbb{K}}$ and $f : V \to W$ is a linear map between $\mathbb{K}$-vector spaces, then

$$\ker f = f^{-1}\big(\{0\}\big) \qquad \text{and} \qquad \operatorname{coker} f = W/f(V)$$

Proposition :
If ker and coker exist, then i is a monomorphism and p is an epimorphism, i.e. for any morphisms g_1, g_2, h_1, h_2 :

$$i \circ g_1 = i \circ g_2 \;\Rightarrow\; g_1 = g_2 \qquad , \qquad h_1 \circ p = h_2 \circ p \;\Rightarrow\; h_1 = h_2$$

b) Let $L_1, L_2 \in \mathtt{LA}(\mathbb{K})$ and $\phi \in \mathrm{Hom}_{\mathtt{LA}}(L_1, L_2)$. We define the kernel and the image of f as

$$\ker \phi := \big\{\, x \in L_1 \;\big|\; \phi(x) = 0 \,\big\}$$
$$\operatorname{im} \phi := \big\{\, y \in L_2 \;\big|\; \exists\, x \in L_1 \text{ such that } y = \phi(x) \,\big\}$$

Then $\ker \phi \leq L_1$ and $\operatorname{im} \phi \leq L_2$ are Lie subalgebras.
Let L be a Lie algebra. A vector subspace $I \subseteq L$ is called a Lie ideal of L if $[L, I] \subseteq I$. We denote $I \trianglelefteq L$.

Examples :
− the center of a Lie algebra $Z(L) := \{\, x \in L \mid [x, y] = 0, \forall\, y \in L \,\} \trianglelefteq L$
− If $\phi : L_1 \to L_2$ is a Lie algebra morphism, then $\ker \phi \trianglelefteq L_1$
− Any intersection of Lie ideals is still a Lie ideal : $I_j \trianglelefteq L, \forall\, j \;\Rightarrow\; \bigcap_j I_j \trianglelefteq L$

c) However, the image of a Lie algebra morphism is in general not a Lie ideal : $\phi \in \mathrm{Hom}_{\mathtt{LA}}(L_1, L_2) \not\Rightarrow \operatorname{im} \phi \trianglelefteq L_2$.
As an example, consider the 3-dimensional Heisenberg algebra $\mathcal{H} = \langle\, p, q, I \,\rangle_{\mathbb{C}}$, where the generators satisfy

$$[p, q] = I \qquad , \qquad [p, I] = 0 \qquad , \qquad [q, I] = 0$$

We define a Lie algebra representation (i.e. a Lie algebra morphism) $\rho : \mathcal{H} \to \mathfrak{gl}(3, \mathbb{C})$ by

$$\rho(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad , \qquad \rho(q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad , \qquad \rho(I) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\operatorname{im}\rho = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{C} \right\} \leq \mathfrak{gl}(3, \mathbb{C})$$

This is a Lie subalgebra of $\mathfrak{gl}(3, \mathbb{C})$, but not a Lie ideal, e.g.

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right]_c = \begin{pmatrix} y & y & y \\ z & z & z \\ 0 & -x & -x \end{pmatrix} \notin \operatorname{im}\rho$$

Proposition :
Let $L \in \mathtt{LA}(\mathbb{K})$. For any subset $A \subseteq L$, there exists a smallest Lie ideal containing it, given by

$$I(A) := \bigcap_{A \subseteq J \trianglelefteq L} J \trianglelefteq L$$

In particular, one can consider the Lie ideal $I(\operatorname{im}\phi) \trianglelefteq L_2$.

d) Quotient spaces : let $L \in \mathtt{LA}(\mathbb{K})$ and $I \trianglelefteq L$.
One can define the usual quotient vector space L/I. We endow it with the following Lie algebra structure :

$$[\bar{x}, \bar{y}] := \overline{[x, y]} \qquad \Leftrightarrow \qquad \big[x + I \,,\, y + I\big] := [x, y] + I$$

This is well-defined since I is a Lie ideal (a Lie subalgebra would not be sufficient).

Proposition :
Let $L_1, L_2 \in \mathtt{LA}(\mathbb{K})$ and $\phi \in \operatorname{Hom}_{\mathtt{LA}}(L_1, L_2)$. Then $L_1/\ker\phi \cong \operatorname{im}\phi$ as Lie algebras, given by $\bar{x} \mapsto \phi(x)$.

e) For a Lie algebra morphism $\phi : L_1 \to L_2$, we define $\operatorname{coker}\phi := L_2 \,/\, I(\operatorname{im}\phi)$.

f) The above definitions of $\ker\phi$ and $\operatorname{coker}\phi$ are kernels and cokernels in the categorical sense.

$$\begin{array}{ccccc} \ker\phi & \xhookrightarrow{\;i\;} & L_1 & \xrightarrow{\;\phi\;} & L_2 \\ & {\scriptstyle \exists!\,\rho}\nwarrow & \uparrow{\scriptstyle \varphi} & \nearrow{\scriptstyle 0} & \\ & & L & & \end{array} \qquad\qquad \begin{array}{ccccc} L_1 & \xrightarrow{\;\phi\;} & L_2 & \xrightarrow{\;p\;} & \operatorname{coker}\phi \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \exists!\,\rho} & \\ & & Z & & \end{array}$$

We take the inclusion $i : \ker\phi \hookrightarrow L_1$, hence $\phi \circ i = 0$. And if $\varphi : L \to L_1$ is a Lie algebra morphism such that $\phi \circ \varphi = 0$, it suffices to take $\rho := \varphi$ since $\operatorname{im}\varphi \subseteq \ker\phi$. Thus $(\ker\phi, i)$ is a kernel in $\mathtt{LA}(\mathbb{K})$.

We also define $p : L_2 \twoheadrightarrow \operatorname{coker}\phi$ to be the canonical projection, thus $p \circ \phi = 0$ since $\operatorname{im}\phi \subseteq I(\operatorname{im}\phi)$. If there is another morphism of Lie algebras $\psi : L_2 \to L$ such that $\psi \circ \phi = 0$, we can define the map

$$\rho \;:\; \operatorname{coker}\phi \longrightarrow L \;:\; \bar{y} \longmapsto \psi(y)$$

This is the only possible choice in order to obtain the above commutative diagram. ρ is in addition well-defined because $\psi \circ \phi = 0 \;\Rightarrow\; \operatorname{im}\phi \subseteq \ker\psi$, thus $\ker\psi$ is a Lie ideal containing $\operatorname{im}\phi \;\Rightarrow\; I(\operatorname{im}\phi) \subseteq \ker\psi$ and

$$\psi(y + \ell) = \psi(y) + \psi(\ell) = \psi(y) + 0\,, \quad \forall\, \ell \in I(\operatorname{im}\phi)$$

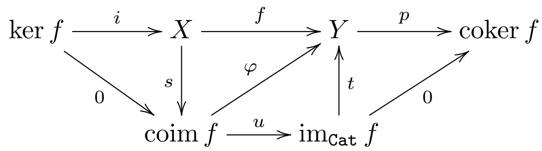
By definition, ρ is also a Lie algebra morphism, hence $(\operatorname{coker}\phi, p)$ is a cokernel in $\mathtt{LA}(\mathbb{K})$.

4 Images and coimages

a) Let $\mathcal{C}$ be a category that admits kernels and cokernels, i.e. $(\ker f, i)$ and $(\operatorname{coker} f, p)$ exist for all $X, Y \in \mathcal{C}$ and any morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$.
The coimage and the categorical image of f are given by $\operatorname{coim} f := \operatorname{coker} i$ and $\operatorname{im}_{\mathtt{Cat}} f := \ker p$. As kernel and

cokernel, they come together with an epimorphism $s : X \to \operatorname{coim} f$ and a monomorphism $t : \operatorname{im}_{\texttt{Cat}} f \to Y$.

Proposition :
Any morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ can be decomposed as follows :



Since i, t are monic and p, s are epic, we moreover obtain that

$$\ker i = \ker t = 0 \quad , \quad \operatorname{coker} p = \operatorname{coker} s = 0 \quad , \quad \ker s \cong \ker f \quad , \quad \operatorname{coker} t \cong \operatorname{coker} f$$

Hence $\ker f$, $\operatorname{coker} f$, $\operatorname{coim} f$ and $\operatorname{im}_{\texttt{Cat}} f$ are all objects that can be created by this process.

b) A category $\mathcal{C}$ is called abelian if
1) $\mathcal{C}$ is additive
2) $\mathcal{C}$ admits kernels and cokernels
3) The canonical morphism $u : \operatorname{coim} f \to \operatorname{im}_{\texttt{Cat}} f$ is an isomorphism for any morphism f in $\mathcal{C}$

Remark :
$\texttt{LA}(\mathbb{K})$ is not abelian since it is not additive. But there are even more problems : let $\phi \in \operatorname{Hom}_{\texttt{LA}}(L_1, L_2)$.

$$\operatorname{im}_{\texttt{Cat}} \phi = \ker p = I(\operatorname{im} \phi) \qquad \text{and} \qquad \operatorname{coim} \phi = \operatorname{coker} i = L_1/I(\operatorname{im} i) = L_1/\ker \phi \cong \operatorname{im} \phi$$

where $i : \ker \phi \hookrightarrow L_1$ and $p : L_2 \twoheadrightarrow \operatorname{coker} \phi$. But $I(\operatorname{im} \phi)$ and $\operatorname{im} \phi$ may be different if $\operatorname{im} \phi$ is not a Lie ideal. Thus in general, $\operatorname{coim} \phi \ncong \operatorname{im}_{\texttt{Cat}} \phi$. In addition, we see that the image of a Lie algebra morphism is not a categorical image, but actually a coimage.

5 Exact sequences

a) A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of Lie algebras and Lie algebra morphisms is called a complex if $g \circ f = 0$, i.e. if $\operatorname{im} f \subseteq \ker g$ as vector spaces. And it is called exact if $\operatorname{im} f = \ker g$ (this is not the categorical image). A short exact sequence of Lie algebras is therefore a sequence

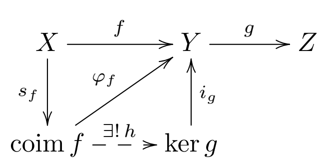
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which means that f is injective, g is surjective and $\operatorname{im} f = \ker g$.

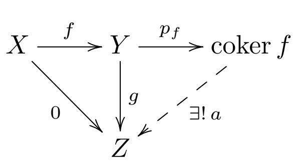
b) Let $\mathcal{C}$ be an abelian category. Similarly,

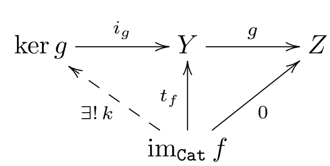
$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{2}$$

is called a complex if $g \circ f = 0$. This still makes sense. But how to replace the condition "$\operatorname{im} f \subseteq \ker g$" ? $(*)$



We have $0 = g \circ f = g \circ \varphi_f \circ s_f \;\Rightarrow\; g \circ \varphi_f = 0$ since s_f is an epimorphism. Hence φ_f uniquely factorizes through $\ker g$ and we obtain a morphism $h : \operatorname{coim} f \to \ker g$. This is the generalized condition $(*)$. Moreover





$g \circ f = 0 \;\Rightarrow\; \exists!\, a : \operatorname{coker} f \to Z$ such that $g = a \circ p_f$
$g \circ t_f = a \circ p_f \circ t_f = a \circ 0 = 0 \;\Rightarrow\; \exists!\, k : \operatorname{im}_{\mathtt{Cat}} f \to \ker g$ such that $t_f = i_g \circ k$. And

$$i_g \circ k \circ u_f = t_f \circ u_f = \varphi_f = i_g \circ h \qquad \Rightarrow\; h = k \circ u_f \text{ since } i_g \text{ is monic}$$

Hence the condition $g \circ f = 0$ induces the following commutative diagram :

$$\begin{array}{ccc} & & \ker g \\ & \overset{h}{\nearrow} & \uparrow{\scriptstyle k} \\ \operatorname{coim} f & \xrightarrow{u_f} & \operatorname{im}_{\mathtt{Cat}} f \end{array} \tag{3}$$

Now the sequence (2) is called exact if h is an isomorphism, i.e. if $\operatorname{coim} f \cong \ker g$.
If $\mathcal{C}$ is abelian, this is equivalent to requiring that k is an isomorphism since u_f will be an isomorphism as well.

c) If $\mathcal{C}$ is not abelian, it is important to specify that h should be an isomorphism, e.g. if $\mathcal{C} = \mathtt{LA}(\mathbb{K})$:
Let $f \in \operatorname{Hom}_{\mathtt{LA}}(A, B)$ and $g \in \operatorname{Hom}_{\mathtt{LA}}(B, C)$. Then

$$\operatorname{im} f \subseteq \ker g \;\Leftrightarrow\; I(\operatorname{im} f) \subseteq \ker g \qquad \text{but} \qquad \operatorname{im} f = \ker g \;\not\Leftrightarrow\; I(\operatorname{im} f) = \ker g$$

i.e. existence of the morphisms $\operatorname{coim} f \to \ker g$ and $\operatorname{im}_{\mathtt{Cat}} f \to \ker g$ (here they are just inclusions) is equivalent. But requiring that they should be isomorphisms is not equivalent ! The condition $\operatorname{im} f = \ker g$ is much stronger than $I(\operatorname{im} f) = \ker g$ since it implies that $\operatorname{im} f$ is a Lie ideal, which is in general not the case. However, $I(\operatorname{im} f)$ is always a Lie ideal by definition.

d) Proposition :
Let $\mathcal{C}$ be an abelian category. Then

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \text{ is exact } \Leftrightarrow\; (X, f) \cong (\ker g, i_g)$$
$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \text{ is exact } \Leftrightarrow\; (Z, g) \cong (\operatorname{coker} f, p_f)$$

This is no longer true for Lie algebras : we only have

$$1)\;\; 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \text{ is exact } \Leftrightarrow\; (A, f) \cong (\ker g, i_g)$$
$$2)\;\; A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \text{ is exact } \Rightarrow\; (C, g) \cong (\operatorname{coker} f, p_f)$$

Proof. 1) $\underline{\Rightarrow}$: $A \cong \operatorname{im} f = \ker g$, $\underline{\Leftarrow}$: because $0 \longrightarrow \ker g \xrightarrow{i_g} B \xrightarrow{g} C$ is exact : $\ker g = \operatorname{im} i_g$
2) $\underline{\Rightarrow}$: $\operatorname{coker} f = B/I(\operatorname{im} f) = B/\ker g \cong \operatorname{im} g = C$

$$\text{but } A \xrightarrow{f} B \xrightarrow{p_f} \operatorname{coker} f \longrightarrow 0 \text{ is not exact in general since } \ker p_f = I(\operatorname{im} f) \neq \operatorname{im} f$$

□

6 The Splitting Lemma

a) Let $\mathcal{C}$ be an abelian category and consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{$*$}$$

We define the 3 following conditions :
(**S1**) : $(*)$ is called split exact if there exists an isomorphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & A & \xrightarrow{\varepsilon_A} & A \oplus C & \xrightarrow{\pi_C} & C & \longrightarrow & 0 \end{array}$$

$(\mathbf{S2})$: $(*)$ is called left split if there exists a morphism $r \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $r \circ f = \mathrm{id}_X$.

$$0 \longrightarrow A \underset{r}{\overset{f}{\rightleftarrows}} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

$(\mathbf{S3})$: $(*)$ is called right split if there exists a morphism $s \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ such that $g \circ s = \mathrm{id}_Z$.

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \underset{s}{\overset{g}{\rightleftarrows}} C \longrightarrow 0$$

b) Theorem :
1) If $\mathcal{C} = \mathtt{Vect}_{\mathbb{K}}$, then any short exact sequence is split exact.
2) If $\mathcal{C}$ is abelian, then $(\mathbf{S1}) \Leftrightarrow (\mathbf{S2}) \Leftrightarrow (\mathbf{S3})$.

Proof. 2) $(\mathbf{S1}) \Rightarrow (\mathbf{S2})$, set $r := \varphi^{-1} \circ \pi_A \circ \phi$, $(\mathbf{S1}) \Rightarrow (\mathbf{S3})$, set $s := \phi^{-1} \circ \varepsilon_C \circ \psi$
For the other implications, we only sketch the proof in the case of vector spaces :
$(\mathbf{S2}) \Rightarrow (\mathbf{S1})$: we have $B = \mathrm{im}\, f \oplus^i \ker r \cong A \oplus^e C$ with the isomorphisms $\varphi := \mathrm{id}_A$, $\psi := \mathrm{id}_C$ and

$$\phi \;:\; B \longrightarrow A \oplus^e C \;:\; b = f(a) + k \longmapsto \big(a, g(k)\big)$$

$(\mathbf{S3}) \Rightarrow (\mathbf{S1})$: we have $B = \ker g \oplus^i \mathrm{im}\, s \cong A \oplus^e C$ with the isomorphisms $\varphi := \mathrm{id}_A$, $\psi := \mathrm{id}_C$ and

$$\phi^{-1} \;:\; A \oplus^e C \longrightarrow B \;:\; (a, c) \longmapsto f(a) + s(c)$$

□

c) But for Lie algebras, we only have $(\mathbf{S1}) \Leftrightarrow (\mathbf{S2}) \Rightarrow (\mathbf{S3})$. Assume that r and s are Lie algebra morphisms.
$(\mathbf{S2})$: if we want to obtain $B \cong A \oplus_L C$, we need that $[A, C] = \{0\}$ in the above computations :

$$[A, C] \cong [\mathrm{im}\, f, \ker r] = [\ker g, \ker r] \subseteq \ker g \cap \ker r = \{0\}$$

since kernels are Lie ideals and $\ker g \oplus^i \ker r$ is an internal direct sum. Hence $B \cong A \oplus_L C$ as Lie algebras.
$(\mathbf{S3})$: here it does not work any more since images of Lie algebra morphisms are not necessarily Lie ideals :

$$[A, C] \cong [\ker g, \mathrm{im}\, s] \subseteq \ker g\,, \text{ but this does not need to be zero}$$

Hence $B \cong A \oplus^e C$ as vector spaces, but this is in general not an isomorphism of Lie algebras.

d) Particular case : we have $(\mathbf{S1}) \Leftrightarrow (\mathbf{S3})$ if A is central in B, i.e. if $f(A) \subseteq Z(B)$. Indeed, this implies that

$$\mathrm{im}\, s \trianglelefteq B \qquad \text{and} \qquad [A, C] \cong [\mathrm{im}\, f, \mathrm{im}\, s] \subseteq \big[f(A), B\big] = \{0\}$$

This is for example the case for cental extensions : if there is a Lie algebra morphism $s : L \to \widehat{L}$ such that

$$0 \longrightarrow \mathbb{K} \longrightarrow \widehat{L} \underset{s}{\rightleftarrows} L \longrightarrow 0$$

then $\widehat{L} \cong \mathbb{K} \oplus_L L$ since $\big[\mathbb{K}, \widehat{L}\big] = \{0\}$.

7 Semi-direct sums

a) Let L be a Lie algebra over $\mathbb{K}$. A $\mathbb{K}$-linear map $\phi : L \to L$ is called a derivation of Lie algebras if

$$\phi\big([x, y]\big) = \big[\phi(x), y\big] + \big[x, \phi(y)\big]\,, \quad \forall\, x, y \in L$$

Example :
The adjoint action $\mathrm{ad}_x : L \to L : y \mapsto [x, y]$ is a derivation by the Jacobi identity. It is called an inner derivation.

Proposition :
The set of all Lie algebra derivations, denoted by $\mathrm{Der}\, L$, is itself a Lie algebra with respect to the commutator

bracket of linear maps. In particular, $\mathrm{Der}\, L \leq \mathfrak{gl}(L)$.

b) Let L, A be Lie algebras and $\theta : L \to \mathrm{Der}\, A : x \mapsto \theta_x$ a given Lie algebra morphism.
The semi-direct sum $A \rtimes_\theta L$ is given by $A \oplus^e L$ as vector spaces, with the Lie algebra structure

$$\big[(a, x), (b, y)\big]_\theta := \big([a, b] + \theta_x(b) - \theta_y(a)\,,\, [x, y]\big)$$

In particular, we recover the direct sum $A \oplus_L L$ for $\theta \equiv 0$.

c) Theorem :
A short exact sequence of Lie algebras is right split $\Leftrightarrow$ it is isomorphic (as complexes) to a semi-direct sum.

$$(\mathbf{S3}) \quad \Leftrightarrow \quad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\varepsilon_A} & A \rtimes_\theta C & \xrightarrow{\pi_C} & C & \longrightarrow & 0 \end{array}$$

Proof. As an exact sequence, we know that $f(A) \trianglelefteq B$ since $\mathrm{im}\, f = \ker g$, so it makes sense to define

$$\theta \ : \ C \to \mathrm{Der}\, A \ : \ c \mapsto \theta_c \qquad \text{where } \theta_c(a) := f^{-1}\big([s(c), f(a)]\big) = \text{“}\mathrm{ad}_{s(c)|_A}\text{”}$$

where $[s(c), f(a)] \in f(A)$. Then $\phi^{-1} : A \rtimes_\theta C \xrightarrow{\sim} B : (a, c) \mapsto f(a) + s(c)$ is a Lie algebra isomorphism. $\square$