Master Thesis in Mathematics
An introduction to Schemes and Moduli Spaces in Geometry

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Summer semester 2011–2012

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Preface

The aim of my thesis is to give an introduction to Schemes and Moduli Spaces in modern Algebraic Geometry. Roughly speaking, algebraic geometry is the study of solutions of systems of polynomial equations in an affine (or projective) space, i.e. the study of algebraic varieties. Guiding problems occurring in this field are the so-called classification problems, whose goal is to classify all algebraic varieties up to isomorphism. However, such problems are usually so difficult that one never expects to solve them completely.
The study of schemes and moduli spaces is a first approach for classifying our geometric objects; one also speaks of moduli problems. The basic idea is to replace the classical geometric space by the algebra of functions on the space, or an even more general set of maps from this space to other spaces, and the geometry corresponds to some algebraic structure on this set of maps. One of the advantages of this generalization is the possibility to extend the techniques of geometry to more general objects that may not be considered as "classical manifolds".
This concept was established by **Alexander Grothendieck** in the 1960s.

The thesis is presented in 4 parts:

In the first chapter, we give a quick overview of some concepts from category theory. Most of this chapter is a summary of the text of [S]. We are not going to study categories in detail but only define the language and introduce a few basic notions, such as functors between categories. Moreover we analyze some of the most important results as for example the Yoneda Lemma, for which we detail the proof sketched in [S], and some of its consequences. The goal is to apply these rather abstract theory to more concrete situations later on.

Chapter 2 deals with elements of classical Commutative Algebra, such as rings, ideals, algebraic varieties, localizations of rings, modules and sheaves. In particular we concentrate on locally ringed spaces and the spectrum of a commutative unital ring, which will be of great importance for the rest of the thesis. We will also prove that the spectrum defines a contravariant equivalence between commutative unital rings and locally ringed spaces. Here we mostly base ourselves on [Ha], sometimes with a more explicit presentation. We also add some ideas from [S], [Sch] and [U1].

In chapter 3, we are going to define the very important concept of schemes, which is connecting the fields of algebraic geometry and commutative algebra. We will give some examples of affine and general schemes (collected from several sources) and explain how sheaves and schemes can be glued together. We also shortly describe how schemes can be seen as "generalized varieties". However, we can only give some ideas of the actual aim of schemes and why they are useful, but it is not yet possible to establish deep results. The main reference here is again [Ha], together with some ideas developped in [Ga], [Ma], [S] and [U2].

The last chapter introduces the moduli problems and their associated moduli spaces. Here we follow the texts of [HM] and [Ho]. Again this will only be an introduction to the whole concept which shall help getting used to the language. We do this by giving several examples of moduli problems and explaining how category theory can be used in order to solve some of them. In particular, it is not always possible to find an "easy" solution. We finally close the thesis by describing the case of $\mathcal{M}_g$, the space of isomorphism classes of compact Riemann surfaces of genus g, as it is presented in [Sch]. This will be an example of a coarse moduli space.

Alain Leytem

Thanks
I especially would like to thank my lecturers Prof. Martin Schlichenmaier and Dr. Oleksandr Iena for their enthusiastic support and for giving me useful tips and advices during the writing of this thesis.

Chapter 1

Categories and functors

In this chapter we introduce some basic notions of category theory, which are of constant use in various fields of Mathematics. Roughly speaking, category theory examines in an abstract way the properties of mathematical concepts by considering them as collections of objects and arrows (called morphisms), these collections satisfying some basic conditions.
We can see a category as a type of mathematical structure and therefore look for "processes" which preserve this structure in some sense. Such a process is called a functor and associates, in a compatible way, to every object of one category an object of another category, and to every morphism in the first category a morphism in the second one. Functors, in particular representable functors, will play an important role in the context of moduli spaces later on.
Category theory was created in 1942−45 by the American mathematicians **Samuel Eilenberg** (1913−1998) and **Saunders Mac Lane** (1909−2005) as part of their work in algebraic topology and homological algebra. Our aim is not to give a course on category theory, but to understand the language of categories in order to apply the concepts to concrete situations in the following. This whole first chapter is based on [S].

1.1 Categories

1.1.1 Definitions

A *category* $\mathcal{C}$ consists of

1) a set $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects* of $\mathcal{C}$
2) $\forall\, X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose elements are called *morphisms* from X to Y
3) $\forall\, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition* map

$$\circ\,:\,\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)\,:\,(f,g) \mapsto g \circ f$$

such that these data satisfy

a) $\circ$ is associative, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$ for any morphisms f, g, h such that composition is defined
b) $\forall\, X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called the *identity morphism on X* such that $f \circ \mathrm{id}_X = f$, $\forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{id}_X \circ g = g$, $\forall\, g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ for all $Y \in \mathrm{Ob}(\mathcal{C})$.

<u>Notation</u> : One also writes $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Y$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. f is called
− an *isomorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.
− a *monomorphism* if for any morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.
− an *epimorphism* if whenever $h_1 \circ f = h_2 \circ f$ for some morphisms $h_1, h_2 : Y \to Z$, then $h_1 = h_2$.
If there exists an isomorphism $f : X \to Y$, we say that X and Y are *isomorphic* and denote $X \cong Y$.

Remarks :

1) Although the most important example of a category is the category of sets, with objects being sets and morphisms being functions from one set to another, it is important to note that, in whole generality, objects of a category do not need to be sets and morphisms are not necessarily functions between sets. Neither composition needs to be the well-known composition of maps. This leads to quite abstract notions.

2) There are some set-theoretical dangers, e.g. for the category of sets, one has to take care because the "set" of all sets is not a set. Indeed one has to specify in which universe we are working (we do not give the definition of a universe here). The crucial point is Grothendieck's axiom which states that any set belongs to some universe. We do not develop this any further.

Opposite category

Let $\mathcal{C}$ be a category. The *opposite category* of $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{op}}$, is given by the data

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}) \quad , \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

and composition map

$$\circ_{\mathrm{op}} \,:\, \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Z) \,:\, (f,g) \mapsto g \circ_{\mathrm{op}} f = f \circ g \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$$

Subcategories

A category $\mathcal{C}$ is called a *subcategory* of another category $\mathcal{C}'$, denoted by $\mathcal{C} \subset \mathcal{C}'$, if it has "less objects" and "less morphisms", i.e. $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{C}')$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}'}(X,Y)$, $\forall\, X,Y \in \mathcal{C}$, such that composition and identities in $\mathcal{C}$ are induced by those in $\mathcal{C}'$.

$\mathcal{C}$ is called a *full subcategory* of $\mathcal{C}'$ if it has "less objects" but with the "same morphisms", i.e. $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{C}')$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{C}'}(X,Y)$, $\forall\, X,Y \in \mathcal{C}$. This means that there are as many $\mathcal{C}'$-morphisms defined on objects of the subcategory than there are $\mathcal{C}$-morphisms.

Full subcategories are characterized by the fact that they only differ from the "bigger" category by additional properties, but no new data is needed to define these properties.

1.1.2 Examples

Here below, we fix notations and give examples of the most common and important categories.

category	objects	morphisms
$\mathtt{Set}$	sets	maps between sets
$\mathtt{Set}^f$	finite sets	maps between finite sets
$\mathtt{Top}$	topological spaces	continuous maps
$\mathtt{Diff}$	real differentiable manifolds	smooth maps
$\mathtt{Man}^p$	real C^p-manifolds	p times continuously differentiable maps
$\mathtt{Ring}$	rings	ring homomorphisms
$\mathtt{Grp}$	groups	group homomorphisms
$\mathtt{Mod}(R)$	modules over a ring R	R-module homomorphisms
$\mathtt{Mod}^f(R)$	finitely generated modules over R	R-module homomorphisms
$\mathtt{Mod}^{\mathrm{free}}(R)$	free modules over a ring R	R-module homomorphisms
$\mathtt{Vect}_{\mathbb{K}} = \mathtt{Mod}(\mathbb{K})$	vector spaces over a field $\mathbb{K}$	$\mathbb{K}$-linear maps
$\mathtt{Ab} = \mathtt{Mod}(\mathbb{Z})$	abelian groups	abelian group homomorphisms
$\mathtt{Ban}(\mathbb{K})$	Banach spaces over a field $\mathbb{K}$	continuous $\mathbb{K}$-linear maps
$\mathtt{Cat}$	categories	functors between categories
$\mathtt{Fct}(\mathcal{C},\mathcal{C}')$	functors between categories $\mathcal{C}$ and $\mathcal{C}'$	natural transformations

The notions of functors and natural transformations will be defined in section 1.2.

We say that a category is *concrete* if it is a subcategory of $\mathtt{Set}$. Hence all of the above examples are concrete categories, except the 2 last ones. Note that $\mathtt{Set}^f \subset \mathtt{Set}$, $\mathtt{Mod}^f(R) \subset \mathtt{Mod}(R)$ and $\mathtt{Ab} \subset \mathtt{Grp}$ are full subcategories. The category of unital rings is an example of a subcategory of $\mathtt{Ring}$ which is not a full subcategory since the unit element gives new data for the existing structure of a ring.

1.1.3 Initial and terminal objects

Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is called *initial* if $\mathrm{Hom}_{\mathcal{C}}(I, X)$ is a singleton, $\forall\, X \in \mathcal{C}$.
An object $T \in \mathcal{C}$ is called *terminal* if it is initial in $\mathcal{C}^{\mathrm{op}}$, i.e. if $\mathrm{Hom}_{\mathcal{C}}(X, T)$ has only 1 element, $\forall\, X \in \mathcal{C}$.
An object $P \in \mathcal{C}$ is called a *zero-object* if it is both initial and terminal. In this case, we denote $P = 0$.

Initial objects are unique up to isomorphism since if $I_1, I_2 \in \mathcal{C}$ are both initial, then

$$\mathrm{Hom}_{\mathcal{C}}(I_1, I_2) = \{\, f : I_1 \to I_2 \,\} \quad , \quad \mathrm{Hom}_{\mathcal{C}}(I_2, I_1) = \{\, g : I_2 \to I_1 \,\} \quad , \quad \mathrm{Hom}_{\mathcal{C}}(I_j, I_j) = \{\, \mathrm{id}_{I_j} : I_j \to I_j \,\}$$

with $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(I_1, I_1)$ and $f \circ g \in \mathrm{Hom}_{\mathcal{C}}(I_2, I_2) \;\Rightarrow\; g \circ f = \mathrm{id}_{I_1}$ and $f \circ g = \mathrm{id}_{I_2}$, thus $I_1 \cong I_2$.

Examples :
– In `Set`, $\emptyset$ is an initial object and singletons, denoted by $\{\mathrm{pt}\}$, are terminal objects.
– $\{0\}$ is a zero-object in $\mathtt{Mod}(R)$.
– $\mathbb{Z}$ is an initial object in the category of unital rings since any unital ring homomorphism $\varphi : \mathbb{Z} \to R$ is completely determined by its value at 1, which has to be 1_R. Thus $\varphi(1) = 1_R$ implies that

$$\varphi(n) = \varphi(1 + \ldots + 1) = \varphi(1) + \ldots + \varphi(1) = 1_R + \ldots + 1_R \quad \text{and} \quad \varphi(-n) = -\varphi(n),\ \forall\, n \geq 0$$

hence $\varphi(p)$ is known for all $p \in \mathbb{Z}$.

1.2 Functors

1.2.1 Definition

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories.
A *covariant functor* $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is made up by 2 maps (which we denote by the same symbol)

$$\begin{aligned} \mathcal{F} &: \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}') \\ \mathcal{F} &: \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}\big(\mathcal{F}(X), \mathcal{F}(Y)\big),\ \forall\, X, Y \in \mathcal{C} \end{aligned}$$

such that they respect the categorical structure, i.e. composition and the identity morphism. In other words, $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$, $\forall\, X \in \mathcal{C}$ and $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all morphisms f, g in $\mathcal{C}$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a covariant functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$, i.e. it satisfies

$$\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)} \quad \text{and} \quad \mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g) \tag{1.1}$$

In the following, the word "functor" always refers to a covariant functor.
The defining properties in (1.1) imply that functors map isomorphisms to isomorphisms.

Functors can be composed naturally since they are made up by 2 usual maps. Indeed, let $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and $\mathcal{G} : \mathcal{C}' \to \mathcal{C}''$ be two functors between categories. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{C} \to \mathcal{C}''$ is defined "pointwise", i.e.

$$\begin{aligned} (\mathcal{G} \circ \mathcal{F})(X) &= \mathcal{G}\big(\mathcal{F}(X)\big) \in \mathrm{Ob}(\mathcal{C}'')\,,\ \forall\, X \in \mathcal{C} \\ (\mathcal{G} \circ \mathcal{F})(f) &= \mathcal{G}\big(\mathcal{F}(f)\big) \in \mathrm{Hom}_{\mathcal{C}''}\big((\mathcal{G} \circ \mathcal{F})(X), (\mathcal{G} \circ \mathcal{F})(Y)\big)\,,\ \forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \end{aligned}$$

and this assignment again defines a functor since $\mathcal{G}\big(\mathcal{F}(f \circ g)\big) = \mathcal{G}\big(\mathcal{F}(f) \circ \mathcal{F}(g)\big) = \mathcal{G}\big(\mathcal{F}(f)\big) \circ \mathcal{G}\big(\mathcal{F}(g)\big)$.

Bifunctors
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. We define the *product category* $\mathcal{C}_1 \times \mathcal{C}_2$ componentwise by

$$\mathrm{Ob}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{Ob}(\mathcal{C}_1) \times \mathrm{Ob}(\mathcal{C}_2) \quad , \quad \mathrm{Hom}_{\mathcal{C}_1 \times \mathcal{C}_2}\big((X_1, X_2), (Y_1, Y_2)\big) = \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathcal{C}_2}(X_2, Y_2)$$

This is again a category. A *bifunctor* $\mathcal{F} : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}'$ is then a functor on the product category.

1.2.2 Example

Let $\mathcal{C}$ be a category and fix an object $W \in \mathcal{C}$.
Then we have the Hom-functors $\mathcal{F} = \mathrm{Hom}_{\mathcal{C}}(W, \cdot\,)$ and $\mathcal{G} = \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, W)$, defined by

$$\begin{aligned}
\mathcal{F} &: \mathcal{C} \to \mathtt{Set} \;:\; X \mapsto \mathrm{Hom}_{\mathcal{C}}(W, X) \\
\mathcal{F} &: \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathtt{Set}}\big(\mathcal{F}(X), \mathcal{F}(Y)\big) \;:\; f \mapsto \mathcal{F}(f) = f\circ \\
&\qquad\qquad \mathcal{F}(f) \;:\; \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y) \;:\; g \mapsto f \circ g \\
\mathcal{G} &: \mathcal{C}^{\mathrm{op}} \to \mathtt{Set} \;:\; X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, W) \\
\mathcal{G} &: \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) \to \mathrm{Hom}_{\mathtt{Set}}\big(\mathcal{G}(X), \mathcal{G}(Y)\big) \;:\; f \mapsto \mathcal{G}(f) = \circ f \\
&\qquad\qquad \mathcal{G}(f) \;:\; \mathrm{Hom}_{\mathcal{C}}(X, W) \to \mathrm{Hom}_{\mathcal{C}}(Y, W) \;:\; g \mapsto g \circ f
\end{aligned}$$

Hence $\mathcal{F}$ is covariant, $\mathcal{G}$ is contravariant and we have a bifunctor $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,,\,\cdot\,) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathtt{Set}$.

1.2.3 The category of functors

Fix two categories $\mathcal{C}$ and $\mathcal{C}'$. Functors from $\mathcal{C}$ to $\mathcal{C}'$ form again a category, denoted by $\mathtt{Fct}(\mathcal{C}, \mathcal{C}')$, with

$$\mathrm{Ob}\big(\mathtt{Fct}(\mathcal{C}, \mathcal{C}')\big) = \big\{ \text{(covariant) functors from } \mathcal{C} \text{ to } \mathcal{C}' \big\}$$

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathtt{Fct}(\mathcal{C}, \mathcal{C}')$. A *morphism of functors* $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ (also called a *natural transformation*) is the data of a morphism $\varphi_X : \mathcal{F}_1(X) \to \mathcal{F}_2(X)$ in $\mathcal{C}'$ for every object $X \in \mathcal{C}$ such that $\forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following diagram commutes :

$$\begin{tikzcd}
\mathcal{F}_1(X) \arrow[r, "\varphi_X"] \arrow[d, "\mathcal{F}_1(f)"'] & \mathcal{F}_2(X) \arrow[d, "\mathcal{F}_2(f)"] \\
\mathcal{F}_1(Y) \arrow[r, "\varphi_Y"] & \mathcal{F}_2(Y)
\end{tikzcd}$$

Hence natural transformations assign, in a compatible way, to every object in the source category a morphism in the target category. If $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ and $\psi : \mathcal{F}_2 \to \mathcal{F}_3$ are two morphisms of functors, their composition $\psi \circ \varphi : \mathcal{F}_1 \to \mathcal{F}_3$ is again given "pointwise" :

$$(\psi \circ \varphi)_X := \psi_X \circ \varphi_X \in \mathrm{Hom}_{\mathcal{C}'}\big(\mathcal{F}_1(X), \mathcal{F}_3(X)\big)\,,\ \forall\, X \in \mathcal{C}$$

This composition is associative and $\psi \circ \varphi$ again defines a morphism of functors since $\forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$:

$$\mathcal{F}_3(f) \circ (\psi \circ \varphi)_X = \mathcal{F}_3(f) \circ \psi_X \circ \varphi_X = \psi_Y \circ \mathcal{F}_2(f) \circ \varphi_X = \psi_Y \circ \varphi_Y \circ \mathcal{F}_1(f) = (\psi \circ \varphi)_Y \circ \mathcal{F}_1(f)$$

An *isomorphism of functors* is thus an isomorphism in the category of functors, i.e. $\mathcal{F}_1 \cong \mathcal{F}_2 \;\Leftrightarrow\;$ there are natural transformations $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ and $\psi : \mathcal{F}_2 \to \mathcal{F}_1$ such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}_1}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{F}_2}$, so that

$$\forall\, X \in \mathcal{C}, \quad (\psi \circ \varphi)_X = \mathrm{id}_{\mathcal{F}_1(X)} \quad , \quad (\varphi \circ \psi)_X = \mathrm{id}_{\mathcal{F}_2(X)}$$

In particular, this implies that φ_X and ψ_X are isomorphisms in $\mathcal{C}'$ and $\mathcal{F}_1(X) \cong \mathcal{F}_2(X)$ in $\mathcal{C}'$, $\forall\, X \in \mathcal{C}$.

The converse is however not true : $\mathcal{F}_1(X) \cong \mathcal{F}_2(X)$, $\forall\, X \in \mathcal{C}$ does not imply that $\mathcal{F}_1 \cong \mathcal{F}_2$. If we want to emphasize that functors are isomorphic in term of objects, we say that $\mathcal{F}_1(X) \cong \mathcal{F}_2(X)$ *functorially* in X.

Definition :
A functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is called *faithful* / *full* / *fully faithful* if the map $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}\big(\mathcal{F}(X), \mathcal{F}(Y)\big)$ is injective / surjective / bijective for all objects $X, Y \in \mathcal{C}$. This leads to the following fact :

If $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a fully faithful functor, then $\mathcal{C}$ can be identified with a full subcategory $\mathcal{D}$ of $\mathcal{C}'$, given by

$$\mathrm{Ob}\big(\mathcal{D}\big) = \big\{ \mathcal{F}(X) \;\big|\; X \in \mathrm{Ob}(\mathcal{C}) \big\} \quad , \quad \mathrm{Hom}_{\mathcal{D}}\big(\mathcal{F}(X), \mathcal{F}(Y)\big) = \big\{ \mathcal{F}(f) \;\big|\; f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \big\}$$

showing that $\mathcal{C}$ and $\mathcal{D}$ have the "same" morphisms since $\mathcal{F}$ is bijective on morphisms.

1.2.4 Proposition

Let $\mathcal{C}, \mathcal{C}'$ be categories, $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ a functor and $f : X \to Y$ a given morphism in $\mathcal{C}$. If $\mathcal{F}$ is fully faithful and $\mathcal{F}(f)$ is an isomorphism, then f is an isomorphism. One also says that $\mathcal{F}$ is *conservative*.

Proof. Assume that $\mathcal{F}(f)$ is an isomorphism, i.e.

$$\exists\, G \;:\; \mathcal{F}(Y) \to \mathcal{F}(X) \;\text{ such that }\; \mathcal{F}(f) \circ G = \mathrm{id}_{\mathcal{F}(Y)} \;\text{ and }\; G \circ \mathcal{F}(f) = \mathrm{id}_{\mathcal{F}(X)}$$

$G \in \mathrm{Hom}_{\mathcal{C}'}(\mathcal{F}(Y), \mathcal{F}(X)) \;\Rightarrow\; \exists!\, g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $G = \mathcal{F}(g)$ since $\mathcal{F}$ is full. Hence

$$\begin{aligned} \mathcal{F}(\mathrm{id}_Y) = \mathrm{id}_{\mathcal{F}(Y)} = \mathcal{F}(f) \circ G = \mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(f \circ g) \;&\Rightarrow\; f \circ g = \mathrm{id}_Y \\ \mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)} = G \circ \mathcal{F}(f) = \mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f) \;&\Rightarrow\; g \circ f = \mathrm{id}_X \end{aligned}$$

by faithfulness of $\mathcal{F}$, so $f : X \to Y$ is isomorphism with inverse $f^{-1} = g$. □

1.3 The Yoneda Lemma

The Yoneda Lemma, named after the Japanese mathematician **Nobuo Yoneda** (1930−1996), is actually the Main Theorem of Category Theory. It states the "embedding" of any category into the category of contravariant set-valued functors defined on that category. Here we develop the ideas of [S] in more detail. Let $\mathcal{C}$ be a category and set $\mathcal{C}^\wedge := \mathtt{Fct}(\mathcal{C}^{\mathrm{op}}, \mathtt{Set})$.

1.3.1 The Yoneda functor

We first fix an object $X \in \mathcal{C}$ and define the functor $h_{\mathcal{C}}(X) : \mathcal{C}^{\mathrm{op}} \to \mathtt{Set}$ by

$$\begin{aligned} h_{\mathcal{C}}(X) \;&:\; \mathcal{C}^{\mathrm{op}} \to \mathtt{Set} \;:\; Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) \\ h_{\mathcal{C}}(X) \;&:\; \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Y') \to \mathrm{Hom}_{\mathtt{Set}}\big(\mathrm{Hom}_{\mathcal{C}}(Y, X), \mathrm{Hom}_{\mathcal{C}}(Y', X)\big) \;:\; f \mapsto \circ f \end{aligned}$$

Hence it is a covariant functor on $\mathcal{C}^{\mathrm{op}}$ and we have $h_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X) \in \mathtt{Fct}(\mathcal{C}^{\mathrm{op}}, \mathtt{Set}) = \mathcal{C}^\wedge$. Now we want to analyze what happens with respect to X. In fact $h_{\mathcal{C}}$ defines a covariant functor $\mathcal{C} \to \mathcal{C}^\wedge$ because

$$\begin{aligned} h_{\mathcal{C}} \;&:\; \mathcal{C} \to \mathcal{C}^\wedge \;:\; X \mapsto h_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X) \\ h_{\mathcal{C}} \;&:\; \mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X), \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X')\big) \;:\; f \mapsto h_{\mathcal{C}}(f) = h^f \end{aligned}$$

where $h^f : h_{\mathcal{C}}(X) \to h_{\mathcal{C}}(X')$ is a morphism of functors such that $h^f_Y = f\circ$ with commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y', X) \arrow[r, "h^f_{Y'} = f\circ"] \arrow[d, "\circ g"'] & \mathrm{Hom}_{\mathcal{C}}(Y', X') \arrow[d, "\circ g"] \\ \mathrm{Hom}_{\mathcal{C}}(Y, X) \arrow[r, "h^f_Y = f\circ"] & \mathrm{Hom}_{\mathcal{C}}(Y, X') \end{tikzcd}$$

for all $Y, Y' \in \mathcal{C}$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ because $h_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X)$ is contravariant for all $X \in \mathcal{C}$. In addition $h_{\mathcal{C}}$ is covariant since $\forall\, Y \in \mathcal{C}$,

$$h^{f_1 \circ f_2}_Y = (f_1 \circ f_2)\circ = (f_1\circ) \circ (f_2\circ) = h^{f_1}_Y \circ h^{f_2}_Y = (h^{f_1} \circ h^{f_2})_Y \;\Rightarrow\; h^{f_1 \circ f_2} = h^{f_1} \circ h^{f_2}$$

The covariant functor $h_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\wedge : X \mapsto h_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X)$ is called the *Yoneda functor*.

Next we define the functor $\Gamma : \mathcal{C} \times \mathcal{C}^\wedge \to \mathtt{Set}$ by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ with action on morphisms defined as follows. If we fix $X \in \mathcal{C}$, then

$$\begin{aligned} \Gamma(X, \,\cdot\,) \;:\; \mathcal{C}^\wedge &\longrightarrow \mathtt{Set} \;:\; \mathcal{F} \longmapsto \mathcal{F}(X) \\ (\varphi : \mathcal{F} \to \mathcal{G}) &\longmapsto \big(\varphi_X : \mathcal{F}(X) \to \mathcal{G}(X)\big) \in \mathrm{Hom}_{\mathtt{Set}}\big(\Gamma(X, \mathcal{F}), \Gamma(X, \mathcal{G})\big) \end{aligned}$$

i.e. Γ is covariant in the second argument.

Now fix a functor $\mathcal{F} \in \mathcal{C}^\wedge$ (i.e. $\mathcal{F}$ is contravariant) and set

$$\begin{aligned} \Gamma(\,\cdot\,,\mathcal{F}) \ :\ \mathcal{C} \longrightarrow \texttt{Set} \ :\ X \longmapsto \mathcal{F}(X) \\ (f \,:\, X \to Y) \longmapsto \big(\mathcal{F}(f) \,:\, \mathcal{F}(Y) \to \mathcal{F}(X)\big) \in \mathrm{Hom}_{\texttt{Set}}\big(\Gamma(Y,\mathcal{F}),\Gamma(X,\mathcal{F})\big) \end{aligned}$$

implying that Γ is contravariant in the first argument, i.e. $\Gamma : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^\wedge \to \texttt{Set}$ is a bifunctor. Now we are ready to state and prove the Yoneda Lemma. The proof is however not very instructive and may be skipped.

1.3.2 Theorem (The Yoneda Lemma)

For $X \in \mathcal{C}$ and $\mathcal{F} \in \mathcal{C}^\wedge$, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big) \cong \mathcal{F}(X)$ functorially in X and $\mathcal{F}$, i.e. $\forall\, X \in \mathcal{C}$, $\forall\, \mathcal{F} \in \mathcal{C}^\wedge$ there are isomorphisms of functors

$$\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\,\cdot\,\big) \cong \Gamma(X,\,\cdot\,) \tag{1.2}$$

$$\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(\,\cdot\,),\mathcal{F}\big) \cong \Gamma(\,\cdot\,,\mathcal{F}) = \mathcal{F} \tag{1.3}$$

This makes sense since both functors in (1.2) are covariant and both functors in (1.3) are contravariant.

Proof. 1) We first show that $\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big) \cong \mathcal{F}(X)$ as sets for any fixed $X \in \mathcal{C}$ and $\mathcal{F} \in \mathcal{C}^\wedge$. Define

$$\begin{aligned} \alpha \ :\ \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big) = \mathrm{Hom}_{\mathcal{C}^\wedge}\big(\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,,X),\mathcal{F}\big) \longrightarrow \mathrm{Hom}_{\texttt{Set}}\big(\mathrm{Hom}_{\mathcal{C}}(X,X),\mathcal{F}(X)\big) \longrightarrow \mathcal{F}(X) \\ \alpha \ :\ \varphi \longmapsto \varphi_X \longmapsto \varphi_X(\mathrm{id}_X) \end{aligned}$$

In order to define $\beta : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big)$, it suffices to set for $s \in \mathcal{F}(X)$ and $Y \in \mathcal{C}$,

$$\beta(s)_Y \in \mathrm{Hom}_{\texttt{Set}}\big(\mathrm{Hom}_{\mathcal{C}}(Y,X),\mathcal{F}(Y)\big)$$

and then to check that $\beta(s)$ defines a natural transformation between contravariant functors. We set

$$\beta(s)_Y \ :\ \mathrm{Hom}_{\mathcal{C}}(Y,X) \longrightarrow \mathrm{Hom}_{\texttt{Set}}\big(\mathcal{F}(X),\mathcal{F}(Y)\big) \longrightarrow \mathcal{F}(Y) \ :\ f \longmapsto \mathcal{F}(f) \longmapsto \mathcal{F}(f)(s)$$

which is well-defined since $\mathcal{F}$ is contravariant. In addition, $\forall\, g \in \mathrm{Hom}_{\mathcal{C}}(Y,Y')$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y',X) \arrow[r, "\beta(s)_{Y'}"] \arrow[d, "\circ g"'] & \mathcal{F}(Y') \arrow[d, "\mathcal{F}(g)"] \\ \mathrm{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "\beta(s)_Y"] & \mathcal{F}(Y) \end{tikzcd}$$

$$f \in \mathrm{Hom}_{\mathcal{C}}(Y',X) \ \Rightarrow\ \beta(s)_Y(f \circ g) = \mathcal{F}(f \circ g)(s) = \mathcal{F}(g)\big(\mathcal{F}(f)(s)\big) = \mathcal{F}(g)\big(\beta(s)_{Y'}(f)\big)$$

hence $\beta(s)$ is indeed a natural transformation and we constructed $\beta : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big)$.

2) Now we have to check that the maps α and β are inverse to each other :

$$\forall\, s \in \mathcal{F}(X),\ (\alpha \circ \beta)(s) = \alpha\big(\beta(s)\big) = \beta(s)_X(\mathrm{id}_X) = \mathcal{F}(\mathrm{id}_X)(s) = \mathrm{id}_{\mathcal{F}(X)}(s) = s \quad \Rightarrow\ \alpha \circ \beta = \mathrm{id}_{\mathcal{F}(X)}$$

For computing $\beta \circ \alpha$, let $\varphi \in \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_{\mathcal{C}}(X),\mathcal{F}\big)$, i.e. $\mathcal{F}(g) \circ \varphi_{Z'} = \varphi_Z \circ (\circ g)$, $\forall\, g \in \mathrm{Hom}_{\mathcal{C}}(Z,Z')$. Then

$$\begin{aligned} (\beta \circ \alpha)(\varphi) = \beta\big(\alpha(\varphi)\big) = \beta\big(\varphi_X(\mathrm{id}_X)\big) \\ \Rightarrow\ \beta\big(\varphi_X(\mathrm{id}_X)\big)_Y(f) = \mathcal{F}(f)\big(\varphi_X(\mathrm{id}_X)\big) = \big(\mathcal{F}(f) \circ \varphi_X\big)(\mathrm{id}_X) = \big(\varphi_Y \circ (\circ f)\big)(\mathrm{id}_X) = \varphi_Y(\mathrm{id}_X \circ f) = \varphi_Y(f) \end{aligned}$$

$\forall\, Y \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ and hence $(\beta \circ \alpha)(\varphi) = \beta\big(\varphi_X(\mathrm{id}_X)\big) = \varphi$. Hence α and β are bijections.

3) Next we fix $X \in \mathcal{C}$ and let vary $\mathcal{F} \in \mathcal{C}^\wedge$. Denote

$$\alpha^X \;:\; \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\,\cdot\,\big) \to \Gamma(X,\,\cdot\,) \quad , \quad \alpha^X_\mathcal{F} \;:\; \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big) \to \Gamma(X,\mathcal{F})$$

We know that $\alpha^X_\mathcal{F}$ is an isomorphism for all $\mathcal{F} \in \mathcal{C}^\wedge$. Hence in order to show that α^X is an isomorphism of functors, it remains to prove that it is a natural transformation, i.e. the diagram below must commute :

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big) \arrow[r, "\alpha^X_\mathcal{F}"] \arrow[d, "\psi\circ"'] & \Gamma(X,\mathcal{F}) \arrow[d, "\psi_X"] \\
\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{G}\big) \arrow[r, "\alpha^X_\mathcal{G}"] & \Gamma(X,\mathcal{G})
\end{tikzcd}$$

for any natural transformation $\psi : \mathcal{F} \to \mathcal{G}$. But this is clear : $\forall\, \varphi \in \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big)$,

$$(\psi_X \circ \alpha^X_\mathcal{F})(\varphi) = \psi_X\big(\alpha^X_\mathcal{F}(\varphi)\big) = \psi_X\big(\varphi_X(\mathrm{id}_X)\big) = (\psi\circ\varphi)_X(\mathrm{id}_X) = \alpha^X_\mathcal{G}(\psi\circ\varphi) = \big(\alpha^X_\mathcal{G}\circ(\psi\circ)\big)(\varphi)$$

Hence α^X is an isomorphism of functors, which proves (1.2), and its inverse is given by

$$\beta^X \;:\; \Gamma(X,\,\cdot\,) \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\,\cdot\,\big) \quad , \quad \beta^X_\mathcal{F} \;:\; \Gamma(X,\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big)$$

4) Finally we fix $\mathcal{F} \in \mathcal{C}^\wedge$ and let vary $X \in \mathcal{C}$. We denote similarly

$$\alpha^\mathcal{F} \;:\; \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(\,\cdot\,),\mathcal{F}\big) \to \Gamma(\,\cdot\,,\mathcal{F}) = \mathcal{F} \quad , \quad \alpha^\mathcal{F}_X \;:\; \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big) \to \mathcal{F}(X)$$

where $\alpha^\mathcal{F}_X$ is again an isomorphism for all $X \in \mathcal{C}$. It remains to show that $\alpha^\mathcal{F}$ is a natural transformation.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(Y),\mathcal{F}\big) \arrow[r, "\alpha^\mathcal{F}_Y"] \arrow[d, "\circ h^g"'] & \mathcal{F}(Y) \arrow[d, "\mathcal{F}(g)"] \\
\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),\mathcal{F}\big) \arrow[r, "\alpha^\mathcal{F}_X"] & \mathcal{G}(X)
\end{tikzcd}$$

Let $g \in \mathrm{Hom}_\mathcal{C}(X,Y)$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(Y),\mathcal{F}\big)$. Then $\mathcal{F}(g)\circ\alpha^\mathcal{F}_Y = \alpha^\mathcal{F}_X \circ (\circ h^g)$ because

$$\mathcal{F}(g)\big(\alpha^\mathcal{F}_Y(\varphi)\big) = \mathcal{F}(g)\big(\varphi_Y(\mathrm{id}_Y)\big) = \big(\mathcal{F}(g)\circ\varphi_Y\big)(\mathrm{id}_Y) = \big(\varphi_X\circ(\circ g)\big)(\mathrm{id}_Y) = \varphi_X(\mathrm{id}_Y\circ g) = \varphi_X(g)$$
$$\big(\alpha^\mathcal{F}_X\circ(\circ h^g)\big)(\varphi) = \alpha^\mathcal{F}_X\big(\varphi\circ h^g\big) = (\varphi\circ h^g)_X(\mathrm{id}_X) = \varphi_X\big(h^g_X(\mathrm{id}_X)\big) = \varphi_X(g\circ\mathrm{id}_X) = \varphi_X(g)$$

Thus $\alpha^\mathcal{F}$ is an isomorphism as well and we showed (1.3). This finishes the proof. □

1.3.3 Corollary

The functor $h_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\wedge$ is fully faithful.

Proof. We have to show that $h_\mathcal{C} : \mathrm{Hom}_\mathcal{C}(X,Y) \to \mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),h_\mathcal{C}(Y)\big)$ is a bijection of sets, $\forall\, X,Y \in \mathcal{C}$. Choose $\mathcal{F} = h_\mathcal{C}(Y)$. This is a contravariant functor, hence by Yoneda :

$$\mathrm{Hom}_{\mathcal{C}^\wedge}\big(h_\mathcal{C}(X),h_\mathcal{C}(Y)\big) \cong h_\mathcal{C}(Y)(X) = \mathrm{Hom}_\mathcal{C}(X,Y)$$

And this isomorphism is indeed given by $\beta = h_\mathcal{C}$ because $\forall\, s \in \mathcal{F}(X) = \mathrm{Hom}_\mathcal{C}(X,Y)$, $f \in \mathrm{Hom}_\mathcal{C}(Y,X)$,

$$\beta(s)_Y(f) = \mathcal{F}(f)(s) = h_\mathcal{C}(Y)(f)(s) = (\circ f)(s) = s\circ f \quad , \quad h_\mathcal{C}(s)_Y(f) = h^s_Y(f) = s\circ f$$

□

1.3.4 Conclusion

It follows that any category $\mathcal{C}$ can be identified with a full subcategory of $\mathcal{C}^\wedge = \texttt{Fct}(\mathcal{C}^{\text{op}}, \texttt{Set})$. This is why $h_\mathcal{C}$ is also called the *Yoneda embedding*. In particular, any category $\mathcal{C}$ inherits a lot of the properties of the category $\texttt{Set}$. Moreover full faithfulness of $h_\mathcal{C}$ implies that any morphism of functors $h_\mathcal{C}(X) \to h_\mathcal{C}(Y)$ is in fact induced by a morphism $X \to Y$ in $\mathcal{C}$:

$$\forall \big(\varphi : \text{Hom}_\mathcal{C}(\,\cdot\,, X) \to \text{Hom}_\mathcal{C}(\,\cdot\,, Y) \big) \ , \ \exists\, f \in \text{Hom}_\mathcal{C}(X, Y) \ \text{ such that } \ \varphi = h_\mathcal{C}(f) = h^f = f\circ \tag{1.4}$$

1.3.5 Representable functors

Let $\mathcal{C}$ be a category. We say that a functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \texttt{Set}$ is *representable* if there exists an object $X \in \mathcal{C}$ such that $F(Y) \cong \text{Hom}_\mathcal{C}(Y, X)$ functorially in $Y \in \mathcal{C}$, i.e. if $\mathcal{F} \cong \text{Hom}_\mathcal{C}(\,\cdot\,, X) = h_\mathcal{C}(X)$ as functors in $\mathcal{C}^\wedge$.

Similarly, a functor $\mathcal{G} : \mathcal{C} \to \texttt{Set}$ is called representable if $\exists\, X \in \mathcal{C}$ such that $\mathcal{G} \cong \text{Hom}_\mathcal{C}(X, \,\cdot\,)$.

Representability of functors will be important for studying moduli problems later on.

Chapter 2

The spectrum of a ring

After the introductory chapter, we start with the main part of the thesis. First we define the notion of a sheaf and give some recalls on Commutative Algebra. Then we introduce the spectrum of a ring, the set of all (proper) prime ideals of the ring, and endow it with the Zariski topology and a structure sheaf, turning it into a locally ringed space. The motivation is to somehow "enrich" the Zariski topology of an affine space by adding non-closed points. Moreover we will see that the spectrum defines a contravariant functor from the category of commutative unital rings to the category of locally ringed spaces. This functor is in addition fully faithful, so that we have a contravariant equivalence between the category of commutative unital rings and a full subcategory of the category of locally ringed spaces. It will turn out that this full subcategory is exactly the category of affine schemes. Most of this chapter is taken from [Ha], [S] and [Sch].

2.1 Sheaves

2.1.1 Presheaves

Let X be a topological space and R a commutative unital ring. Here we follow the notations of [S].
We denote by OP_X the set of all open sets of X and turn it into a category by defining as morphisms

$$\mathrm{Hom}_{\mathrm{OP}_X}(U,V) = \begin{cases} \{U \hookrightarrow V\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

i.e. there is a unique morphism $U \to V$ (the inclusion) $\Leftrightarrow$ $U \subseteq V$. This is indeed a category.

A *presheaf* of R-modules on X is a functor $\mathcal{F}$ from $\mathrm{OP}_X^{\mathrm{op}}$ to $\mathtt{Mod}(R)$ and we denote

$$\mathcal{F} \in \mathtt{PSh}(R_X) := \mathtt{Fct}\left(\mathrm{OP}_X^{\mathrm{op}}, \mathtt{Mod}(R)\right)$$

Translating into "concrete terms", this means : A presheaf $\mathcal{F}$ of R-modules on X is the data of

1) an R-module $\mathcal{F}(U)$ for every open subset $U \subseteq X$
2) an R-module homomorphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, called *restriction morphism*, for every inclusion of open sets $V \subseteq U$

such that $\mathcal{F}(\emptyset) = \{0\}$ and the restriction morphisms behave functorially, i.e.

a) $\rho_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map $\mathrm{id}_{\mathcal{F}(U)}$
b) for every inclusion of open sets $W \subseteq V \subseteq U$, we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.

Elements of $\mathcal{F}(U)$ are also called *sections* of the presheaf $\mathcal{F}$ over the open subset U and we sometimes denote $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$. Note that sections do not need to be functions and that restriction morphisms are not necessarily restrictions of functions. Nevertheless if $s \in \mathcal{F}(U)$, we often write $s_{|V} := \rho_V^U(s) \in \mathcal{F}(V)$.

Remark :
One can define more general presheaves by dropping the condition that the $\mathcal{F}(U)$ are modules over some ring. Indeed one can as well define presheaves as functors $\mathtt{Fct}\left(\mathrm{OP}_X^{\mathrm{op}}, \mathtt{Set}\right)$ or even $\mathtt{Fct}\left(\mathrm{OP}_X^{\mathrm{op}}, \mathcal{C}\right)$ for some arbitrary category $\mathcal{C}$. We will e.g. encounter (pre)sheaves of rings (i.e. $\mathcal{C} = \mathtt{Ring}$) in the context of schemes later on. Modules over a ring R (in particular vector spaces if R is a field or abelian groups if $R = \mathbb{Z}$) will be sufficient for the moment.

Morphisms of presheaves
Being functors, presheaves on X form a category by definition. In particular, we can speak of morphism between presheaves : let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is thus given by a family of R-module homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, $\forall\, U \subseteq X$ open, which commute with the restriction morphisms of $\mathcal{F}$ and $\mathcal{G}$, i.e. for any inclusion of open sets $V \subseteq U$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_V^U"'] & \mathcal{G}(U) \arrow[d, "\rho_V'^U"] \\
\mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

Again we say that $\varphi : \mathcal{F} \to \mathcal{G}$ is an *isomorphism of presheaves* if there is a morphism of presheaves $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$. In particular, $\mathcal{F}(U) \cong \mathcal{G}(U)$, $\forall\, U \subseteq X$ open in that case.

Restriction of presheaves
The *restriction* of a presheaf $\mathcal{F} \in \mathtt{PSh}(R_X)$ to an open subset $U \subseteq X$, denoted by $\mathcal{F}_{|U}$, is defined by $V \mapsto \mathcal{F}(V)$ for $V \subseteq U$ open with the same restriction morphisms as $\mathcal{F}$ (whenever defined). Hence $\mathcal{F}_{|U}$ is a presheaf on U and restriction of presheaves defines a functor

$$(\,\cdot\,)_{|U} \;:\; \mathtt{PSh}(R_X) \to \mathtt{PSh}(R_U)$$

2.1.2 Germs and stalks

Let $\mathcal{F}$ be a presheaf on a topological space X and $x \in X$. Consider two open neighborhoods U, V of x and sections $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$. We define the equivalence relation

$$(s, U) \sim_x (t, V) \;\Leftrightarrow\; \exists\, W \subseteq U \cap V \text{ open such that } x \in W \text{ and } \rho_W^U(s) = \rho_W^V(t) \;\Leftrightarrow\; s_{|W} = t_{|W}$$

i.e. we say that s and t are equivalent with respect to x if they coincide on some smaller neighborhood of x.

The equivalence classes of this relation are called *germs* of sections of $\mathcal{F}$ at x and denoted by

$$[s]_x := \left\{ (t, V) \;\middle|\; (s, U) \sim_x (t, V) \right\}$$

Usually the domain of a germ is not specified since we only consider small open neighborhoods of the given point $x \in X$. Representatives of germs are however given by pairs (s, U) where $s \in \mathcal{F}(U)$ and two representatives are identified if the corresponding sections coincide on some smaller open neighborhood of x. The set of all germs is called the *stalk* of $\mathcal{F}$ at x and denoted by

$$\mathcal{F}_x := \left\{ [s]_x \;\middle|\; s \text{ is a section of } \mathcal{F} \text{ over an open neighborhood of } x \right\}$$

$\mathcal{F}_x$ is again an R-module for all $x \in X$ with respect to the following definitions :
Let $[s]_x, [t]_x \in \mathcal{F}_x$ be represented by $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ where $x \in U \cap V$. Then we set

$$[s]_x + [t]_x := \left[\rho_{U\cap V}^U(s) + \rho_{U\cap V}^V(t)\right]_x$$

If $r \in R$ and $[s]_x \in \mathcal{F}_x$ is represented by $s \in \mathcal{F}(U)$, we also define $r \cdot [s]_x := [r \cdot s]_x$. One checks that both definitions are indeed independent of the chosen representatives.

Moreover any morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ induces an R-module homomorphism on the stalks :

$$\varphi_x \;:\; \mathcal{F}_x \to \mathcal{G}_x \;:\; [s]_x \mapsto \big[\varphi_U(s)\big]_x$$

where $s \in \mathcal{F}(U)$ is a representative of $[s]_x$. This is well-defined since φ commutes with restrictions.

Remark :
This definition of germs and stalks is somehow the translation of the actual definition into "concrete terms". For example, [S] and [Ha] define the stalk of a presheaf as the direct limit $\mathcal{F}_x = \varinjlim \mathcal{F}(U)$ with respect to the direct system of open neighborhoods of x in X and ordered by reversed inclusion. Direct limits are a quite general tool for constructing new objects and we will not explain how they are defined here.

2.1.3 Sheaves

Let $\mathcal{F} \in \mathtt{PSh}(R_X)$ be a presheaf. We say that $\mathcal{F}$ is a *sheaf* if it satisfies in addition the axioms

S1 For any open subset $U \subseteq X$, any open covering $U = \bigcup_i U_i$ and $\forall\, s \in \mathcal{F}(U)$ such that $s_{|U_i} = 0$, $\forall\, i$, we have $s = 0$.

S2 For any open subset $U \subseteq X$, any open covering $U = \bigcup_i U_i$ and any family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$, $\forall\, i, j$, there exists a section $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$, $\forall\, i$.

S1 implies that the section s in S2 is necessarily unique. Condition S1 is called *local identity* and requires that every section that is locally zero is indeed zero. S2 is the *gluing property* and says that every family of locally defined sections that glue on intersections defines a global section.

Hence sheaves are presheaves whose sections are determined by local data. The set of all sheaves on X is denoted by $\mathtt{Sh}(R_X)$. In particular, we see that sheaves differ from presheaves only by additional properties (the axioms S1 and S2) and it follows that $\mathtt{Sh}(R_X)$ is a full subcategory of $\mathtt{PSh}(R_X)$. Thus sheaves and presheaves admit exactly the same morphisms :

$$\forall\, \mathcal{F}, \mathcal{G} \in \mathtt{Sh}(R_X) \;:\; \mathrm{Hom}_{\mathtt{Sh}(R_X)}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathtt{PSh}(R_X)}(\mathcal{F}, \mathcal{G})$$

However there exist presheaves that are not sheaves, e.g. the presheaf of bounded real-valued functions on X or the presheaf of constant functions on X, hence $\mathtt{Sh}(R_X) \subsetneq \mathtt{PSh}(R_X)$.
Obviously, if $\mathcal{F}$ is a sheaf on X, then every restriction $\mathcal{F}_{|U}$ for any open subset $U \subseteq X$ is a sheaf as well.

The following proposition, again taken from [S], illustrates the local nature of a sheaf.

2.1.4 Proposition

Let $\mathcal{F}, \mathcal{G}$ be sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Then φ is an isomorphism of sheaves if and only if the induced map on the stalk $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism of R-modules, $\forall\, x \in X$.

<u>Remark</u> : This is false if $\mathcal{F}$ and $\mathcal{G}$ are presheaves only.

Proof. <u>$\Rightarrow$</u> : If $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism, then $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism, $\forall\, U \subseteq X$ open. Let $[s]_x \in \mathcal{F}_x$ be such that $\varphi_x\big([s]_x\big) = 0$, i.e. if $s \in \mathcal{F}(U)$ is a representative of $[s]_x$, then $[\varphi_U(s)]_x = 0$, which means that $\exists\, W \subseteq U$ open with $x \in W$ such that $\varphi_U(s)_{|W} = 0$. But φ commutes with restrictions :

$$0 = \varphi_U(s)_{|W} = \varphi_W(s_{|W}) \;\Rightarrow\; s_{|W} = 0 \;\text{ since } \varphi_W \text{ is injective}$$

and it follows that $[s]_x = 0$ since s coincides with 0 on W, hence φ_x is injective. Now let $[t]_x \in \mathcal{G}_x$ be represented by $t \in \mathcal{G}(U)$. Then $\exists\, s \in \mathcal{F}(U)$ such that $\varphi_U(s) = t$ by surjectivity of φ_U and

$$\varphi_x\big([s]_x\big) = [\varphi_U(s)]_x = [t]_x \;\Rightarrow\; \varphi_x \text{ is also surjective}$$

<u>$\Leftarrow$</u> : Assume that $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, $\forall\, x \in X$. Since φ is a morphism of sheaves (i.e. it already commutes with restrictions), it suffices to show that each $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

– φ_U is injective : let $s \in \mathcal{F}(U)$ such that $\varphi_U(s) = 0$, hence for every $x \in U$, $\varphi_x\big([s]_x\big) = 0$, implying that $[s]_x = 0$ by injectivity of φ_x and there is an open neighborhood U_x of x in U such that $s_{|U_x} = 0$. This can be done for any $x \in U$, so we can write

$$U = \bigcup_{x \in U} U_x$$

with $s_{|U_x} = 0$, $\forall\, x \in U$. $\mathcal{F}$ satisfying S1, it follows that $s = 0$, hence that φ_U is injective.

– φ_U is surjective : let $t \in \mathcal{G}(U)$. Then $[t]_x \in \mathcal{G}_x$, $\forall\, x \in U$ and $\exists\, [s]_x \in \mathcal{F}_x$ such that $\varphi_x\big([s]_x\big) = [t]_x$ since φ_x is surjective. Let $[s]_x$ be represented by $\sigma \in \mathcal{F}(V_x)$ for some open neighborhood V_x of x. Then

$$[t]_x = \varphi_x\big([s]_x\big) = \big[\varphi_{V_x}(\sigma)\big]_x$$

i.e. t and $\varphi_{V_x}(\sigma)$ are sections of $\mathcal{F}$ (over different open subsets) whose germs at x are the same, hence there is an open neighborhood $U_x \subseteq U \cap V_x$ of x such that $t_{|U_x} = \varphi_{V_x}(\sigma)_{|U_x} = \varphi_{U_x}(\sigma_{|U_x})$ with $\sigma_{|U_x} \in \mathcal{F}(U_x)$. Summarizing and changing notations (write U_i instead of U_x), we get :
There is an open covering $U = \bigcup_i U_i$ and $\exists\, s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t_{|U_i}$, $\forall\, i$. Denote $U_{ij} = U_i \cap U_j$:

$$\varphi_{U_{ij}}(s_{i|U_{ij}}) = \big(\varphi_{U_i}(s_i)\big)_{|U_{ij}} = (t_{|U_i})_{|U_{ij}} = t_{|U_{ij}} = (t_{|U_j})_{|U_{ij}} = \big(\varphi_{U_j}(s_j)\big)_{|U_{ij}} = \varphi_{U_{ij}}(s_{j|U_{ij}})$$

thus $s_{i|U_{ij}} = s_{j|U_{ij}}$ since we already know that $\varphi_{U_{ij}}$ is injective, $\forall\, i, j$. So the s_i glue on intersections and $\exists\, s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$, $\forall\, i$ since $\mathcal{F}$ satisfies S2. Then $\varphi_U(s) = t$ because

$$\forall\, i,\ t_{|U_i} = \varphi_{U_i}(s_i) = \varphi_{U_i}(s_{|U_i}) = \big(\varphi_U(s)\big)_{|U_i} \ \Rightarrow\ \varphi_U(s) = t \text{ by local identity}$$

since $\mathcal{G}$ satisfies S1. Finally s is a preimage of t and we obtain surjectivity of φ_U. □

2.1.5 Proposition

Let $\mathcal{F}, \mathcal{G}$ be sheaves on X and $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ two morphisms of sheaves. Then $\varphi = \psi \ \Leftrightarrow\ \varphi_x = \psi_x$, $\forall\, x \in X$.

Proof. $\underline{\Rightarrow}$: clear by definition
$\underline{\Leftarrow}$: We have to show that $\varphi_U = \psi_U$, $\forall\, U \subseteq X$ open. Let $s \in \mathcal{F}(U)$. Then $\forall\, x \in U$,

$$\begin{aligned} \varphi_x\big([s]_x\big) = \psi_x\big([s]_x\big) \ &\Leftrightarrow\ \big[\varphi_U(s)\big]_x = \big[\psi_U(s)\big]_x \\ &\Leftrightarrow\ \exists\, W_x \subseteq U \text{ open with } x \in W_x \text{ such that } \varphi_U(s)_{|W_x} = \psi_U(s)_{|W_x} \end{aligned}$$

Hence we have an open cover $U = \bigcup_{x \in U} W_x$ and it follows from S1 that $\varphi_U(s) = \psi_U(s)$. □

Remark :
One may interpret this result as follows : $\varphi_x = \psi_x$ means that φ and ψ coincide in a small neighborhood U_x of any point $x \in X$, hence by S1 they coincide everywhere since $X = \bigcup_{x \in X} U_x$.

2.1.6 Lemma

Let $\mathcal{F}$ be a sheaf on X and $U_1, U_2 \subseteq X$ open subsets. Denote $U_{12} = U_1 \cap U_2$. Then the sequence

$$0 \longrightarrow \mathcal{F}(U_1 \cup U_2) \xrightarrow{\alpha} \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \xrightarrow{\beta} \mathcal{F}(U_1 \cap U_2)$$

where $\alpha(s) = (s_{|U_1}, s_{|U_2})$ and $\beta(s_1, s_2) = s_{1|U_{12}} - s_{2|U_{12}}$ is exact, i.e. α is injective and $\operatorname{im}\alpha = \ker\beta$.

Proof. – α injective : if $s_{|U_1} = s_{|U_2} = 0$, then $s = 0$ by S1.
– $\operatorname{im}\alpha \subset \ker\beta$: $(\beta \circ \alpha)(s) = (s_{|U_1})_{|U_{12}} - (s_{|U_2})_{|U_{12}} = s_{|U_{12}} - s_{|U_{12}} = 0$.
– $\ker\beta \subset \operatorname{im}\alpha$: if $\beta(s_1, s_2) = 0$, i.e. $s_{1|U_{12}} = s_{2|U_{12}}$, then S2 implies that $\exists\, s \in \mathcal{F}(U_1 \cup U_2)$ such that $s_{|U_1} = s_1$ and $s_{|U_2} = s_2 \ \Rightarrow\ \alpha(s) = (s_1, s_2)$. □

As a consequence, we obtain that $\mathcal{F}(U_1 \cup U_2) \cong \operatorname{im}\alpha = \ker\beta = \big\{ (s_1, s_2) \;\big|\; s_i \in \mathcal{F}(U_i),\; s_{1|U_{12}} = s_{2|U_{12}} \big\}$.
More generally, we have

$$\mathcal{F}\Big(\bigcup_{i\in I} U_i\Big) \;\cong\; \Big\{ \{s_i\}_i \in \prod_{i\in I} \mathcal{F}(U_i) \;\Big|\; s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j},\, \forall\, i,j \Big\} \tag{2.1}$$

Note that (2.1) is actually a direct consequence of the defining axioms S1 and S2.

2.1.7 Direct image of a sheaf

So far we only considered sheaves on a fixed topological space. Now we define an operation on sheaves that is associated with a continuous map between 2 topological spaces. Here we also follow the notations of [S], however with a more explicit presentation, e.g. by writing out formulas or developing proofs in more detail.

Let $f : X \to Y$ be a continuous map of topological spaces. We define the functor $f^t : \mathrm{OP}_Y^{\mathrm{op}} \to \mathrm{OP}_X^{\mathrm{op}}$ by

$$V \subseteq Y \text{ open}\,,\; f^t(V) := f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\}$$

and f^t assigns to any inclusion of open sets in Y the corresponding inclusion of preimages in X, i.e.

$$\begin{aligned} f^t \;:\; \operatorname{Hom}_{\mathrm{OP}_Y^{\mathrm{op}}}(V,W) &\longrightarrow \operatorname{Hom}_{\mathrm{OP}_X^{\mathrm{op}}}\big(f^t(V), f^t(W)\big) \\ (W \hookrightarrow V) &\longmapsto \big(f^{-1}(W) \hookrightarrow f^{-1}(V)\big) \end{aligned}$$

where $f^{-1}(V)$ and $f^{-1}(W)$ are open in X since f is continuous. Moreover it shows that f^t is covariant since taking preimages respects inclusion.

Now let $\mathcal{F} \in \mathtt{PSh}(R_X)$ be a presheaf on X. The *direct image* of $\mathcal{F}$ by f, denoted by $f_*\mathcal{F}$, is given by

$$f_*\mathcal{F}(V) := \mathcal{F}\big(f^t(V)\big) = \mathcal{F}\big(f^{-1}(V)\big),\; \forall\, V \subseteq Y \text{ open}$$

Hence $f_*\mathcal{F} \in \mathtt{PSh}(R_Y)$ because it is a composition of 2 functors :

$$\begin{tikzcd} \mathrm{OP}_X^{\mathrm{op}} \arrow[r, "\mathcal{F}"] & \mathtt{Mod}(R) \\ \mathrm{OP}_Y^{\mathrm{op}} \arrow[u, "f^t"] \arrow[ur, "f_*\mathcal{F}"'] & \end{tikzcd}$$

In particular, if $W \subseteq V$ is an inclusion of open sets in Y, the restriction morphisms of $f_*\mathcal{F}$ are

$$\rho^V_{*W} = (\mathcal{F} \circ f^t)(W \hookrightarrow V) = \mathcal{F}\big(f^{-1}(W) \hookrightarrow f^{-1}(V)\big) = \rho^{f^{-1}(V)}_{f^{-1}(W)} \;:\; f_*\mathcal{F}(V) \to f_*\mathcal{F}(W) \tag{2.2}$$

Now let $\mathcal{F}$ be a sheaf on X. Then $f_*\mathcal{F}$ is a sheaf on Y, so we have an assignment $f_* : \mathtt{Sh}(R_X) \to \mathtt{Sh}(R_Y)$.

Proof. If $\{V_i\}_i$ is an open covering of $V \subseteq Y$, then $\{f^{-1}(V_i)\}_i$ is an open covering of $f^{-1}(V) \subseteq X$.
– S1 : let $s \in f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$ such that $s_{|V_i} = 0$, $\forall\, i$, which means by definition that

$$s_{|V_i} = \rho^V_{*V_i}(s) = \rho^{f^{-1}(V)}_{f^{-1}(V_i)} = s_{|f^{-1}(V_i)} = 0,\; \forall\, i \quad\Rightarrow\quad s = 0 \text{ since } \mathcal{F} \text{ satisfies S1}$$

– S2 : let $s_i \in f_*\mathcal{F}(V_i) = \mathcal{F}\big(f^{-1}(V_i)\big)$, $\forall\, i$, such that $s_{i|V_i\cap V_j} = s_{j|V_i\cap V_j}$, $\forall\, i,j$, which again means

$$s_{i|f^{-1}(V_i\cap V_j)} = s_{j|f^{-1}(V_i\cap V_j)} \;\Leftrightarrow\; s_{i|f^{-1}(V_i)\cap f^{-1}(V_j)} = s_{j|f^{-1}(V_i)\cap f^{-1}(V_j)},\; \forall\, i,j$$

$\mathcal{F}$ satisfies S2 $\Rightarrow$ $\exists\, s \in \mathcal{F}\big(f^{-1}(V)\big)$ such that $s_{|f^{-1}(V_i)} = s_i = s_{|V_i}$, $\forall\, i$. $\square$

2.1.8 Remark

One would expect a relation on the stalks like $(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x$, $\forall\, x \in X$. But this is not true in general! However there is a natural map $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$. Indeed,

$$(f_*\mathcal{F})_{f(x)} = \left\{ [s]^*_{f(x)} \;\middle|\; s \in f_*\mathcal{F}(V),\ V \text{ is an open neighborhood of } f(x) \right\}$$

where the classes $[\ ]^*$ are defined with respect to the restrictions ρ_*. Let $s \in f_*\mathcal{F}(V)$ be a representative of $[s]^*_{f(x)}$, i.e. $s \in \mathcal{F}\big(f^{-1}(V)\big)$ and V is an open neighborhood of $f(x)$ in Y. If $t \in f_*\mathcal{F}(V')$ is another representative, this means that

$$\begin{aligned}
[s]^*_{f(x)} = [t]^*_{f(x)} \;&\Leftrightarrow\; \exists\, W \subseteq V \cap V' \text{ open with } f(x) \in W \text{ such that } \rho^V_{*W}(s) = \rho^{V'}_{*W}(t) \\
&\Rightarrow\; x \in f^{-1}(W) \subseteq f^{-1}(V) \cap f^{-1}(V') \text{ with } s_{|f^{-1}(W)} = t_{|f^{-1}(W)} \;\Rightarrow\; [s]_x = [t]_x
\end{aligned}$$

We see that sections of the sheaf $f_*\mathcal{F}$ over an open set $V \subseteq Y$ which represent the same germ in $(f_*\mathcal{F})_{f(x)}$ are sections of the original sheaf $\mathcal{F}$ over the open set $f^{-1}(V)$ that define the same germ in $\mathcal{F}_x$. So the map

$$(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x \;:\; [s]^*_{f(x)} \mapsto [s]_x \tag{2.3}$$

actually "does nothing". Moreover it is an R-module homomorphism, but in general neither injective nor surjective. It is only an isomorphism if f is a homeomorphism.

2.2 Recalls from Commutative Algebra

The goal of this section is to fix notations and to recall some important results from commutative algebra. These results being well-known, no proofs will be given. They can be found in any book about commutative algebra, for example in [Sch].

2.2.1 Algebraic sets and varieties

Let $\mathbb{K}$ be a field (not necessarily algebraically closed) and denote by $\mathbb{A}^n(\mathbb{K})$ the n-dimensional *affine space* over $\mathbb{K}$, i.e. we have $\mathbb{A}^n(\mathbb{K}) \cong \mathbb{K}^n$ but not canonically. Denote by $R_n := \mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring in n variables with coefficients in $\mathbb{K}$.
Classical algebraic geometry is interested in the study of the sets of points where a given set of polynomials have a common zero (if we plug in coordinates), i.e. a subset $A \subseteq \mathbb{K}^n$ should be a geometric object if there exist finitely many polynomials $f_1, \ldots, f_s \in R_n$ such that

$$(x_1, \ldots, x_n) \in A \;\Leftrightarrow\; f_1(x_1, \ldots, x_n) = \ldots = f_s(x_1, \ldots, x_n) = 0$$

Hence we say that a subset $A \subseteq \mathbb{K}^n$ is an *(affine) algebraic set* if $\exists\, f_1, \ldots, f_s \in R_n$ such that

$$A = Z(f_1, \ldots, f_m) := \left\{ (x_1, \ldots, x_n) \in \mathbb{K}^n \;\middle|\; f_1(x_1, \ldots, x_n) = \ldots = f_s(x_1, \ldots, x_n) = 0 \right\}$$

where $Z(\ )$ is the *common zero set* of the polynomials $f_1, \ldots, f_s$. Obviously, $\emptyset = Z(\mathbf{1})$, $\mathbb{K}^n = Z(\mathbf{0})$ and

$$Z(f_1, \ldots, f_s) = Z(f_1) \cap \ldots \cap Z(f_s) \;\Rightarrow\; Z(f_1, \ldots, f_s) \cap Z(g_1, \ldots, g_r) = Z(f_1, \ldots, f_s, g_1, \ldots, g_r)$$
$$Z(f_1, \ldots, f_s) \cup Z(g_1, \ldots, g_r) = Z(f_i \cdot g_j,\ i = 1, \ldots, s,\ j = 1, \ldots, r)$$

where $\mathbf{1}$ and $\mathbf{0}$ are the constant polynomials, showing that $\emptyset$ and $\mathbb{K}^n$ are algebraic sets and that finite unions and finite intersections of algebraic sets are again algebraic sets.
Moreover every point is an algebraic set : if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$, let $f_i = X_i - \alpha_i$, $\forall\, i \in \{1, \ldots, n\}$, then

$$\{\alpha\} = Z(f_1, f_2, \ldots, f_n) = Z(X_1 - \alpha_1\,,\, X_2 - \alpha_2\,, \ldots,\, X_n - \alpha_n)$$

This implies that every finite set of points in $\mathbb{K}^n$ is an algebraic set as well.

An algebraic set $A \subseteq \mathbb{K}^n$ is called *irreducible*
$\Leftrightarrow$ if one can write $A = A_1 \cup A_2$ for some algebraic sets $A_1, A_2 \subseteq \mathbb{K}^n$, then either $A_1 = A$ or $A_2 = A$.
An irreducible algebraic set is also called a *variety*. Otherwise we say that A is *reducible*.

2.2.2 Ideals

Let R be a commutative unital ring. A subset $I \subseteq R$ is called an *ideal* of R, denoted by $I \trianglelefteq R$, if I is closed under addition and under multiplication with the whole ring, i.e. if $0 \in I$, $I + I \subseteq I$ and $R \cdot I \subseteq I$. This implies that $I = R$ whenever $1 \in I$.
Given a (non-empty) family of elements $\{a_\lambda\}_{\lambda\in\Lambda}$ with $a_\lambda \in R$, we denote the *ideal generated by* this family

$$\big\langle \{a_\lambda\}_{\lambda\in\Lambda} \big\rangle := \Big\{ \sum_{\lambda\in\Lambda}{}' r_\lambda \, a_\lambda \;\Big|\; r_\lambda \in R \Big\}$$

where $\sum'$ means that only finitely many coefficients in the sum are non-zero. In particular, $\forall\, a \in R$,

$$\langle\, a \,\rangle := \{\, r \cdot a \mid r \in R \,\}$$

Ideals that are generated by a single element are called *principal ideals*. If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a family of ideals, we define their *sum* as

$$\sum_{\lambda\in\Lambda} I_\lambda := \Big\{ \sum_{\lambda\in\Lambda}{}' b_\lambda \;\Big|\; b_\lambda \in I_\lambda \Big\}$$

Moreover $\bigcap\limits_{\lambda\in\Lambda} I_\lambda$ is again an ideal, but $\bigcup\limits_{\lambda\in\Lambda} I_\lambda$ is in general not an ideal and we have $\sum\limits_{\lambda\in\Lambda} I_\lambda = \big\langle \bigcup\limits_{\lambda\in\Lambda} I_\lambda \big\rangle$.

Finally, the *product* of finitely many ideals $I_1, \dots, I_m$ is given by

$$I_1 \cdot \ldots \cdot I_m = \Big\{ \sum{}' a_1 \cdot \ldots \cdot a_m \;\Big|\; a_i \in I_i,\ \forall\, i \Big\}$$

Radical ideals

The *radical* of an ideal $I \trianglelefteq R$ is defined by $\mathrm{Rad}(I) := \big\{\, r \in R \mid \exists\, n \in \mathbb{N} \text{ such that } r^n \in I \,\big\}$.
Then $I \trianglelefteq \mathrm{Rad}(I) \trianglelefteq R$. Hence $\mathrm{Rad}(R) = R$ and if I, J are ideals in R, we have

$$I \subseteq J \;\Rightarrow\; \mathrm{Rad}(I) \subseteq \mathrm{Rad}(J) \quad , \quad \mathrm{Rad}\big(\mathrm{Rad}(I)\big) = \mathrm{Rad}(I) \quad , \quad \mathrm{Rad}(I) + \mathrm{Rad}(J) \subseteq \mathrm{Rad}(I + J)$$

A special case is given by

$$\mathrm{nil}(R) := \mathrm{Rad}\big(\{0\}\big) = \big\{\, r \in R \mid \exists\, n \in \mathbb{N} \text{ such that } r^n = 0 \,\big\}$$

$\mathrm{nil}(R)$ is called the *nil-radical* of R and contains all nilpotent elements in R, which are in particular zero divisors. R is called a *reduced ring* if $\mathrm{nil}(R) = \{0\}$, i.e. if R has no non-trivial nilpotent elements. Hence integral domains are reduced rings.
$I \trianglelefteq R$ is called a *radical ideal* if $I = \mathrm{Rad}(I)$. Thus $\mathrm{Rad}(I)$ is a radical ideal. Moreover we have that

$$I \text{ is a radical ideal} \quad \Leftrightarrow \quad R/I \text{ is a reduced ring}$$

Prime ideals

An ideal $P \trianglelefteq R$ is called a *prime ideal* $\Leftrightarrow$ 1) $P \neq R$, i.e. P is a proper ideal
2) if $a, b \in R$ are such that $a \cdot b \in P$, then either $a \in P$ or $b \in P$.

Prime ideals are in particular radical ideals and $I \trianglelefteq R$ is prime $\Leftrightarrow$ R/I is an integral domain.

Maximal ideals

An ideal $M \trianglelefteq R$ is called a *maximal ideal* $\Leftrightarrow$
1) $M \neq R$
2) if $\exists\, M' \trianglelefteq R$ such that $M \subseteq M' \subseteq R$, then either $M' = M$ or $M' = R$.

Maximal ideal are in particular prime ideals and $I \trianglelefteq R$ is maximal $\Leftrightarrow$ R/I is a field. Hence we have

$$\{\text{ maximal ideals }\} \subsetneq \{\text{ prime ideals }\} \subsetneq \{\text{ radical ideals }\} \subsetneq \{\text{ ideals }\}$$

All inclusions are strict in general. However if R is a principal ideal domain (i.e. R has no zero divisors and every ideal in R is principal), then any non-zero prime ideal is also maximal.
Using *Zorn's Lemma*, one can show that any proper ideal of a ring R is contained in a maximal ideal.

2.2.3 Relations between ideals and algebraic sets

Let $\mathbb{K}$ be a field, $\mathbb{K}^n$ the affine space and $R_n = \mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring in n variables. Let $I \trianglelefteq R_n$ be an ideal and $A \subseteq \mathbb{K}^n$ an algebraic set. We define the operators $\mathcal{V}$ and $\mathcal{J}$ by

$$\mathcal{V}(I) := \left\{ r \in \mathbb{K}^n \mid f(r) = 0, \ \forall f \in I \right\} \subseteq \mathbb{K}^n$$
$$\mathcal{J}(A) := \left\{ f \in R_n \mid f(r) = 0, \ \forall r \in A \right\} \trianglelefteq R_n$$

Hilbert's Basissatz implies that R_n is a Noetherian ring (i.e. every ideal is finitely generated), thus we get

$$A \subseteq \mathbb{K}^n \text{ is an algebraic set } \Leftrightarrow \exists I \trianglelefteq R_n \text{ such that } A = \mathcal{V}(I)$$

and if $I = \langle f_1, \ldots, f_s \rangle$, then $\mathcal{V}(I) = \mathcal{V}\big(\langle f_1, \ldots, f_s \rangle\big) = Z(f_1, \ldots, f_s)$. Now we have the following results :

1) $\mathcal{J}(\emptyset) = R_n$ and $\mathcal{V}(R_n) = \emptyset$.
2) $\mathcal{V}\big(\{\mathbf{0}\}\big) = \mathbb{K}^n$ and if $\mathbb{K}$ is not a finite field, then $\mathcal{J}(\mathbb{K}^n) = \{\mathbf{0}\}$.
3) $\mathrm{Rad}\big(\mathcal{J}(A)\big) = \mathcal{J}(A)$, i.e. $\mathcal{J}(A)$ is always a radical ideal.
4) A is a variety (i.e. an irreducible algebraic set) $\Leftrightarrow$ $\mathcal{J}(A)$ is a prime ideal.
5) $A = \mathcal{V}\big(\mathcal{J}(A)\big)$, hence $\mathcal{V} \circ \mathcal{J} = \mathrm{id}$, showing that $\mathcal{J}$ is injective.
6) $I \subseteq \mathcal{J}\big(\mathcal{V}(I)\big)$, but in general $\mathcal{J}\big(\mathcal{V}(I)\big) \nsubseteq I$.
7) $\mathcal{J}$ and $\mathcal{V}$ are inclusion reversing, i.e. $A_1 \subseteq A_2 \Rightarrow \mathcal{J}(A_1) \supseteq \mathcal{J}(A_2)$ and $I_1 \subseteq I_2 \Rightarrow \mathcal{V}(I_1) \supseteq \mathcal{V}(I_2)$.
8) $\mathcal{J}$ moreover satisfies $A_1 \subseteq A_2 \Leftrightarrow \mathcal{J}(A_1) \supseteq \mathcal{J}(A_2)$ and $A_1 \subsetneq A_2 \Leftrightarrow \mathcal{J}(A_1) \supsetneq \mathcal{J}(A_2)$.
9) $\mathcal{V}(I_1 \cap \ldots \cap I_n) = \mathcal{V}(I_1) \cup \ldots \cup \mathcal{V}(I_n)$ and $\mathcal{J}\big(\bigcup_i A_i\big) = \bigcap_i \mathcal{J}(A_i)$.
10) $A_1 \cup \ldots \cup A_n = \mathcal{V}\big(\mathcal{J}(A_1) \cdot \ldots \cdot \mathcal{J}(A_n)\big)$ and $\bigcap_i A_i = \mathcal{V}\big(\sum_i \mathcal{J}(A_i)\big)$.

The last formulas allow in particular to define the *Zariski topology* on $\mathbb{K}^n$ by choosing the algebraic sets of $\mathbb{K}^n$ as closed sets of the topology. Indeed, $\emptyset = \mathcal{V}(R_n)$ and $\mathbb{K}^n = \mathcal{V}\big(\{\mathbf{0}\}\big)$ are closed and 10) shows that finite unions and arbitrary intersections of closed sets (algebraic sets) are again closed.
In addition, points and finite sets are also closed because they are algebraic sets (as claimed in section 2.2.1).

2.2.4 Hilbertscher Nullstellensatz

The operators $\mathcal{J}$ and $\mathcal{V}$ now give well-defined maps

$$\begin{aligned} \left\{ \text{algebraic sets in } \mathbb{K}^n \right\} &\longleftrightarrow \left\{ \text{radical ideals in } R_n \right\} \\ A &\overset{\mathcal{J}}{\longmapsto} \mathcal{J}(A) \\ \mathcal{V}(I) &\overset{\mathcal{V}}{\longleftarrow\!\shortmid} I = \mathrm{Rad}(I) \end{aligned}$$

where $\mathcal{J}$ is injective and both operations are inclusion reversing. *Hilbert's Nullstellensatz* now states that :

- If the field $\mathbb{K}$ is algebraically closed, then $\mathcal{V}(I) \neq \emptyset$ for any proper ideal $I \trianglelefteq R_n$, i.e.

$$\forall I \trianglelefteq R_n \text{ such that } I \neq R_n, \ \exists x \in \mathbb{K}^n \text{ such that } f(x) = 0, \forall f \in I$$

This strong statement has a lot of important consequences.

Always under the assumption that $\mathbb{K}$ is algebraically closed, we have :
– $\mathcal{V}$ and $\mathcal{J}$ define a 1-to-1 correspondence between affine algebraic sets in $\mathbb{K}^n$ and radical ideals of R_n.
– If $I = \langle f_1, \ldots, f_s \rangle$, then $\mathcal{V}(I) \neq \emptyset \;\Leftrightarrow\; \mathbf{1}$ cannot be written as an R_n-linear combination of the f_i.
– For any ideal $I \trianglelefteq R_n$, we have the formula : $\mathcal{J}\big(\mathcal{V}(I)\big) = \mathrm{Rad}(I)$.
– If $I_1, I_2 \trianglelefteq R_n$ are two ideals, then $\mathcal{V}(I_1) = \mathcal{V}(I_2) \;\Leftrightarrow\; \mathrm{Rad}(I_1) = \mathrm{Rad}(I_2)$.
– Let $I \trianglelefteq R_n$ be a radical ideal. Then I is a prime ideal $\;\Leftrightarrow\; \mathcal{V}(I)$ is a variety.
– Restricting the 1-to-1 correspondence, we get a bijection between prime ideals in R_n and varieties in $\mathbb{K}^n$.
– Let $M \trianglelefteq R_n$ be a maximal ideal. Then there is a point $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$ such that

$$M = \langle X_1 - \alpha_1 , \ldots , X_n - \alpha_n \rangle \quad \text{and} \quad \mathcal{V}(M) = \{\alpha\} = \big\{(\alpha_1, \ldots, \alpha_n)\big\}$$

– Again by restriction, we obtain a 1-to-1 correspondence between maximal ideals in R_n and points in $\mathbb{K}^n$.
– In particular, for algebraically closed fields we know all the maximal ideals of their polynomial ring.

Particular case :
Let $\mathbb{K}$ be any field and $f \in R_n$ such that f is not a unit (i.e. f is not a non-zero constant). Then
1) $\langle f \rangle$ is a prime ideal in $R_n \;\Leftrightarrow\; f$ is an irreducible polynomial.
2) If $n = 1$, then f is irreducible $\;\Leftrightarrow\; \langle f \rangle$ is a maximal ideal.
3) If $\mathbb{K}$ is algebraically closed, then $Z(f) \subset \mathbb{K}^n$ is a variety $\;\Leftrightarrow\; f$ is irreducible.

Coordinate ring of an algebraic set
Let $A \subseteq \mathbb{K}^n$ be an algebraic set. We say that $V \subseteq A$ is a *subvariety* of A if V is a closed subset in A with respect to the induced Zariski topology. Note that subvarieties do not need to be irreducible.

Now let $\mathbb{K}$ be algebraically closed and $A = \mathcal{V}(I)$ for some ideal $I \trianglelefteq R_n$. The *coordinate ring* of A is

$$\mathbb{K}[A] := R_n \,/\, I = R_n \,/\, \mathcal{J}(A)$$

where $\mathcal{J}(A) = \mathcal{J}\big(\mathcal{V}(I)\big) = \mathrm{Rad}(I) = I$ since I is a radical ideal by Hilbert's Nullstellensatz. In order to know which ideals correspond to the points and subvarieties of A, we define the operators

$$\mathcal{V}_A(J) = \big\{\, \alpha \in A \;\big|\; \varphi(\alpha) = 0, \ \forall\, \varphi \in J \,\big\} \quad \text{and} \quad \mathcal{J}_A(W) = \big\{\, \varphi \in \mathbb{K}[A] \;\big|\; \varphi(\alpha) = 0, \ \forall\, \alpha \in W \,\big\}$$

for an ideal $J \trianglelefteq \mathbb{K}[A]$ and a subvariety $W \subseteq A$. Applying the above results modulo I then yields :
– $\mathcal{V}_A$ and $\mathcal{J}_A$ define a 1-to-1 correspondence between radical ideals in $\mathbb{K}[A]$ and subvarieties of A.
– Restricting, we have a bijection between the irreducible subvarieties of A and the prime ideals in $\mathbb{K}[A]$.
– The points of A correspond exactly to the maximal ideals in $\mathbb{K}[A]$.
So we see that the coordinate ring $\mathbb{K}[A]$ encodes all the geometry of the algebraic set A.

2.2.5 Localizations

A commutative unital ring R is called a *local ring* $\;\Leftrightarrow\; R$ contains only 1 maximal ideal. We denote by $R^\times$ the group of units in R. Then we have the criteria :

$$R \text{ is a local ring} \quad \Leftrightarrow \quad R \setminus R^\times \text{ is an ideal in } R$$

Moreover $R \setminus R^\times$ will be the unique maximal ideal of R in this case.

A subset $S \subseteq R$ is called a *multiplicative set* if $1 \in S$ and $a, b \in S \;\Rightarrow\; a \cdot b \in S$. If S is such a multiplicative set, we define an equivalence relation on $R \times S$ by

$$(r_1, s_1) \sim (r_2, s_2) \;\Leftrightarrow\; \exists\, t \in S \text{ such that } t \cdot (r_1 s_2 - r_2 s_1) = 0$$

We denote the equivalence class of $(r,s) \in R \times S$ by $\frac{r}{s}$ and the set of equivalence classes by $S^{-1}R$. This is a ring with respect to the well-defined pointwise operations

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}$$

Moreover we have the ring homomorphism $i_S : R \to S^{-1}R : r \mapsto \frac{r}{1}$. In general, i_S is neither injective nor surjective, but i_S is injective if and only if $0 \notin S$ and S does not contain zero divisors. Hence we have an embedding $R \hookrightarrow S^{-1}R$ if R is an integral domain (and $0 \notin S$).
If I is an ideal in R, then $S^{-1}I$ is an ideal in $S^{-1}R$ and it is a proper ideal if and only if $I \cap S = \emptyset$. If we denote $\pi : R \to R/I$ and $S' = \pi(S)$, then $I \cap S = \emptyset \iff 0 \notin S'$. Moreover we have the isomorphism

$$S^{-1}R/S^{-1}I \cong S'^{-1}(R/I) \qquad \text{via} \ \left[\frac{r}{s}\right] \longmapsto \frac{\bar{r}}{\bar{s}} \tag{2.4}$$

Particular cases :
1) If R is an integral domain, then $S := R \setminus \{0\}$ is a multiplicative set and $S^{-1}R$ will be a field (hence a local ring). We call this field the *quotient field* of R and denote it by $\mathrm{Quot}(R)$.
2) If $s \in R$ is not nilpotent, i.e. $s^n \neq 0, \forall n \in \mathbb{N}_0$, then the set $S = \{ s^n \mid n \in \mathbb{N}_0 \}$ with $s^0 := 1$ is a multiplicative set and we call $R_s := S^{-1}R$ the *localized ring at s*. However R_s is in general not a local ring.
3) If $P \trianglelefteq R$ is a prime ideal, then $S := R \setminus P$ is multiplicative and the ring $R_P := S^{-1}R$ is a local ring with maximal ideal $S^{-1}P$. R_P is called the *localization of R at P*.

Proposition :
Let $S \subseteq R$ be a multiplicative subset with $0 \notin S$ and such that S does not contain zero divisors. Then there is a surjective map

$$e \ : \ \{ I \mid I \trianglelefteq R \} \to \{ J \mid J \trianglelefteq S^{-1}R \} \ : \ I \mapsto S^{-1}I \tag{2.5}$$

Moreover e maps prime ideals to prime ideals and there is a 1-to-1 correspondence between prime ideals in $S^{-1}R$ and prime ideals in R which do not intersect S.
Remark : We will see in section 3.1.5 that this result still holds true if the condition of S having no zero divisors is dropped (this requires some more advanced tools).

2.2.6 Corollary

Let $I \trianglelefteq R$ be an ideal in a commutative unital ring. Then $\mathrm{Rad}(I)$ is equal to the intersection of all prime ideals of R containing I :

$$\mathrm{Rad}(I) = \bigcap_{\substack{P \text{ prime} \\ I \subseteq P \trianglelefteq R}} P \tag{2.6}$$

In particular, the nil-radical of R is equal to the intersection of all prime ideals in R.

Proof. $\subseteq$: since prime ideals are radical, we have $I \subseteq P \Rightarrow \mathrm{Rad}(I) \subseteq \mathrm{Rad}(P) = P$ for any prime ideal P
$\supseteq$: by contraposition, assume that $r \notin \mathrm{Rad}(I)$. Then r is not nilpotent since $r^n \neq 0, \forall n \in \mathbb{N}_0$ (however r may be a zero divisor) and the set $S = \{ r^n \,|\, n \in \mathbb{N}_0 \}$ is multiplicative, so we can consider the localized rings $R_r = S^{-1}R$ and $I_r = S^{-1}I$ at r. Forming the quotient $Q := R_r/I_r$, we get the ring homomorphism

$$f \ : \ R \xrightarrow{i_S} S^{-1}R \xrightarrow{\pi} Q \ , \ f(s) = \left[\tfrac{s}{1}\right]$$

By Zorn's Lemma, there exists a maximal ideal $M \trianglelefteq Q$. Denote $P := f^{-1}(M)$. P is a prime ideal in R as preimage of a prime ideal under a ring homomorphism (maximal ideals are prime).
P also contains I since $f(I) \subseteq \{0\} \subset M$ and $P \cap S = \emptyset$, otherwise $\exists \, s \in S$ such that $f(s) \in M$, where

$$f(s) = \left[\tfrac{s}{1}\right] \quad \text{with } \tfrac{1}{s} \in S^{-1}R \ \Rightarrow \ \left[\tfrac{1}{s}\right] \cdot f(s) = [1] \in M$$

which is impossible since M is maximal (hence proper). $P \cap S = \emptyset$ in particular implies that P does not contain r, thus r does not belong to the intersection of all prime ideals in R containing I. $\square$

2.3 The spectrum and its Zariski topology

The main references we have used for this section are [Ha], [Ku], [Sch] and [U1]. However most of the results can be found in any book about commutative algebra that discusses the spectrum of a ring.

2.3.1 Definitions

Let R be a commutative unital ring. As a set, we define the *spectrum* of R, denoted by $\operatorname{Spec} R$, as the set of all prime ideals in R (recall our convention that prime ideals are always proper) :

$$\operatorname{Spec} R := \left\{ P \trianglelefteq R \;\middle|\; P \text{ is a prime ideal of } R \right\}$$

Recalling the correspondence between ideals and varieties in 2.2.4, we thus may say that $\operatorname{Spec} R$ contains in some sense all "irreducible subvarieties" of the "geometric model" of R (see section 2.3.9 for a more rigorous interpretation). We also define the *maximal spectrum* of R by

$$\operatorname{Max} R := \left\{ M \trianglelefteq R \;\middle|\; M \text{ is a maximal ideal of } R \right\}$$

so that $\operatorname{Max}(R)$ consists of all "points" of R.

Let S be an arbitrary subset of R. We denote by $V(S)$ the associated subset of $\operatorname{Spec} R$ consisting of all prime ideals that contain S :

$$V(S) := \left\{ P \in \operatorname{Spec} R \;\middle|\; S \subseteq P \right\}$$

$V(S)$ is called the *zero set of S* in $\operatorname{Spec} R$. Of course, $S \subseteq T \;\Rightarrow\; V(T) \subseteq V(S)$. The converse however is not true in general. Moreover we observe that $V(S)$ only depends on the ideal generated by S since

$$V(S) = \left\{ P \in \operatorname{Spec} R \;\middle|\; S \subseteq P \right\} = \left\{ P \in \operatorname{Spec} R \;\middle|\; \langle S \rangle \subseteq P \right\} = V\big(\langle S \rangle\big)$$

Hence it suffices to consider $V(I)$ for ideals $I \trianglelefteq R$ only.

For a subset $A \subseteq \operatorname{Spec} R$, we define the *ideal of A* in R as the intersection of all prime ideals contained in A :

$$J(A) := \bigcap_{P \in A} P \qquad \Rightarrow \; J(A) \trianglelefteq R$$

Again, $A \subseteq B \subseteq \operatorname{Spec} R \;\Rightarrow\; J(B) \subseteq J(A)$, i.e. V and J are both inclusion-reversing.

2.3.2 Lemma

1) For any ideal $I \trianglelefteq R$, we have $V\big(\operatorname{Rad}(I)\big) = V(I)$.
2) Let I, J be ideals of R. Then $V(I) \subseteq V(J) \;\Leftrightarrow\; \operatorname{Rad}(J) \subseteq \operatorname{Rad}(I)$.
3) If I and J are ideals of R, then $V(I \cdot J) = V(I) \cup V(J) = V(I \cap J)$.
4) If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in R, then $V\big(\sum_{\lambda \in \Lambda} I_\lambda\big) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Proof. 1) follows from the fact that $I \subseteq P \;\Leftrightarrow\; \operatorname{Rad}(I) \subseteq \operatorname{Rad}(P) = P$ for any prime ideal $P \trianglelefteq R$.
2) We use formula (2.6) from corollary 2.2.6 :
$\underline{\Rightarrow}$: Assume that $V(I) \subseteq V(J)$, i.e. all prime ideals containing I also contain J. Then

$$\operatorname{Rad}(I) = \bigcap_{\substack{P \text{ prime} \\ I \subseteq P \trianglelefteq R}} P \;\supseteq\; \bigcap_{\substack{Q \text{ prime} \\ J \subseteq Q \trianglelefteq R}} Q = \operatorname{Rad}(J)$$

$\underline{\Leftarrow}$: If $\operatorname{Rad}(J) \subseteq \operatorname{Rad}(I)$, then $I \subseteq P$ implies that $J \subseteq \operatorname{Rad}(J) \subseteq \operatorname{Rad}(I) \subseteq \operatorname{Rad}(P) = P$ for any prime ideal P, i.e. any prime ideal containing I also contains J. Thus $V(I) \subseteq V(J)$.

3a) $\underline{\supseteq}$: if $I \subseteq P$ or $J \subseteq P$ for some prime ideal P, then certainly $I \cdot J \subseteq P$.
$\underline{\subseteq}$: Let P be a prime ideal containing $I \cdot J$ and assume e.g. that $J \nsubseteq P$, i.e. $\exists\, j \in J$ such that $j \notin P$. If $i \in I$ is arbitrary, then $i \cdot j \in P$ necessarily implies that $i \in P$ since P is prime. Hence $I \subseteq P$.

3b) $\subseteq$: if P is a prime ideal such that $I \subseteq P$ or $J \subseteq P$, then certainly $I \cap J \subseteq P$.
$\supseteq$: Let P be prime with $I \cap J \subseteq P$ and assume that $I \nsubseteq P$ and $J \nsubseteq P$, i.e. $\exists\, i \in I$, $\exists\, j \in J$ such that $i \notin P$ and $j \notin P$. But $i \cdot j \in I \cap J \subseteq P \;\Rightarrow\;$ either $i \in P$ or $j \in P$: contradiction, so $I \subseteq P$ or $J \subseteq P$.

4) Recall that $\sum\limits_{\lambda \in \Lambda} I_\lambda$ is the ideal generated by $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$, i.e. it is the smallest ideal containing all ideals I_λ :

$$P \in V\Big(\sum_{\lambda \in \Lambda} I_\lambda\Big) \;\Leftrightarrow\; \sum_{\lambda \in \Lambda} I_\lambda \subseteq P \;\Leftrightarrow\; I_\lambda \subseteq P,\ \forall\, \lambda \in \Lambda \;\Leftrightarrow\; P \in V(I_\lambda),\ \forall\, \lambda \in \Lambda \;\Leftrightarrow\; P \in \bigcap_{\lambda \in \Lambda} V(I_\lambda)$$

□

2.3.3 Definition of the Zariski topology

The previous results allow to define a topology on the set $\operatorname{Spec} R$, thus turning the spectrum of a ring into a topological space. In fact, we define a subset $A \subseteq \operatorname{Spec} R$ to be *closed* if there is an ideal $I \trianglelefteq R$ such that $A = V(I)$. This indeed defines a topology :

$$\emptyset = V(R) \qquad , \qquad \operatorname{Spec} R = V\big(\{0\}\big)$$

$\emptyset$ is closed since there are no prime ideals containing the whole ring and $\operatorname{Spec} R$ is closed since any (prime) ideal must contain 0. Moreover lemma 2.3.2 shows that finite unions and arbitrary intersections of sets of the form $V(\ )$ are again of that form. Hence the $V(\ )$ do form the set of closed subsets for a topology on $\operatorname{Spec} R$, called the *Zariski topology* on $\operatorname{Spec} R$.
This construction parallels the construction of the Zariski topology on affine spaces, except that the points of $\operatorname{Spec} R$ correspond to all prime ideals of R, and not just the maximal ideals.
Both topologies are named after the Russian mathematician **Oscar Zariski** (1899–1986).

2.3.4 Proposition

1) Let $A \subseteq \operatorname{Spec} R$. Then $V\big(J(A)\big) = \overline{A}$, the topological closure of A in $\operatorname{Spec} R$.
2) If $A \subseteq \operatorname{Spec} R$ is closed, then $J(A)$ is a radical ideal.
3) Let $I \trianglelefteq R$ be an ideal. Then $J\big(V(I)\big) = \operatorname{Rad}(I)$. This is an analogue to Hilbert's Nullstellensatz.
4) There is a 1-to-1 correspondence between closed subsets of $\operatorname{Spec} R$ and radical ideals in R.

Proof. 1) By definition, $A \subseteq V\big(J(A)\big)$ since prime ideals in A always contain $J(A)$. Note that $V\big(J(A)\big)$ is closed. Now consider an arbitrary closed subset $V(I)$ that contains A for some $I \trianglelefteq R$:

$$A \subseteq V(I) \;\Rightarrow\; I \subseteq P,\ \forall\, P \in A \;\;\Rightarrow\; I \subseteq \bigcap_{P \in A} P = J(A)$$

so that $V\big(J(A)\big) \subseteq V(I)$. This holds for any $V(I)$ containing A, hence $V\big(J(A)\big)$ is the smallest closed subset of $\operatorname{Spec} R$ that contains A. So by definition of closure : $\overline{A} = V\big(J(A)\big)$.

2) Using again formula (2.6), we get :

$$\operatorname{Rad}\big(J(A)\big) = \bigcap_{\substack{P \text{ prime} \\ J(A) \subseteq P}} P \;=\; \bigcap_{P \in V(J(A))} P \;=\; \bigcap_{P \in \bar{A}} P = J(\bar{A}) = J(A)$$

3)

$$J\big(V(I)\big) = \bigcap_{P \in V(I)} P = \bigcap_{\substack{P \text{ prime} \\ I \subseteq P}} P = \operatorname{Rad}(I)$$

4) Consider

$$\begin{aligned} \big\{\text{radical ideals in } R\big\} &\longleftrightarrow \big\{\text{closed subsets of } \operatorname{Spec} R\big\} \\ \operatorname{Rad}(I) = I \;&\xmapsto{\ V\ }\; V(I) \\ J(A) \;&\xleftarrow{\ J\ }\!\shortmid\; A = \overline{A} \end{aligned}$$

with $J\big(V(I)\big) = \operatorname{Rad}(I) = I$ and $V\big(J(A)\big) = \overline{A} = A$, i.e. $J \circ V = \mathrm{id}$ and $V \circ J = \mathrm{id}$. □

2.3.5 Definition

We introduce some additional notation. Let $X := \operatorname{Spec} R$ and $r \in R$. We denote $V(r) := V\big(\langle r \rangle\big)$ and

$$U(r) := X \setminus V(r) = \big\{ P \in \operatorname{Spec} R \bigm| P \notin V(r) \big\} = \big\{ P \in X \bigm| \langle r \rangle \nsubseteq P \big\} = \big\{ P \in X \bigm| r \notin P \big\}$$

so that $U(r)$ is open in X and consists of all prime ideals in R which do not contain r. $U(r)$ is called the *distinguished open set* or *standard open set* associated to r. Open sets of this form satisfy :

1) $\forall\, r, s \in R$, $U(r) \cap U(s) = U(r \cdot s)$
2) $\forall\, r \in R$, $U(r^n) = U(r)$, $\forall\, n \in \mathbb{N}$
3) $U(r) = \emptyset \;\Leftrightarrow\; r$ is nilpotent
4) $U(r) = X \;\Leftrightarrow\; r$ is a unit
5) if $r = u \cdot s$ where u is a unit, then $U(r) = U(s)$
6) $\forall\, r, s \in R$, $U(s) \subseteq U(r) \;\Leftrightarrow\; U(s) = U(r \cdot s)$

Proof. 1) By lemma 2.3.2 and using that $\langle r \rangle \cdot \langle s \rangle = \langle r \cdot s \rangle$, we get

$$\begin{aligned} U(r) \cap U(s) &= \big(X \setminus V(r)\big) \cap \big(X \setminus V(s)\big) = X \setminus \big(V(r) \cup V(s)\big) = X \setminus V\big(\langle r \rangle \cdot \langle s \rangle\big) \\ &= X \setminus V\big(\langle r \cdot s \rangle\big) = X \setminus V(r \cdot s) = U(r \cdot s) \end{aligned}$$

Alternatively, this can be seen by the fact that prime ideals containing neither r nor s do not contain $r \cdot s$ neither and vice-versa :

$$\big(r \cdot s \in P \;\Leftrightarrow\; r \in P \text{ or } s \in P \big) \qquad \Rightarrow \qquad \big(r \cdot s \notin P \;\Leftrightarrow\; r \notin P \text{ and } s \notin P \big)$$

2) A prime ideal contains an element if any only if it contains a power of that element :

$$\begin{gathered} r \in P \;\Rightarrow\; r^n \in P,\ \forall\, n \geq 1 \\ r^n \in P \;\Rightarrow\; r \in P \;\text{ or }\; r^{n-1} \in R \;\Rightarrow\; \ldots \;\Rightarrow\; r \in R \;\text{ or }\; r^2 \in R \;\Rightarrow\; r \in R \end{gathered}$$

3) $\underline{\Leftarrow}$: r nilpotent $\;\Rightarrow\; r^n = 0$ for some $n \in \mathbb{N}$, so by 2) $U(r) = U(0) = \emptyset$ since every prime ideal contains 0
$\underline{\Rightarrow}$: $U(r) = \emptyset$ means that $V(r) = X$, i.e. every prime ideal contains r. Hence r is nilpotent by corollary 2.2.6 since the intersection of all prime ideals of a ring is equal to its nil-radical.

4) $\underline{\Leftarrow}$: if r is a unit, then $\exists\, s \in R$ such that $r \cdot s = 1$, i.e. a prime ideal containing r also contains 1. This is impossible, hence $V(r) = \emptyset \;\Rightarrow\; U(r) = X$.
$\underline{\Rightarrow}$: if $U(r) = X$, i.e. $V(r) = \emptyset$, there does not exist a prime ideal containing $\langle r \rangle$. But if $\langle r \rangle \neq R$, there is a maximal (hence prime) ideal containing it. So $\langle r \rangle$ cannot be proper and $\langle r \rangle = R$ means that $1 \in \langle r \rangle$, i.e. $\exists\, s \in R$ such that $r \cdot s = 1$, so r is a unit.

5) By contraposition, it suffices to prove that a prime ideal P contains r if and only if it contains s :

$$\begin{aligned} &s \in P \;\Rightarrow\; r = u \cdot s \in P \text{ since } P \text{ is an ideal} \\ &r \in P \;\Rightarrow\; s \in P \text{ or } u \in P, \text{ but } u \notin P, \text{ otherwise } 1 \in P \text{ since } u \text{ is a unit} \end{aligned}$$

6) $\underline{\Leftarrow}$: follows from 1), $U(s) = U(r \cdot s) \subseteq U(r)$ is always true
$\underline{\Rightarrow}$: $U(s) \subseteq U(r)$ implies that $U(s) = U(s) \cap U(r) = U(r \cdot s)$ □

2.3.6 Proposition

The open subsets $U(r)$ form a basis for the Zariksi topology on $\operatorname{Spec} R$, i.e. every open set in $\operatorname{Spec} R$ can be written as a union of sets of the form $U(r)$. More precisely: if $U \subseteq X$ is open and given by $U = X \setminus V(I)$ for some $I \trianglelefteq R$, then

$$U = \bigcup_{i \in J} U(a_i)$$

where $\{a_i\}_{i \in J}$ is a set of generators of the ideal I.

Proof. General topology shows that $\mathcal{A} \subset \mathcal{P}(X)$ is a basis for the topology of X if and only if X can be written as a union of sets in $\mathcal{A}$ and any intersection of 2 sets in $\mathcal{A}$ is again a union of sets in $\mathcal{A}$.
Since no prime ideal contains 1, we get $\operatorname{Spec} R = U(1)$ and it suffices to show that the intersection of 2 sets of the form $U(\ )$ is again of that form. This is shown in 2.3.5 : $U(r) \cap U(s) = U(r \cdot s)$, $\forall\, r, s \in R$.
Now let $U = X \setminus V(I)$ be open and $\{a_i\}_{i\in J}$ a set of generators of $I \trianglelefteq R$ (this always exists : in the worst case, take as generators $\{r\}_{r\in I}$). Then

$$V(I) = \bigcap_{i\in J} V(a_i)$$

because a prime ideal $P \trianglelefteq R$ satisfies $I \subseteq P \;\Leftrightarrow\; \langle\, a_i \,\rangle \subseteq P$, $\forall\, i \in J$. It follows that

$$U = X \setminus V(I) = X \setminus \bigcap_i V(a_i) = \bigcup_i \big(X \setminus V(a_i)\big) = \bigcup_i U(a_i) \qquad \square$$

Remark :
Since the covering of an open set $U \subseteq X$ by basis open sets only depends on the chosen set of generators, we obtain that every open set can be covered by finitely many $U(r)$ if R is a Noetherian ring.

Before stating the main theorem about the Zariski topology on $\operatorname{Spec} R$, recall that a topological space X is said to be a *Kolmogorov space* if $\forall\, x, y \in X$ such that $x \neq y$, there either exists an open set containing x but not y or an open set containing y but not x. Note that this condition is much weaker than the condition of being Hausdorff (two distinct points can be separated by two disjoint neighborhoods). Equivalently,

$$X \text{ is a Kolmogorov space} \quad \Leftrightarrow \quad \Big(\forall\, x, y \in X \;:\; x \neq y \;\Rightarrow\; x \notin \overline{\{y\}} \text{ or } y \notin \overline{\{x\}}\Big) \tag{2.7}$$

because $a \notin \overline{\{b\}}$ means that there exists an open neighborhood of a that does not intersect $\{b\}$, i.e.

$$a \notin \overline{\{b\}} \;\Leftrightarrow\; \exists\, U \subset X \text{ open such that } a \in U \text{ and } b \notin U$$

2.3.7 Theorem

$\operatorname{Spec} R$ is a compact Kolmogorov space.

Proof. 1) Let $\{U_i\}_{i\in J}$ be an open cover of $X = \operatorname{Spec} R$ and denote $V_i = X \setminus U_i$. Then $\bigcap_i V_i = \emptyset$ because

$$X = \bigcup_i U_i = \bigcup_i \big(X \setminus V_i\big) = X \setminus \bigcap_i V_i$$

Since $\{\, U(r) \,|\, r \in R \,\}$ is a basis of topology, we can write $U_i = \bigcup_j U(r_{ij})$ for some $r_{ij} \in R$, $\forall\, i \in J$, thus

$$V_i = X \setminus U_i = X \setminus \bigcup_j U(r_{ij}) = \bigcap_j \big(X \setminus U(r_{ij})\big) = \bigcap_j V(r_{ij})$$

Changing notations, it follows that $\emptyset = \bigcap_i V_i = \bigcap_{k\in K} V(r_k)$. This means that there is no prime ideal in R which belongs to all the $V(r_k)$, i.e. $\nexists\, P \trianglelefteq R$ such that P is prime and $r_k \in P$, $\forall\, k \in K$.
Now let $I = \langle\, \{r_k\}_{k\in K} \,\rangle$ be the ideal generated by all elements $r_k \in R$. If I is proper, there is a maximal (hence prime) ideal $M \trianglelefteq R$ such that $I \subseteq M$, which is not possible since there are no prime ideals in R that contain all the r_k. Thus $I = R$, so $1 \in I$ and there exist finitely many coefficients $c_l \in R$ such that

$$1 = c_1 \cdot r_{k_1} + c_2 \cdot r_{k_2} + \ldots + c_n \cdot r_{k_n}$$

Then $\{U(r_{k_1}), U(r_{k_2}), \ldots, U(r_{k_n})\}$ is a finite covering of $\operatorname{Spec} R$. Indeed if $P \in \operatorname{Spec} R$, then $\exists\, l \in \{1, \ldots, n\}$ such that $r_{k_l} \notin P$, otherwise $1 \in P$ and P would not be prime. By definition, $\forall\, l \in \{1, \ldots, n\}$, $\exists\, i \in J$ such that $U(r_{k_l}) \subseteq U_i$. These finitely many U_i then form a finite subcover of $\operatorname{Spec} R$.

2) To show that $\operatorname{Spec} R$ is a Kolmogorov space, we first have to find expressions for the (topological) closure of points, i.e. of singleton sets. Let $P \in \operatorname{Spec} R$. Since the closure of $\{P\}$ is the smallest closed subset of $\operatorname{Spec} R$ containing $\{P\}$, we have

$$\overline{\{P\}} = V(P) = \Big\{\, Q \in \operatorname{Spec} R \;\Big|\; P \subseteq Q \,\Big\} \tag{2.8}$$

Applying criterion (2.7), the result now follows immediately : let $P_1 \neq P_2$ be 2 distinct prime ideals. If one of them is included in the other one, say $P_i \subsetneq P_j$, then $P_i \notin V(P_j)$. And if no one contains the other one, then $P_1 \notin V(P_2)$ and $P_2 \notin V(P_1)$. This finishes the proof. $\square$

2.3.8 Remarks

1) We say that a topological space X is *compact* if every open cover of X has a finite subcover, which is the case here. Some authors however, as e.g. [Ha], [Ma] and [Sch], call such spaces only *quasi-compact* and require compact spaces to be in addition Hausdorff. We do not use this convention. In fact, $\operatorname{Spec} R$ is almost never Hausdorff.

In a Hausdorff space, all points are closed : let $x \in X$ and $y \in X \setminus \{x\}$. Hence $y \neq x$ and there is an open neighborhood U_y of y which does not contain x. This can be done for any such y, so

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y \quad \Rightarrow \quad X \setminus \{x\} \text{ is open}$$

In particular, we have that $\{x\} = \overline{\{x\}}$, $\forall\, x \in X$. (2.8) however shows that the only closed points in $\operatorname{Spec} R$ are the maximal ideals. In fact, maximality of an ideal M implies that it cannot be strictly included in some other prime ideal, hence $V(M) = \{M\}$, and for any prime ideal P that is not maximal, there is a maximal ideal M_P strictly containing P (by Zorn), so that $\{P\} \subsetneq V(P)$.

Hence non-closed points in $\operatorname{Spec} R$ correspond to prime ideals which are not maximal. But there are a lot of rings containing prime ideals which are not necessarily maximal, for example

− the polynomial rings in $n \geq 2$ variables, where e.g. $\langle\, X_1 \,\rangle$ is a prime ideal, but not maximal

− any integral domain with at least one non-zero proper ideal (so $\{0\}$ is prime, but not a maximal ideal)

If one wants $\operatorname{Spec} R$ to be a Hausdorff space, we necessarily need a ring R where every prime ideal is maximal (but this is still not sufficient). A trivial example where $\operatorname{Spec} R$ is Hausdorff occurs if $R = \mathbb{K}$ is a field :

$$\operatorname{Spec} \mathbb{K} = \big\{\{0\}\big\} \quad \text{with} \quad \{0\} = \overline{\{0\}}$$

2) Let X be an arbitrary topological space. A point $x \in X$ is called a *generic point* if $\{x\}$ is dense in X, i.e.

$$x \in X \text{ is a generic point} \quad \Leftrightarrow \quad \overline{\{x\}} = X$$

In particular, $\overline{\{x\}} = X$ means that every neighborhood of every point in X must contain x. Thus

− In a trivial topological space (i.e. $\emptyset$ and X are the only open sets), every point is generic.

− The only Hausdorff space that has a generic point is the singleton set.

− If R is an integral domain, then $\{0\}$ is prime and a generic point of $\operatorname{Spec} R$ since every ideal contains 0 :

$$\overline{\{0\}} = V\big(\{0\}\big) = \big\{\, P \in \operatorname{Spec} R \;\big|\; \{0\} \subseteq P \,\big\} = \operatorname{Spec} R$$

3) Fix $r \in R$ such that r is not nilpotent. Hence $U(r) \neq \emptyset$ by 2.3.5 and $S = \{\, r^n \mid n \in \mathbb{N}_0 \,\}$ is a multiplicative set, so we can form the localized ring at r :

$$R_r = S^{-1}R = \left\{\, \frac{a}{r^n} \;\middle|\; a \in R,\ n \in \mathbb{N}_0 \,\right\}$$

We know that there is a 1-to-1 correspondence between prime ideals in R_r and prime ideals in R which do not intersect S (even if r is a zero divisor, see sections 2.2.5 and 3.1.5). But if $P \trianglelefteq R$ is prime, then

$$P \cap S = \emptyset \;\Leftrightarrow\; r \notin P \;\Leftrightarrow\; P \in U(r)$$

since $U(r)$ is exactly the set of all prime ideals in R which do not contain r. Thus we have a bijection

$$U(r) \xrightarrow{\sim} \operatorname{Spec} R_r \;:\; P \mapsto S^{-1}P \tag{2.9}$$

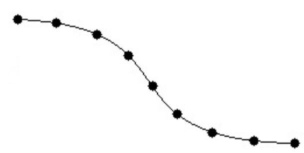
We will show in 3.1.6 that this bijection is also a homeomorphism with respect to the corresponding Zariski topologies, i.e. $U(r) \cong \operatorname{Spec} R_r$ as topological spaces. In particular, $U(r)$ is also compact since the spectrum of any ring is compact (this includes the case where r is nilpotent since $\emptyset$ is compact).

Hence we have showed that, for any ring R, the distinguished open sets $U(r)$ are compact as subspaces of $\operatorname{Spec} R$, $\forall\, r \in R$, so that $\operatorname{Spec} R$ has a basis of compact open sets.

2.3.9 Interpretation and examples

The definition of the spectrum of a ring and its Zariski topology is motivated by the study of algebraic sets. If $\mathbb{K}$ is algebraically closed (so $\mathbb{K}$ cannot be a finite field), we already know that the points of an algebraic set $A \subseteq \mathbb{K}^n$ correspond exactly to the maximal ideals of the coordinate ring $\mathbb{K}[A]$. Moreover we have 1-to-1 correspondences between radical (resp. prime) ideals in $\mathbb{K}[A]$ and (irreducible) subvarieties of A. Also recall that this correspondence is inclusion-reversing.
The spectrum of $\mathbb{K}[A]$ consisting of all prime ideals in $\mathbb{K}[A]$, we thus may say that $\operatorname{Spec}\mathbb{K}[A]$ contains not only all points of A, but also all other irreducible subvarieties of A. Formula (2.8) shows that the points of A (the maximal ideals) are closed in the spectrum. Moreover it allows to see the closure of an irreducible subvariety V of A (a prime ideal) as consisting of V and all other irreducible subvarieties contained in V. Summarizing : Irreducible subvarieties of A have a closure consisting of themselves and all their irreducible subvarieties (this includes all the points). For example in dimension 2, this means that the closure of a curve (in the usual sense, see figure below) consists of the curve itself as a geometric object and all points lying on that curve.

Figure 2.1: the closure of a curve



If one only considers the points of A, i.e. the maximal ideals in $\mathbb{K}[A]$, then the (induced) Zariski topology on $\operatorname{Max}\mathbb{K}[A] \subset \operatorname{Spec}\mathbb{K}[A]$ coincides with the (induced) Zariski topology on the algebraic set $A \subset \mathbb{K}^n$, which has precisely the algebraic subsets as closed sets. Indeed (see section 2.2.4) :
Let $\mathcal{M} \subseteq \operatorname{Max}\mathbb{K}[A]$ be a set of maximal ideals. To each $M \in \mathcal{M}$ corresponds exactly one point in A, given by $\alpha = \mathcal{V}_A(M) \in A$. Let $C := \{\, \mathcal{V}_A(M) \mid M \in \mathcal{M} \,\}$ be the subset of points in A corresponding to $\mathcal{M}$. Then

$$\mathcal{M} \text{ is closed in } \operatorname{Max}\mathbb{K}[A] \;\Leftrightarrow\; C \text{ is closed in } A \tag{2.10}$$

Proof. $\underline{\Leftarrow}$: Let $C \subseteq A$ be closed, i.e. C is an algebraic subset of A and can thus be written as $C = \mathcal{V}_A(I)$ for some radical ideal $I \trianglelefteq \mathbb{K}[A]$ with $I = \mathcal{J}_A(C)$ by 1-to-1 correspondence. Hence $C = \mathcal{V}_A\big(\mathcal{J}_A(C)\big)$ and

$$M \text{ maximal } \;\Rightarrow\; \Big(M \in \mathcal{M} \;\Leftrightarrow\; \mathcal{V}_A(M) \in C \;\Leftrightarrow\; \mathcal{J}_A\big(\mathcal{V}_A(M)\big) \supseteq \mathcal{J}_A(C) \;\Leftrightarrow\; I \subseteq M \Big) \tag{2.11}$$

because $M = \mathcal{J}_A(\{\alpha\})$ by 1-to-1 correspondence again, implying that

$$\mathcal{J}_A\big(\mathcal{V}_A(M)\big) = \mathcal{J}_A\big(\mathcal{V}_A\big(\mathcal{J}_A(\{\alpha\})\big)\big) = \mathcal{J}_A(\{\alpha\}) = M$$

Thus (2.11) shows that $\mathcal{M} = V(I) \cap \operatorname{Max}\mathbb{K}[A]$, so $\mathcal{M}$ is closed in $\operatorname{Max}\mathbb{K}[A]$.

$\underline{\Rightarrow}$: Let $\mathcal{M}$ be closed in $\operatorname{Max}\mathbb{K}[A]$, i.e. $\exists\, I \trianglelefteq \mathbb{K}[A]$ such that $\mathcal{M} = V(I) \cap \operatorname{Max}\mathbb{K}[A]$. We may even assume that I is a radical ideal because $I \subseteq P \Rightarrow \operatorname{Rad}(I) \subseteq P$ for any prime ideal P : $V(I) = V\big(\operatorname{Rad}(I)\big)$.
I being radical, it therefore corresponds to a subvariety $S \subseteq A$, given by $S = \mathcal{V}_A(I)$. Now $\mathcal{M}$ consists of all maximal ideals containing I, hence corresponds to all points lying on the subvariety S. But $\mathcal{M}$ also corresponds to all points in C by definition, thus $C = S$ as sets $\;\Rightarrow\; C = \mathcal{V}_A(I)$ and C is closed in A. $\square$

Thus one can view the topological space $\operatorname{Spec}\mathbb{K}[A]$ as an "enrichment" of the topological space A (with induced Zariski topology), in the sense that we add an additional non-closed point for every irreducible subvariety of A (which is not a point of A), and this point "keeps track" of the corresponding subvariety. Moreover this point is then a generic point for the corresponding subvariety. By studying spectra of polynomial rings, one can generalize these concepts to fields that are not algebraically closed. This will eventually lead to the language of schemes.

We illustrate the above concepts by the following important examples.

Example 1
Let $\mathbb{K}$ be a field. The *affine line* over $\mathbb{K}$ is defined by $\mathbb{A}^1_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[X]$. Since $\mathbb{K}[X]$ is a principal ideal domain, every ideal in $\mathbb{K}[X]$ is of the form $I = \langle f \rangle$ for some $f \in \mathbb{K}[X]$. Hence (see section 2.2.4)

$$\mathbb{A}^1_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[X] = \big\{ P \trianglelefteq \mathbb{K}[X] \bigm| P \text{ is a prime ideal} \big\} = \big\{ \langle f \rangle \bigm| f \in \mathbb{K}[X] \text{ is irreducible, but not a unit} \big\}$$

The zero ideal $\{0\}$ is a generic point of $\mathbb{A}^1_{\mathbb{K}}$ (its closure is equal to the whole space) since $\mathbb{K}[X]$ is an integral domain. In particular, it is non-closed. All other points $\langle f \rangle$ in $\mathbb{A}^1_{\mathbb{K}}$ ($\deg f \geq 1$) are however closed since every non-zero prime ideal in a principal ideal domain is automatically maximal.
If $\mathbb{K}$ is moreover algebraically closed, the only irreducible polynomials of degree ≥ 1 are the linear polynomials $X - \alpha$ for $\alpha \in \mathbb{K}$. Thus the maximal ideals of $\mathbb{K}[X]$ (the closed points in $\mathbb{A}^1_{\mathbb{K}}$) are indeed in 1-to-1 correspondence with the geometric points in $\mathbb{K}$ (the varieties in $\mathbb{K}$), while the non-closed point corresponds to the whole affine line.

Example 2
Let $\mathbb{K}$ be an algebraically closed field. The *affine plane* over $\mathbb{K}$ is defined by $\mathbb{A}^2_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[X, Y]$. Here a study is more complicated since $\mathbb{K}[X, Y]$ is not a principal ideal ring. However it is still an integral domain, thus the zero ideal $\{0\}$ is again a generic point of $\mathbb{A}^2_{\mathbb{K}}$ and hence not closed.
We know that the closed points of $\mathbb{A}^2_{\mathbb{K}}$ are given by the maximal ideals in $\mathbb{K}[X, Y]$. Since $\mathbb{K}$ is algebraically closed, it follows from Hilbert's Nullstellensatz that every maximal ideal $M \trianglelefteq \mathbb{K}[X, Y]$ is of the form

$$M = \langle X_1 - \alpha_1 \,,\, X_2 - \alpha_2 \rangle \;\text{ for some } \alpha = (\alpha_1, \alpha_2) \in \mathbb{K}^2$$

Hence the closed points in $\mathbb{A}^2_{\mathbb{K}}$ are in 1-to-1 correspondence with ordered pairs of elements in $\mathbb{K}$, i.e. with the points of the algebraic set $\mathbb{K}^2$. Moreover it follows from (2.10) that $\operatorname{Max} \mathbb{K}[X, Y]$ (the set of closed points in $\mathbb{A}^2_{\mathbb{K}}$) and $\mathbb{K}^2$ (the set of all geometric points) are homeomorphic with respect to their Zariski topologies.

Now let $f \in \mathbb{K}[X, Y]$ be an irreducible polynomial of degree ≥ 1. Then the algebraic set $Z(f)$ is a variety (an irreducible curve) in $\mathbb{K}^2$, hence there exists a non-zero prime ideal $P \in \mathbb{A}^2_{\mathbb{K}}$ such that $Z(f) = \mathcal{V}(P)$. This P cannot be maximal since $Z(f)$ is not a single point in $\mathbb{K}^2$, so the point $\{P\}$ is not closed and its closure in $\mathbb{A}^2_{\mathbb{K}}$ consists of P itself and all maximal ideals containing P because the closure of $Z(f)$ consists of $Z(f)$ itself and all points lying on $Z(f)$, i.e. all points $(a, b) \in \mathbb{K}^2$ such that $f(a, b) = 0$.

Example 3
We come back to the case of the affine line $\mathbb{A}^1_{\mathbb{K}}$, but now we drop the condition that $\mathbb{K}$ is algebraically closed. Consider e.g. $\mathbb{R}[X]$, in which we have 2 different types of irreducible polynomials :
1) the linear polynomials $f_\alpha(X) = X - \alpha$ for some $\alpha \in \mathbb{R}$.
2) quadratic polynomials $h_{bc}(X) = X^2 + 2bX + c$ with negative discriminant, i.e. $b, c \in \mathbb{R}$ and $b^2 - c < 0$.

The polynomials in 1) being irreducible, we know that $\langle f_\alpha \rangle$ is a maximal ideal in $\mathbb{R}[X]$ for any $\alpha \in \mathbb{R}$. Hence the maximal ideals generated by the linear polynomials f_α define again geometric points of $\mathbb{R}$: $Z(f_\alpha) = \{\alpha\}$, $\forall\, \alpha \in \mathbb{R}$. But since $\mathbb{R}$ is not algebraically closed, not all maximal ideals are of this type. Indeed $\langle h_{bc} \rangle$ is also a maximal ideal for all $b, c \in \mathbb{R}$ such that $b^2 - c < 0$, but there is no correspondence as in 1) because $Z(h_{bc}) = \emptyset$, i.e. there is no subvariety of $\mathbb{R}$ at all that is associated to these ideals.

Now we compute the corresponding coordinate ring of each subvariety :

$$\begin{aligned} \mathbb{R} = \mathcal{V}\big(\{\mathbf{0}\}\big) \quad &\Rightarrow \quad \mathbb{R}[\mathbb{R}] = \mathbb{R}[X]/\{\mathbf{0}\} = \mathbb{R}[X] \\ \{\alpha\} = \mathcal{V}\big(\langle f_\alpha \rangle\big) \quad &\Rightarrow \quad \mathbb{R}[\{\alpha\}] = \mathbb{R}[X]/\langle f_\alpha \rangle \cong \mathbb{R}(\alpha) = \mathbb{R} \end{aligned}$$

since f_α is the minimal polynomial of $\alpha \in \mathbb{R}$.
If we consider a "non-existing" subvariety A of $\mathbb{R}$ given by a maximal ideal of the type $\langle h_{bc} \rangle$, we get

$$\mathbb{R}[A] = \mathbb{R}[X]/\langle h_{bc} \rangle = \mathbb{R}[X]/\langle X^2 + 2bX + c \rangle \cong \mathbb{R} \oplus \mathbb{R}\bar{X}$$

with the relation $\bar{X}^2 = -2b\bar{X} - c$.

Thus $\mathbb{R}[A]$ is a 2-dimensional vector space over $\mathbb{R}$ and actually isomorphic to $\mathbb{C}$ by the isomorphism

$$\varphi \;:\; \mathbb{R}[A] \longrightarrow \mathbb{C} \;:\; \bar{X} \longmapsto -b + \sqrt{c-b^2}\cdot i$$

with $b^2 - c < 0$, $\varphi(1) = 1$ and extended by linearity. Indeed :

$$\begin{aligned}\big(-b + \sqrt{c-b^2}\cdot i\big)^2 + 2b\cdot\big(-b+\sqrt{c-b^2}\cdot i\big) + c \\ = b^2 - (c-b^2) - 2b\sqrt{c-b^2}\, i - 2b^2 + 2b\sqrt{c-b^2}\, i + c = 0\end{aligned}$$

Hence instead of describing the subvariety A as "non-existing", we should rather describe it as a subvariety of the affine real line which is $\mathbb{C}$-valued. This indeed corresponds to the fact that the polynomial h_{bc} splits over the algebraically closed field $\mathbb{C}$ into 2 factors

$$h_{bc}(X) = X^2 + 2bX + c = \big(X + b + \sqrt{c-b^2}\cdot i\big)\cdot\big(X + b - \sqrt{c-b^2}\cdot i\big)$$

Thus the ideal $\langle\, h_{bc}\,\rangle$ corresponds to a (reducible) subvariety consisting of 2 conjugate complex numbers.

2.4 Structure sheaf of the spectrum

2.4.1 Motivation

Let R be a (commutative unital) ring. So far we defined $\operatorname{Spec} R$ as the set of all prime ideals in R and endowed it with the Zariski topology, thus turning X into a topological space. These data are however not sufficient for our purposes yet. Consider the following example, which is taken from [Sch] :

Let $\mathbb{K}$ be a field and $R_1 := \mathbb{K}$, $R_2 := \mathbb{K}[X]/\langle\, X^2\,\rangle$.
R_1 being a field, $\{0\}$ is the only proper ideal in R_1 and it is also prime since fields are integral domains :

$$\operatorname{Spec} R_1 = \big\{\{0\}\big\}$$

i.e. $\operatorname{Spec} R_1$ only consists of one point. But the same holds for R_2 : as vector spaces, we have

$$R_2 = \mathbb{K}[X]/\langle\, X^2\,\rangle \cong \mathbb{K} \oplus \mathbb{K}\bar{X}$$

with the relation $\bar{X}^2 = 0$, so that R_2 is a 2-dimensional vector spaces over $\mathbb{K}$. Note that $\{0\}$ is not a prime ideal in R_2 since $\bar{X}\cdot\bar{X} \in \{0\}$, but $\bar{X} \neq 0$. A prime ideal however is $I = \langle\, \bar{X}\,\rangle = \mathbb{K}\bar{X}$. Indeed, $I \neq R_2$ and if $r = \alpha + \beta\bar{X}$ and $s = a + b\bar{X}$ are elements in R_2 such that $r\cdot s \in I$, then

$$r\cdot s = (\alpha+\beta\bar{X})\cdot(a + b\bar{X}) = a\alpha + (\alpha b + a\beta)\bar{X} \in I \qquad \Rightarrow\; a\cdot\alpha = 0 \;\Rightarrow\; a = 0 \text{ or } \alpha = 0$$

since $\mathbb{K}$ has no zero divisors (field), i.e. either $r \in I$ or $s \in I$. But I is even the only non-zero proper ideal in R_2 : let $J \trianglelefteq R_2$ be a proper ideal that contains some non-zero element $r = \alpha + \beta\bar{X}$. Then

$$r\cdot(\alpha - \beta\bar{X}) = (\alpha+\beta\bar{X})\cdot(\alpha-\beta\bar{X}) = \alpha^2 - \beta^2\bar{X}^2 = \alpha^2 \in J$$

If $\alpha \neq 0$, then $J = R_2$ since $\alpha^2 \in \mathbb{K}$ and non-zero elements in a field are always units (invertible). Hence we need that $\alpha = 0 \;\Rightarrow\; r = \beta\bar{X} \in J$ with $\beta \neq 0 \;\Rightarrow\; \langle\, \bar{X}\,\rangle = I \subseteq J$. And J cannot be strictly bigger than I, otherwise it contains an element of the form $a + b\bar{X}$ with $a \neq 0$ and would not be proper. So $I = J$ and

$$\operatorname{Spec} R_2 = \big\{\langle\, \bar{X}\,\rangle\big\}$$

Thus $\operatorname{Spec} R_1$ and $\operatorname{Spec} R_2$ both consist of just one point and are therefore homeomorphic topological spaces.

However, the reason why they are singletons are quite different and we want the spectrum to encode this information. For this, we will consider "functions" on the spectrum, defined as follows.

Definition :
Let R be a ring, $r \in R$ and $P \in \operatorname{Spec} R$. Since P is prime, R/P is an integral domain and we can construct its quotient field $\kappa(P) := \operatorname{Quot}(R/P)$ together with the injection $i_P : R/P \hookrightarrow \kappa(P)$. We define

$$f^r \;:\; P \mapsto i_P(\bar{r}) \quad \text{ where } \bar{r} = r \bmod P \in R/P$$

i.e. f^r is a function on $\operatorname{Spec} R$ such that $f^r(P) \in \kappa(P)$, $\forall\, P \in \operatorname{Spec} R$. Hence we may see $r \in R$ as a function on $\operatorname{Spec} R$. $f^r(P)$ is called the *value of r at P*. We see that f^r has values in the field $\kappa(P)$, called the *residue field* of R at P, but note that $\kappa(P)$ changes together with P! The values generally lie in different fields.

Remark :
For a subset $S \subseteq R$, we have that $f^r(P) = 0, \forall\, r \in S \;\Leftrightarrow\; r \bmod P = 0, \forall\, r \in S \;\Leftrightarrow\; S \subseteq P$.

Application
We apply this definition to the above example : for $R_1 = \mathbb{K}$ the only prime ideal is $\{0\}$, so we get

$$R_1/\{0\} = R_1 \quad \text{ and } \quad \operatorname{Quot}(R_1) \cong R_1$$

with $f^r\big(\{0\}\big) = r, \forall\, r \in R_1$: each element in R_1 acts as the identity.

This is not the case for R_2. Consider the unique prime ideal $\langle\, \bar{X}\, \rangle$ and the element $\bar{X} \in R_2$. Then

$$f^{\bar{X}}\big(\langle\, \bar{X}\,\rangle\big) = i_{\bar{X}}\big(\bar{X} \bmod \langle\, \bar{X}\,\rangle\big) = i_{\bar{X}}(0) = 0$$

We see that the function associated to an element in the ring is not determined by its values. Thus R_1 and R_2 still have different properties, even if $\operatorname{Spec} R_1 \cong \operatorname{Spec} R_2$. The notion of the spectrum as topological space is therefore not enough and we shall consider "additional data" to distinguish them.

2.4.2 Definition

In the motivation, the idea was to consider functions that are defined on the spectrum. Now we make this precise by defining a sheaf of rings on the topological space $X = \operatorname{Spec} R$, i.e. to every open set $U \subseteq X$, we want to associated a commutative unital ring. This sheaf will be denoted by $\mathcal{O}_R$. Here we follow the construction as it is done in [Ha]. Other authors may have a different approach.

Let $U \subseteq X$ be open and consider for each $P \in U$ the localization R_P of R at P, which is a local ring.

$$R_P = \left\{ \frac{a}{r} \;\middle|\; a \in R,\, r \notin P \right\} = S^{-1}R \quad \text{ with } S = R \setminus P$$

We define $\mathcal{O}_R(U)$ to be the set of all functions $f : U \to \coprod_{P\in U} R_P$ satisfying the following 2 conditions :

1) $\forall\, P \in U,\ f(P) \in R_P$.
2) f is locally a quotient of elements of R : more precisely, $\forall\, P \in U$, there is a neighborhood V_P of P in U and $\exists\, a, r \in R$ (depending on P) such that $\forall\, Q \in V_P$, $r \notin Q$ and $f(Q) = \frac{a}{r} \in R_Q$.

Thus we may see f as a "locally constant" function with values in varying local rings. Such functions exist, consider e.g. $f(P) = \frac{a}{1}$ for some $a \in R$, $\forall\, P \in U$ (this is well-defined since $1 \notin Q$, $\forall\, Q \in X$).
Sums and products of such functions are again of that type, e.g. if $f, g \in \mathcal{O}_R(U)$ and $P \in U$ with

$$f(Q) = \tfrac{a}{r}, \forall\, Q \in V_P\ ,\ g(Q) = \tfrac{b}{s}, \forall\, Q \in V'_P \qquad \Rightarrow\ (f+g)(Q) = \frac{a\cdot s + b \cdot r}{r \cdot s}\, , \forall\, Q \in V_P \cap V'_P$$

where $r \cdot s \notin Q$ since Q is prime (otherwise r or s belongs to Q), thus $f + g \in \mathcal{O}_R(U)$. The same argument also shows that $f \cdot g \in \mathcal{O}_R(U)$ and this product is commutative since R is commutative. Hence $\mathcal{O}_R(U)$ is closed under addition and multiplication. Moreover $\mathcal{O}_R(U)$ has a unit element given by the constant function $f(P) = \frac{1}{1} \in R_P$, $\forall\, P \in U$. This finally shows that $\mathcal{O}_R(U)$ is a commutative unital ring.

If $V \subseteq U$ are two open sets, the restriction $\rho_V^U : \mathcal{O}_R(U) \to \mathcal{O}_R(V)$ is nothing but the usual restriction of maps and hence a unital ring homomorphism, so that $\mathcal{O}_R : U \to \mathcal{O}_R(U)$ is a presheaf of rings.
This also implies that S1 is trivially satisfied. For S2, it suffices to see that the defining properties of the ring $\mathcal{O}_R(U)$ are local. Indeed, it $U \subseteq X$ is open with an open covering $U = \bigcup_i U_i$ and sections $f_i \in \mathcal{O}_R(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$, $\forall\, i, j$, it suffices to define $f(P) := f_i(P)$ if $P \in U_i \subset U$. This is well-defined since the f_i coincide on intersections, $f(P) \in R_P$, $\forall\, P \in U$ and f is locally a quotient since each f_i is locally a quotient, so $f \in \mathcal{O}_R(U)$ with $f_{|U_i} = f_i$, $\forall\, i$. Hence both sheaf axioms are satisfied.

Finally $\mathcal{O}_R$ is a sheaf of commutative unital rings, called the *structure sheaf* of $\operatorname{Spec} R$. In the following, we refer to the *spectrum* of the ring R not only as the topological space $\operatorname{Spec} R$, but the pair $(\operatorname{Spec} R, \mathcal{O}_R)$.

2.4.3 Proposition

Let R be a ring and $(\operatorname{Spec} R, \mathcal{O}_R)$ its spectrum. For any prime ideal $P \in \operatorname{Spec} R$, the stalk $\mathcal{O}_{R,P}$ of the sheaf $\mathcal{O}_R$ at P is isomorphic to the localization R_P (which is a local ring).

Proof. taken from [Ha]. Recall that $\forall\, P \in X$,

$$\mathcal{O}_{R,P} = \left\{ [f]_P \;\middle|\; f \sim_P g \;\Leftrightarrow\; f \text{ and } g \text{ coincide on some smaller neighborhood of } P \right\}$$

We define the map $\varphi : \mathcal{O}_{R,P} \to R_P : [f]_P \mapsto f(P)$, which is well-defined since $f(P) \in R_P$ by definition and all representatives of $[f]_P$ coincide on P. Moreover φ is a unital ring homomorphism.
$-$ φ is surjective : let $p \in R_P$, i.e. $\exists\, a, r \in R$ with $r \notin P$ such that $p = \frac{a}{r}$. We look for some $[f]_P \in \mathcal{O}_{R,P}$ such that $\varphi([f]_P) = p$. $U(r)$ is an open neighborhood of P since $r \notin P$ and for any $Q \in U(r)$, we define

$$f(Q) := \frac{a}{r} \in R_Q \qquad \Rightarrow \;\; f \in \mathcal{O}_R\big(U(r)\big)$$

so that f is defined on an open neighborhood of P (thus $[f]_P$ exists). In particular, $f(P) = \frac{a}{r} = p$ as well and we have $\varphi([f]_P) = f(P) = p$.
$-$ φ is injective : let $[f]_P, [g]_P \in \mathcal{O}_{R,P}$ be such that $f(P) = g(P)$, where $f, g \in \mathcal{O}_R(U)$ are representatives of the germs on a neighborhood $U \subset X$ of P small enough such that $f = \frac{a}{r}$ and $g = \frac{b}{s}$ on U with $a, b, r, s \in R$ and $r, s \notin P$. $f(P) = g(P)$ in R_P means by definition of localization that

$$f(P) = g(P) \;\Leftrightarrow\; \frac{a}{r} = \frac{b}{s} \;\Leftrightarrow\; \exists\, c \in R \setminus P \text{ such that } c \cdot (a \cdot s - b \cdot r) = 0 \text{ in } R$$

Hence we have $\frac{a}{r} = \frac{b}{s}$ in any localization R_Q for any prime ideal $Q \trianglelefteq R$ such that $r, s, c \notin Q$. But

$$\left\{ Q \in \operatorname{Spec} R \;\middle|\; r, s, c \notin Q \right\} = U(r) \cap U(s) \cap U(c) =: V \;\;\Rightarrow\;\; P \in V$$

thus V is an open neighborhood of P and $f(Q) = g(Q)$, $\forall\, Q \in U \cap V$. This means that f and g coincide in a whole open neighborhood of P, so that their germs at P are equal : $[f]_P = [g]_P$, so φ is injective.
Finally we have showed that φ is an isomorphism of rings and it follows that $\mathcal{O}_{R,P} \cong R_P$. $\square$

2.4.4 Theorem

Let R be a ring with spectrum $(\operatorname{Spec} R, \mathcal{O}_R)$. Then :
$\forall\, r \in R$, the ring $\mathcal{O}_R\big(U(r)\big)$ is isomorphic to the localized ring R_r at r. In particular, $\mathcal{O}_R(\operatorname{Spec} R) \cong R$.

Proof. taken from [Ha]. The particular case is obtained for $r = 1$ since $U(1) = \operatorname{Spec} R$ and $R_1 \cong R$.
For the general case, we define the ring homomorphism

$$\psi \;:\; R_r \longrightarrow \mathcal{O}_R\big(U(r)\big) \;:\; \frac{a}{r^n} \longmapsto \left(f \;:\; P \mapsto \frac{a}{r^n} \in R_P \right)$$

which is well-defined since P is prime : $r \notin P \;\Rightarrow\; r^n \notin P$, $\forall\, n \in \mathbb{N}_0$.

$-$ ψ is injective : assume that $\psi\big(\frac{a}{r^n}\big)=\psi\big(\frac{b}{r^m}\big)$. This means that $\frac{a}{r^n}=\frac{b}{r^m}$ in R_P for all $P\in U(r)$, i.e.

$$\forall\,P\in U(r),\ \exists\,h\in R\setminus P \text{ such that } h\cdot(a\cdot r^m-b\cdot r^n)=0 \text{ in } R$$

Let A be the annihilator of the element ar^m-br^n, which is clearly an ideal in R :

$$A=\big\{\,x\in R\ \big|\ x\cdot(ar^m-br^n)=0\,\big\}\trianglelefteq R$$

So $h\in A$, but $h\notin P$ implies that $A\nsubseteq P$ and hence that $P\notin V(A)$. This holds for any $P\in U(r)$, i.e. we conclude that $V(A)\cap U(r)=\emptyset$. Recall that $U(r)=X\setminus V(r)=X\setminus V\big(\langle\,r\,\rangle\big)$, so we obtain

$$V(A)\cap\Big(X\setminus V\big(\langle\,r\,\rangle\big)\Big)=\emptyset\qquad\Rightarrow\ V(A)\subseteq V\big(\langle\,r\,\rangle\big)\ \Leftrightarrow\ \mathrm{Rad}\big(\langle\,r\,\rangle\big)\subseteq\mathrm{Rad}(A)$$

by lemma 2.3.2. Thus $r\in\mathrm{Rad}(A)$, which means that $\exists\,l\in\mathbb{N}$ such that $r^l\in A$, hence $r^l\cdot(ar^m-br^n)=0$. But r^l is an allowed denominator in the localized ring R_r, hence we get $\frac{a}{r^n}=\frac{b}{r^m}$ in R_r.

$-$ ψ is surjective : this is the hard part of the proof. In fact, it is even surprising since $\mathrm{im}\,\psi$ only consists of "constant" functions on $U(r)$, whereas $\mathcal{O}_R\big(U(r)\big)$ contains in general "locally constant" functions.
So let $f\in\mathcal{O}_R\big(U(r)\big)$. By definition of the sheaf $\mathcal{O}_R$, we can cover $U(r)$ by open sets $V_k\subseteq X$ such that f is represented on each V_k by a quotient $\frac{a_k}{g_k}$ with $a_k,g_k\in R$ and $g_k\notin P$, $\forall\,P\in V_k$, implying that $V_k\subseteq U(g_k)$, $\forall\,k$. Now the open sets of the form $U(\ )$ form a basis for the Zariski topology, hence $V_k=\bigcup_i U(s_{ki})$, $\forall\,k$, for some $s_{ki}\in R$. Changing notations, we obtain :
$U(r)$ is covered by open sets $U(s_i)$ on which f is a quotient $\frac{a_i}{g_i}$ with $s_i,a_i,g_i\in R$ and $U(s_i)\subseteq U(g_i)$, $\forall\,i$.

$$\forall\,i,\ U(s_i)\subseteq U(g_i)\ \Leftrightarrow\ X\setminus U(g_i)\subseteq X\setminus U(s_i)\ \Leftrightarrow\ V\big(\langle\,g_i\,\rangle\big)\subseteq V\big(\langle\,s_i\,\rangle\big)\ \Leftrightarrow\ \mathrm{Rad}\big(\langle\,s_i\,\rangle\big)\subseteq\mathrm{Rad}\big(\langle\,g_i\,\rangle\big)$$

In particular, $\exists\,n_i\in\mathbb{N}$ such that $s_i^{n_i}\in\langle\,g_i\,\rangle$, i.e. $\exists\,c_i\in R$ with $s_i^{n_i}=c_i\cdot g_i$. Note that $c_i\neq 0$ since if s_i was nilpotent, then $U(s_i)\subseteq U(0)=\emptyset$. So

$$\frac{a_i}{g_i}=\frac{c_i\cdot a_i}{c_i\cdot g_i}=\frac{c_i\cdot a_i}{s_i^{n_i}}$$

Now set $b_i:=c_i\cdot a_i$ and $h_i:=s_i^{n_i}$, $\forall\,i$. Note that $U(h_i)=U(s_i)$ by 2.3.5. Thus we obtain that $U(r)$ is covered by open sets $U(h_i)$ on which f is represented by a quotient $\frac{b_i}{h_i}$. The goal of these modifications was to obtain a covering of $U(r)$ such that f is given on each $U(h_i)$ by a quotient with the same denominator $h_i\notin P$, $\forall\,P\in U(h_i)$.
Next we observe that $U(r)$ can be covered by finitely many of these $U(h_i)$. Indeed :

$$U(r)\subseteq\textstyle\bigcup_i U(h_i)\ \Leftrightarrow\ V\big(\langle\,r\,\rangle\big)\supseteq\bigcap_i V\big(\langle\,h_i\,\rangle\big)=V\big(\sum_i\langle\,h_i\,\rangle\big)\ \Leftrightarrow\ \mathrm{Rad}\big(\langle\,r\,\rangle\big)\subseteq\mathrm{Rad}\big(\sum_i\langle\,h_i\,\rangle\big)$$

by lemma 2.3.2 again. Hence $r\in\mathrm{Rad}\big(\sum_i\langle\,h_i\,\rangle\big)$, i.e. $\exists\,n\in\mathbb{N}$ such that $r^n\in\sum_i\langle\,h_i\,\rangle$, which means that r^n can be written as a finite sum $r^n=r_{i_1}h_{i_1}+\ldots+r_{i_m}h_{i_m}$ for some $r_{i_j}\in R$, $m\in\mathbb{N}$. It follows that

$$U(r)\subseteq U(h_{i_1})\cup\ldots\cup U(h_{i_m})$$

since a prime ideal containing $h_{i_1},\ldots,h_{i_m}$ also contains r^n and thus r as well. Hence from now on we may assume that there is a finite set $\{h_1,\ldots,h_m\}$ in R such that $U(r)\subseteq U(h_1)\cup\ldots\cup U(h_m)$.
Recall that $U(h_i)\cap U(h_j)=U(h_i\cdot h_j)$. Since the representation of f on any of the open sets $U(h_i)$ is unique, we need that the 2 representations of f on $U(h_i)\cap U(h_j)$ coincide : $f(P)=\frac{b_i}{h_i}=\frac{b_j}{h_j}\in R_P$, $\forall\,P\in U(h_ih_j)$. Due to injectivity of ψ proved above, this means that $\frac{b_i}{h_i}=\frac{b_j}{h_j}$ as elements in $R_{h_ih_j}$. Thus $\exists\,n\in\mathbb{N}$ such that

$$(h_ih_j)^n\cdot(b_i\cdot h_j-b_j\cdot h_i)=0 \tag{2.12}$$

This n depends a priori on i and on j, but since only finitely many indices are involved, we may choose n large enough so that (2.12) holds for all i,j. Rewriting (2.12) yields that

$$h_j^{n+1}\cdot(h_i^n\,b_i)-h_i^{n+1}\cdot(h_j^n\,b_j)=0\ ,\ \forall\,i,j\in\{1,\ldots,m\}$$

Then we set $h'_i := h_i^{n+1}$ and $b'_i := h_i^n\, b_i$, so that $U(h_i) = U(h'_i)$ and f is represented on $U(h'_i)$ by

$$\frac{b_i}{h_i} = \frac{h_i^n \cdot b_i}{h_i^{n+1}} = \frac{b'_i}{h'_i}$$

with the relation $h'_j \cdot b'_i = h'_i \cdot b'_j$, $\forall\, i, j$. Since $\{U(h'_1), \ldots, U(h'_m)\}$ is still a finite open cover of $U(r)$, we can write again $r^n = r_1 h'_1 + \ldots + r_m h'_m$ for some $n \in \mathbb{N}$ and $r_i \in R$. Define $a := r_1 b'_1 + \ldots + r_m b'_m$. Then

$$\forall\, j \;:\; h'_j \cdot a = \sum_{i=1}^{m} r_i \cdot \big(h'_j \cdot b'_i\big) = \sum_{i=1}^{m} r_i \cdot \big(h'_i \cdot b'_j\big) = \sum_{i=1}^{m} \big(r_i \cdot h'_i\big) \cdot b'_j = r^n \cdot b'_j \qquad \Rightarrow\; \frac{a}{r^n} = \frac{b'_j}{h'_j}\,,\; \forall\, j$$

so finally

$$\forall\, i \in \{1, \ldots, m\},\; \forall\, P \in U(h'_i) \;:\; f(P) = \frac{b'_i}{h'_i} = \frac{a}{r^n}$$

i.e. f coincides with $\frac{a}{r^n}$ on each $U(h'_i)$, thus everywhere on $U(r)$. It follows that $f = \psi\big(\frac{a}{r^n}\big)$ and hence that ψ is surjective. So we showed that ψ is indeed a ring isomorphism and that $\mathcal{O}_R\big(U(r)\big) \cong R_r$. □

Remark :
Some authors, for example [Sch], use this property as the actual definition of the structure sheaf by setting

$$\mathcal{O}_R(\operatorname{Spec} R) := R \qquad , \qquad \mathcal{O}_R\big(U(r)\big) := R_r,\; \forall\, r \in R$$

and for $U(r \cdot s) = U(r) \cap U(s) \subseteq U(r)$, the restriction morphisms are given by

$$\rho^r_{rs} \;:\; R_r \to R_{r \cdot s} \cong (R_r)_s \;:\; g \mapsto \frac{g}{1} \tag{2.13}$$

One can then show that these are compatible on the intersections of the basis open sets (hence they indeed define restrictions). The hard part however is to check that the sheaf axioms S1 and S2 are satisfied for the $U(r)$ with respect to their intersections. If $r \in R$ and $\{U(r_i)\}_{i \in J}$ is an open covering of $U(r)$, we have

- S1 If $g, h \in R_r = \mathcal{O}_R\big(U(r)\big)$ are such that $g = h$ as elements in $R_{r_i} = \mathcal{O}_R\big(U(r_i)\big)$, $\forall\, i \in J$, then $g = h$ in R_r as well.
- S2 If $g_i \in R_{r_i} = \mathcal{O}_R\big(U(r_i)\big)$ are such that $g_i = g_j$ in $R_{r_i r_j} = \mathcal{O}_R\big(U(r_i r_j)\big)$, $\forall\, i, j \in J$, then $\exists\, g \in R_r$ such that $g = g_i$ in R_{r_i}, $\forall\, i \in J$.

A proof of these properties is also given in [Sch]. It actually suffices to define the sheaf on these open sets since they are a basis for the topology on $\operatorname{Spec} R$. On an arbitrary open subset $U \subseteq \operatorname{Spec} R$, $\mathcal{O}_R(U)$ is then defined by some general concept, called projective limit. Formally,

$$\mathcal{O}_R(U) := \operatorname*{proj\,lim}_{U(r) \subseteq U} \mathcal{O}_R\big(U(r)\big)$$

$\mathcal{O}_R(U)$ is then again a commutative unital ring. This will define the whole sheaf $\mathcal{O}_R$.

2.5 Ringed spaces and locally ringed spaces

To each (commutative unital) ring R, we can thus associate its spectrum $(\operatorname{Spec} R, \mathcal{O}_R)$, where $\operatorname{Spec} R$ is a topological space and $\mathcal{O}_R$ is a sheaf on this topological space. We would like to say that this assignment is functorial. For this, we first need an appropriate category in which such a requirement makes sense. Some references for this section are [Ha], [Ja] and [U2].

2.5.1 Definitions

A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, called the *structure sheaf* on X, i.e. $\forall\, U \subseteq X$ open, $\mathcal{O}_X(U)$ is a commutative unital ring. In particular, $\forall\, x \in X$, the stalk $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ at x is a ring as well.
A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if the stalk $\mathcal{O}_{X,x}$ is a local ring, $\forall\, x \in X$, i.e. every $\mathcal{O}_{X,x}$ admits a unique maximal ideal $\mathfrak{M}_x \trianglelefteq \mathcal{O}_{X,x}$.

Examples :
1) $(\operatorname{Spec} R, \mathcal{O}_R)$ is a ringed space for any ring R.
2) Let M be a real differentiable manifold and consider its sheaf C^∞_M of smooth functions on M :

$$\forall\, U \subseteq M,\ C^\infty_M(U) := \left\{\, f\, :\, U \to \mathbb{R} \;\middle|\; f \text{ is smooth on } U \,\right\}$$

$C^\infty_M(U)$ is a commutative unital $\mathbb{R}$-algebra (so in particular a ring) and hence (M, C^∞_M) is a ringed space. And it is even a locally ringed space because for all $m \in M$, the unique maximal ideal of $C^\infty_{M,m}$ is

$$\mathfrak{M}_m = \left\{\, [f]_m \in C^\infty_{M,m} \;\middle|\; f(m) = 0 \,\right\} \trianglelefteq C^\infty_{M,m}$$

because every germ of smooth functions at m which does not vanish at m is invertible :

$$f(m) \neq 0 \;\Rightarrow\; f \neq 0 \text{ in an neighborhood of } m \;\Rightarrow\; \left[\tfrac{1}{f}\right]_m \text{ exists and } \left[\tfrac{1}{f}\right]_m \cdot [f]_m = 1$$

Thus $\mathfrak{M}_m$ is the set consisting of all germs which are not units. This is an ideal, hence $C^\infty_{M,m}$ is a local ring by the criterion in section 2.2.5.

Remarks :
1) If $(X, \mathcal{O}_X)$ is a locally ringed space, $U \subseteq X$ an open subset and if we denote $\mathcal{O}_U := \mathcal{O}_{X|U}$, then $(U, \mathcal{O}_U)$ is a locally ringed space as well because $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$, $\forall\, x \in U$.
2) There exist ringed spaces which are not locally ringed spaces. A trivial example is obtained if $X = \{x\}$ is a singleton ; hence giving a sheaf of rings on X is equivalent to giving a ring. Define $\mathcal{O}_X(\{x\}) := R$, where R is any ring that is not local. Then $(X, \mathcal{O}_X)$ is not a locally ringed space.

2.5.2 Lemma

Let R be a local ring with unique maximal ideal $M \trianglelefteq R$ and let $r \in R$. Then $r \notin M \;\Leftrightarrow\; r$ is a unit.

Proof. $\underline{\Leftarrow}$: if r is a unit, then $\exists\, s \in R$ such that $r \cdot s = 1$, so $1 \in M$: contradiction.
$\underline{\Rightarrow}$: if $r \notin M$, then $r \neq 0$ and $\langle\, r\,\rangle$ is therefore a non-zero ideal in R. If it is proper, it would be contained in some maximal ideal, hence $\langle\, r\,\rangle \subseteq M$ since M is the only maximal one. But this is not possible since $r \notin M$, so $\langle\, r\,\rangle = R$, implying that r is a unit because $1 \in \langle\, r\,\rangle$. ☐

Corollary :
This holds in particular in the case where $(X, \mathcal{O}_X)$ is a locally ringed space. For $x \in X$, denote the unique maximal ideal in the stalk by $\mathfrak{M}_x \trianglelefteq \mathcal{O}_{X,x}$. Then $s \in \mathcal{O}_{X,x} \setminus \mathfrak{M}_x \;\Leftrightarrow\; s$ is a unit in $\mathcal{O}_{X,x}$.

2.5.3 Local homomorphisms

Let R_1 and R_2 be local rings with unique maximal ideals $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. We say that a ring homomorphism $\varphi : R_1 \to R_2$ is a *local homomorphism* if is preserves the maximal ideals, i.e. if

$$\varphi(\mathfrak{M}_1) \subseteq \mathfrak{M}_2 \quad \text{or equivalently,} \;\; \mathfrak{M}_1 = \varphi^{-1}(\mathfrak{M}_2)$$

Proof. $\underline{\Leftarrow}$: let $r' = \varphi(r)$ for some $r \in \mathfrak{M}_1 = \varphi^{-1}(\mathfrak{M}_2) \;\Rightarrow\; r' = \varphi(r) \in \mathfrak{M}_2$
$\underline{\Rightarrow}$: let $r \in \mathfrak{M}_1$, then $\varphi(r) \in \mathfrak{M}_2 \;\Rightarrow\; r \in \varphi^{-1}(\mathfrak{M}_2)$
Conversely, let $r \in \varphi^{-1}(\mathfrak{M}_2)$, i.e. $\varphi(r) \in \mathfrak{M}_2$. We want to show that $r \in \mathfrak{M}_1$. Consider the ideal $I = \langle\, r\,\rangle$ generated by r. Then $I \neq R$, otherwise we can write $1 = s \cdot r$ for some $s \in R_1$ and hence

$$1 = \varphi(1) = \varphi(s) \cdot \varphi(r) \text{ with } \varphi(r) \in \mathfrak{M}_2 \;\;\Rightarrow\;\; 1 \in \mathfrak{M}_2 \;:\; \text{contradiction}$$

So I is proper and thus contained in some maximal ideal of R_1. But $\mathfrak{M}_1$ is the unique maximal ideal of R_1, hence $\langle\, r\,\rangle = I \subseteq \mathfrak{M}_1 \;\Rightarrow\; r \in \mathfrak{M}_1$ as well. ☐

2.5.4 Morphisms of ringed spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A *morphism of ringed spaces* $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $\Phi = (\phi, \phi^*)$ where $\phi : X \to Y$ is a continuous map, called the *base map* of Φ, and $\phi^* : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ is a morphism of sheaves, called the *pullback morphism* of Φ. Recall that this means

$$\forall\, V \subseteq Y \text{ open}\,, \;\; \phi_V^* \;:\; \mathcal{O}_Y(V) \to \mathcal{O}_X\big(\phi^{-1}(V)\big)$$

such that ϕ_V^* is a unital ring homomorphism and ϕ^* commutes with the restrictions of $\mathcal{O}_Y$ and $\phi_* \mathcal{O}_X$, i.e.

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\phi_V^*"] \arrow[d] & \mathcal{O}_X\big(\phi^{-1}(V)\big) \arrow[d] \\ \mathcal{O}_Y(W) \arrow[r, "\phi_W^*"] & \mathcal{O}_X\big(\phi^{-1}(W)\big) \end{tikzcd} \qquad \phi_W^* \circ \rho'^V_W = \rho^{\phi^{-1}(V)}_{\phi^{-1}(W)} \circ \phi_V^*$$

for any inclusion of open sets $W \subseteq V \subseteq Y$. Thus Φ respects both the topological and the sheaf structure of the objects.

Ringed spaces and morphisms of ringed spaces form a category, denoted `RS`, with respect to the following composition. If $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\Psi = (\psi, \psi^*) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, then

$$\Psi \circ \Phi = (\psi, \psi^*) \circ (\phi, \phi^*) = (\psi \circ \phi, \phi^* \circ \psi^*) \;\; : \;\; (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z) \tag{2.14}$$

where $\phi^* \circ \psi^* : \mathcal{O}_Z \to \phi_* \psi_* \mathcal{O}_X = (\psi \circ \phi)_* \mathcal{O}_X$ because $\phi^{-1}\big(\psi^{-1}(W)\big) = (\psi \circ \phi)^{-1}(W)$, $\forall\, W \subseteq Z$ open.

2.5.5 Morphisms of locally ringed spaces

Now assume that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces and let $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then the morphism of sheaves $\phi^* : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ induces a ring homomorphism

$$\phi_y^* \;:\; \mathcal{O}_{Y,y} \to (\phi_* \mathcal{O}_X)_y$$

on the stalks for all $y \in Y$ (see section 2.1.2), thus in particular for those of the form $y = \phi(x)$ with $x \in X$. Composing with the natural map in (2.3), we obtain a chain of ring homomorphisms

$$\mathcal{O}_{Y,\phi(x)} \longrightarrow (\phi_* \mathcal{O}_X)_{\phi(x)} \longrightarrow \mathcal{O}_{X,x}$$

By abuse of notation, we denote this map by $\phi_x^* : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$. Note that ϕ_x^* is a ring homomorphism between local rings.

We say that a morphism of ringed spaces $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a *morphism of locally ringed spaces* if all induced ring homomorphisms on the stalks are local homomorphisms, i.e. $\forall\, x \in X$, ϕ_x^* satisfies

$$\phi_x^*\big(\mathfrak{M}_{\phi(x)}\big) \subseteq \mathfrak{M}_x$$

Locally ringed spaces and morphisms of locally ringed spaces again form a category, denoted by `LRS`, with the same composition as above. However `LRS` is not a full subcategory of `RS` since it has additional data, namely the unique maximal ideals of the stalks and morphisms of locally ringed spaces need to preserve these additional data. An example of a morphism of ringed spaces which is not a morphism of locally ringed spaces is given in section 3.1.2.

As usual, we say that $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism of (locally) ringed spaces if there exists a morphism of (locally) ringed spaces $\Psi : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$. Hence by definition of the composition in (2.14), we have that $\Phi = (\phi, \phi^*)$ is an isomorphism if and only if the base map $\phi : X \to Y$ is a homeomorphism of the underlying topological spaces and $\phi^* : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ is an isomorphism of sheaves (as defined in 2.1.1).

Finally we have the following nice property :

2.5.6 Proposition

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces and $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ an isomorphism of ringed spaces. Then Φ is an isomorphism of locally ringed spaces.

Proof. If $\Phi = (\phi, \phi^*)$ is an isomorphism of ringed spaces, then ϕ^* is an isomorphism of sheaves and ϕ^*_x is an isomorphism of rings, $\forall\, x \in X$ by proposition 2.1.4. Thus the statement follows if we prove that any ring isomorphism $\varphi : R \to T$ between local rings R and T is also a local homomorphism.
Let $\mathfrak{M}_R \trianglelefteq R$ and $\mathfrak{M}_T \trianglelefteq T$ be the unique maximal ideals. Then $\varphi^{-1}(\mathfrak{M}_T)$ is a prime ideal in R, hence $\varphi^{-1}(\mathfrak{M}_T) \subseteq \mathfrak{M}_R$ since any proper ideal in R is contained in a maximal ideal, but $\mathfrak{M}_R$ is the only maximal one. If $\varphi^{-1}(\mathfrak{M}_T) \subsetneq \mathfrak{M}_R$, then $\mathfrak{M}_T \subsetneq \varphi(\mathfrak{M}_R) \subsetneq \varphi(R) = T$ since φ is an isomorphism, which is not possible since $\mathfrak{M}_T$ is maximal. Hence $\varphi^{-1}(\mathfrak{M}_T) = \mathfrak{M}_R$. □

2.6 The Spec–functor

We know that $(\operatorname{Spec} R, \mathcal{O}_R)$ is a ringed space for any ring R. But it is even a locally ringed space because of proposition 2.4.3, which states that $\mathcal{O}_{R,P} \cong R_P$, $\forall\, P \in \operatorname{Spec} R$. The localization R_P of R at P is always a local ring, hence so is the stalk $\mathcal{O}_{R,P}$.
In this section we base ourselves again on ideas from [Ha], but we will be more explicit at some points, e.g. by putting definitions into formulas and filling in details that have been omitted in the textbook.

2.6.1 Definition

Let $\varphi : R \to T$ be a unital ring homomorphism. We can define a map $\phi : \operatorname{Spec} T \to \operatorname{Spec} R$ by setting

$$\phi(P) := \varphi^{-1}(P)\,,\ \forall\, P \in \operatorname{Spec} T$$

This is well-defined since the preimage under a ring homomorphism of a prime ideal in T is a prime ideal in R. We will show in 2.6.2 that ϕ is continuous. Denote $\phi = \operatorname{Spec} \varphi$; the assignment $\varphi \mapsto \operatorname{Spec} \varphi$ is functorial (in the contravariant sense) because for $\varphi : R \to T$, $\psi : T \to K$ and $\forall\, P \in \operatorname{Spec} K$, we have

$$\operatorname{Spec}(\psi \circ \varphi)(P) = (\psi \circ \varphi)^{-1}(P) = \varphi^{-1}\big(\psi^{-1}(P)\big) = \operatorname{Spec} \varphi\big(\operatorname{Spec} \psi(P)\big)$$

i.e. $\operatorname{Spec}(\psi \circ \varphi) = \operatorname{Spec} \varphi \circ \operatorname{Spec} \psi$. Hence we have a contravariant functor $\operatorname{Spec} : \texttt{Ring} \to \texttt{Top}$.

Now let $U \subseteq \operatorname{Spec} R$ be open. Then φ moreover induces a ring homomorphism $\phi^*_U : \mathcal{O}_R(U) \to \phi_* \mathcal{O}_T(U)$ defined as follows :
First note that for any prime ideal $P \trianglelefteq T$, φ defines a ring homomorphism φ_P on the localizations by

$$\varphi_P \ :\ R_{\varphi^{-1}(P)} \to T_P \ :\ \frac{a}{r} \longmapsto \frac{\varphi(a)}{\varphi(r)}$$

This is well-defined because $\varphi(r) \notin P$ whenever $r \notin \varphi^{-1}(P)$ and if $\frac{a}{r} = \frac{b}{s}$ in $R_{\varphi^{-1}(P)}$, then

$$\exists\, c \in R \setminus \varphi^{-1}(P) \text{ such that } c \cdot (as - br) = 0 \ \Rightarrow\ \varphi(c) \cdot \big(\varphi(a) \cdot \varphi(s) - \varphi(b) \cdot \varphi(r)\big) = 0$$

with $\varphi(c) \in T \setminus P$, hence $\frac{\varphi(a)}{\varphi(r)} = \frac{\varphi(b)}{\varphi(c)}$ in T_P.
Moreover φ_P is a local homomorphism since the unique maximal ideals of the localizations are (see 2.2.5)

$$\mathfrak{M}_{\varphi^{-1}(P)} = \left\{ \frac{a}{r} \in R_{\varphi^{-1}(P)} \;\middle|\; a \in \varphi^{-1}(P) \right\} \quad \text{and} \quad \mathfrak{M}_P = \left\{ \frac{a'}{r'} \in T_P \;\middle|\; a' \in P \right\}$$

so that $\varphi_P\big(\mathfrak{M}_{\varphi^{-1}(P)}\big) \subseteq \mathfrak{M}_P$. Finally, if $\varphi' : T \to K$ is another ring homomorphism, then $\forall\, P \in \operatorname{Spec} K$

$$(\varphi' \circ \varphi)_P = \varphi'_P \circ \varphi_{\varphi'^{-1}(P)} \ :\ R_{\varphi^{-1}(\varphi'^{-1}(P))} \to T_{\varphi'^{-1}(P)} \to K_P$$

Now let $f \in \mathcal{O}_R(U)$, i.e. $f : U \to \coprod_{Q\in U} R_Q$ such that $f(Q) \in R_Q$, $\forall\, Q \in U$ and f is locally given by a quotient. We denote $W := \phi^{-1}(U)$ and define a map $g : W \to \coprod_{P\in W} T_P$ by the chain of maps

$$g \ : \ P \overset{\phi}{\longmapsto} \underbrace{\phi(P)}_{\in U} \overset{f}{\longmapsto} \underbrace{f\big(\phi(P)\big)}_{\in R_{\phi(P)}} \overset{\varphi_P}{\longmapsto} \varphi_P\Big(f\big(\phi(P)\big)\Big) \in T_P \tag{2.15}$$

This g is also locally a quotient : let $P \in W$ and assume that V is an open neighborhood of $\phi(P)$ in U over which f is a quotient, i.e. $f(Q') = \frac{a}{r}$ with $r \notin Q'$, $\forall\, Q' \in V$. So $W' := \phi^{-1}(V)$ is an open neighborhood of P in W and

$$\forall\, Q \in W' \ : \ g(Q) = \varphi_P\Big(f\big(\phi(Q)\big)\Big) = \varphi_P\big(\tfrac{a}{r}\big) = \tfrac{\varphi(a)}{\varphi(r)}$$

since $\phi(Q) \in V$ and $\varphi(r) \notin Q$, $\forall\, Q \in W'$, otherwise $r \in \varphi^{-1}(Q) = \phi(Q) \in V$, which contradicts the local expression of f on V. Hence we obtain that $g \in \mathcal{O}_T(W) = \mathcal{O}_T\big(\phi^{-1}(U)\big) = \phi_*\mathcal{O}_T(U)$. This defines

$$\phi^*_U \ : \ \mathcal{O}_R(U) \to \phi_*\mathcal{O}_T(U) \ : \ f \mapsto \phi^*_U(f)$$

which is a unital ring homomorphism by definition (2.15). This formula also shows that

$$\forall\, U \subseteq \operatorname{Spec} R,\ f \in \mathcal{O}_R(U),\ P \in \phi^{-1}(U) \ : \ \phi^*_U(f)(P) = \varphi_P\Big(f\big(\phi(P)\big)\Big) = \varphi_P\Big(f\big(\varphi^{-1}(P)\big)\Big) \tag{2.16}$$

The assignment $\varphi \mapsto \phi^*_U$ behaves again functorially because for $\varphi : R \to T$ and $\varphi' : T \to K$, we have

$$\begin{aligned}(\phi' \circ \phi)^*_U(f)(P) &= (\varphi' \circ \varphi)_P\Big(f\big((\varphi' \circ \varphi)^{-1}(P)\big)\Big) = \big(\varphi'_P \circ \varphi_{\varphi'^{-1}(P)}\big)\Big(f\big(\varphi^{-1}\big(\varphi'^{-1}(P)\big)\big)\Big) \\ &= \varphi'_P\Big(\phi^*_U(f)\big(\varphi'^{-1}(P)\big)\Big) = \phi'^*_{\phi^{-1}(U)}\big(\phi^*_U(f)\big)(P)\end{aligned}$$

for all $f \in \mathcal{O}_R(U)$ and $P \in (\phi' \circ \phi)^{-1}(U)$. It follows that $(\phi' \circ \phi)^*_U = \phi'^*_{\phi^{-1}(U)} \circ \phi^*_U$.

We are now able to state the main theorem about the functorial behaviour of the spectrum of a ring. The proof of this theorem is taken from [Ha], but we extended the presentation by adding some more details, diagrams, formulas and explicit computations.

2.6.2 Theorem

Let $\varphi : R \to T$ be a unital ring homomorphism. Then the above induced maps define a morphism of locally ringed spaces

$$(\phi, \phi^*) \ : \ (\operatorname{Spec} T, \mathcal{O}_T) \to (\operatorname{Spec} R, \mathcal{O}_R)$$

where $\phi = \operatorname{Spec}\varphi$. We denote $(\phi, \phi^*) = \mathtt{Spec}(\varphi)$. Hence we have a contravariant functor $\mathtt{Spec} : \mathtt{Ring} \to \mathtt{LRS}$ that assigns to a ring R its spectrum $(\operatorname{Spec} R, \mathcal{O}_R)$ and to any ring homomorphism φ a corresponding morphism of locally ringed spaces $\mathtt{Spec}(\varphi)$ in the opposite direction.
Moreover the functor $\mathtt{Spec}$ is fully faithful, i.e. any morphism of locally ringed spaces from $\operatorname{Spec} T$ to $\operatorname{Spec} R$ is induced by a unique ring homomorphism $\varphi : R \to T$. Thus for any rings R, T, we have a bijection

$$\operatorname{Hom}_{\mathtt{Ring}}(R, T) \xrightarrow{\sim} \operatorname{Hom}_{\mathtt{LRS}}\big((\operatorname{Spec} T, \mathcal{O}_T), (\operatorname{Spec} R, \mathcal{O}_R)\big)$$

Proof. 1) a) First we need to show that $\phi : \operatorname{Spec} T \to \operatorname{Spec} R$ is continuous. This is true because $\forall\, I \trianglelefteq R$,

$$\phi^{-1}\big(V(I)\big) = V\big(\varphi(I)\big) = V\big(\langle\, \varphi(I)\,\rangle\big)$$

where $Q \in \phi^{-1}\big(V(I)\big) \ \Leftrightarrow\ \phi(Q) \in V(I) \ \Leftrightarrow\ I \subseteq \phi(Q) = \varphi^{-1}(Q) \ \Leftrightarrow\ \varphi(I) \subseteq Q \ \Leftrightarrow\ Q \in V\big(\varphi(I)\big)$ for any prime ideal $Q \trianglelefteq T$, i.e. preimages of closed sets under ϕ are again closed.

b) Next we need that the ring homomorphisms $\phi_U^* : \mathcal{O}_R(U) \to \phi_*\mathcal{O}_T(U)$ commute with restrictions, i.e.

$$\begin{tikzcd} \mathcal{O}_R(U) \arrow[r, "\phi_U^*"] \arrow[d] & \mathcal{O}_T\big(\phi^{-1}(U)\big) \arrow[d] \\ \mathcal{O}_R(V) \arrow[r, "\phi_V^*"] & \mathcal{O}_T\big(\phi^{-1}(V)\big) \end{tikzcd}$$

for any inclusion of open sets $V \subseteq U \subseteq \operatorname{Spec} R$. But this is clear since the restrictions of the structure sheaf of a spectrum are nothing but the usual restrictions of maps (see section 2.4.2) :

$$\forall\, f \in \mathcal{O}_R(U) \ : \ \phi_U^*(f)_{|\phi^{-1}(V)} = \phi_V^*(f_{|V})$$

c) Finally the induced ring homomorphisms on the stalks $\phi_P^* : \mathcal{O}_{R,\phi(P)} \to \mathcal{O}_{T,P}$ have to be local homomorphisms, $\forall\, P \in \operatorname{Spec} T$. This follows from the identification of the stalks of the structure sheaf given in proposition 2.4.3. Indeed, we have by construction

$$\phi_P^* \ : \ \mathcal{O}_{R,\phi(P)} \to \mathcal{O}_{T,P} \ : \ [f]_{\phi(P)} \mapsto \big[\phi_U^*(f)\big]_P$$

with $\mathcal{O}_{R,\phi(P)} \cong R_{\phi(P)} = R_{\varphi^{-1}(P)}$, $\mathcal{O}_{T,P} \cong T_P$ and the isomorphism is given by evaluating the germ at the considered point. Hence we have the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_{R,\phi(P)} \arrow[r, "\phi_P^*"] \arrow[d, "\cong"'] & \mathcal{O}_{T,P} \arrow[d, "\cong"] \\ R_{\varphi^{-1}(P)} \arrow[r, "\varphi_P"] & T_P \end{tikzcd} \qquad \begin{tikzcd} {[f]_{\phi(P)}} \arrow[r, mapsto] \arrow[d, mapsto] & \big[\phi_U^*(f)\big]_P \arrow[d, mapsto] \\ f\big(\phi(P)\big) \arrow[r, mapsto] & \phi_U^*(f)(P) \end{tikzcd} \tag{2.17}$$

where $\phi_U^*(f)(P) = \varphi_P\big(f\big(\phi(P)\big)\big)$. So after identification, ϕ_P^* is nothing but $\varphi_P : R_{\varphi^{-1}(P)} \to T_P$, which is a local homomorphism as shown above. It follows now that (ϕ, ϕ^*) is a morphism of locally ringed spaces.

2) To show that the functor `Spec` : `Ring` $\to$ `LRS` is faithful, we have to prove that 2 ring homomorphisms $\varphi, \varphi' : R \to T$ that define the same morphism $(\phi, \phi^*) = \operatorname{Spec}\varphi = \operatorname{Spec}\varphi' = (\phi', \phi'^*)$ are actually equal. Denote $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} T$. Then we have the global pullback morphisms

$$\phi_X^* \ : \ \mathcal{O}_R(X) \to \mathcal{O}_T\big(\phi^{-1}(X)\big) = \mathcal{O}_T(Y) \quad \text{and} \quad \phi_X'^* \ : \ \mathcal{O}_R(X) \to \mathcal{O}_T(Y)$$

By theorem 2.4.4, we have that $\mathcal{O}_R(X) = \mathcal{O}_R(\operatorname{Spec} R) \cong R$ and $\mathcal{O}_T(Y) = \mathcal{O}_T(\operatorname{Spec} T) \cong T$, thus ϕ_X^* defines a ring homomorphism $R \to T$. And this is exactly φ because of the identification

$$\begin{tikzcd} \mathcal{O}_R(X) \arrow[r, "\phi_X^*"] & \mathcal{O}_T(Y) \\ R \arrow[r, "\varphi"] \arrow[u, "\cong"] & T \arrow[u, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} \big(f : P \mapsto \tfrac{r}{1}\big) \arrow[r, mapsto] & \big(g : P \mapsto \varphi_P\big(\tfrac{r}{1}\big)\big) \\ r \arrow[u, mapsto] \arrow[r, mapsto] & \varphi(r) \arrow[u, mapsto] \end{tikzcd}$$

where $\varphi_P\big(\frac{r}{1}\big) = \frac{\varphi(r)}{\varphi(1)} = \frac{\varphi(r)}{1}$. But $\phi_X^* = \phi_X'^*$, hence we recover the same homomorphisms : $\varphi = \varphi'$.

3) So it remains to show that `Spec` : `Ring` $\to$ `LRS` is full, i.e. that the injective map

$$\operatorname{Hom}_{\texttt{Ring}}(R, T) \longrightarrow \operatorname{Hom}_{\texttt{LRS}}\big((\operatorname{Spec} T, \mathcal{O}_T), (\operatorname{Spec} R, \mathcal{O}_R)\big)$$

is also surjective. Let (ψ, ψ^*) be a morphism of locally ringed spaces from $\operatorname{Spec} T$ to $\operatorname{Spec} R$. As above, the global pullback defines a (unique) ring homomorphism

$$\psi_X^* \ : \ \mathcal{O}_R(\operatorname{Spec} R) \to \mathcal{O}_T(\operatorname{Spec} T) \quad \Rightarrow \quad \exists\, \varphi : R \to T$$

In particular, φ now defines the morphism $\texttt{Spec}(\varphi) = (\phi, \phi^*)$ from $\operatorname{Spec} T$ to $\operatorname{Spec} R$ as usual. We want to show that $(\psi, \psi^*) = (\phi, \phi^*)$. $\forall\, P \in \operatorname{Spec} T$, we also have the induced maps on the stalks

$$\psi_P^* \ : \ \mathcal{O}_{R,\psi(P)} \to \mathcal{O}_{T,P} \quad \Rightarrow \ \exists\, h_P \ : \ R_{\psi(P)} \to T_P$$

Since all these ring homomorphisms are induced by the morphism of sheaves $\psi^* : \mathcal{O}_R \to \psi_* \mathcal{O}_T$, they have to be compatible (in the sense that they commute with the localization homomorphisms) because we identified the commutative diagrams

$$\begin{tikzcd} \mathcal{O}_R(X) \arrow[r, "\psi_X^*"] \arrow[d] & \mathcal{O}_T(Y) \arrow[d] \\ \mathcal{O}_{R,\psi(P)} \arrow[r, "\psi_P^*"] & \mathcal{O}_{T,P} \end{tikzcd} \quad \cong \quad \begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & T \arrow[d, "\ell_P"] \\ R_{\psi(P)} \arrow[r, "h_P"] & T_P \end{tikzcd} \tag{2.18}$$

where $\ell_P : T \to T_P$ is the localization homomorphism and h_P is a local homomorphism since ψ_P^* is one. But the local homomorphism $\varphi_P : R_{\varphi^{-1}(P)} \to T_P$ induced by φ satisfies a similar property :

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "l_P"'] & T \arrow[d, "\ell_P"] \\ R_{\psi(P)} \arrow[r, "h_P"] & T_P \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "l'_P"'] & T \arrow[d, "\ell_P"] \\ R_{\varphi^{-1}(P)} \arrow[r, "\varphi_P"] & T_P \end{tikzcd}$$

Let $\mathfrak{M}_{\psi(P)}$, $\mathfrak{M}_{\varphi^{-1}(P)}$ and $\mathfrak{M}_P$ be the unique maximal ideals of $R_{\psi(P)}$, $R_{\varphi^{-1}(P)}$ and T_P respectively. Then

$$\mathfrak{M}_{\psi(P)} = h_P^{-1}\big(\mathfrak{M}_P\big) \quad \text{ and } \quad \mathfrak{M}_{\varphi^{-1}(P)} = \varphi_P^{-1}\big(\mathfrak{M}_P\big)$$

with $\psi(P) = l_P^{-1}\big(\mathfrak{M}_{\psi(P)}\big)$ and $\varphi^{-1}(P) = l_P'^{-1}\big(\mathfrak{M}_{\varphi^{-1}(P)}\big)$. This follows immediately from the expression of the unique maximal ideal of the localization ring (see section 2.2.5 : the preimage of $S^{-1}P$ is P). Thus

$$\begin{aligned} \psi(P) = l_P^{-1}\big(\mathfrak{M}_{\psi(P)}\big) &= l_P^{-1}\Big(h_P^{-1}\big(\mathfrak{M}_P\big)\Big) = (h_P \circ l_P)^{-1}\big(\mathfrak{M}_P\big) = (\ell_P \circ \varphi)^{-1}\big(\mathfrak{M}_P\big) \\ &= (\varphi_P \circ l'_P)^{-1}\big(\mathfrak{M}_P\big) = l_P'^{-1}\Big(\varphi_P^{-1}\big(\mathfrak{M}_P\big)\Big) = l_P'^{-1}\big(\mathfrak{M}_{\varphi^{-1}(P)}\big) = \varphi^{-1}(P) \end{aligned}$$

i.e. $\psi(P) = \varphi^{-1}(P) = \phi(P)$, which means that $\psi = \phi = \operatorname{Spec} \varphi$ is induced by $\varphi : R \to T$. As a consequence $l_P = l'_P$ and $h_P = \varphi_P$ as well. To show that $\psi^* = \varphi^*$, we use the identifications in (2.17) and (2.18) :

$$\begin{aligned} (2.17) \quad &\Rightarrow \quad (\, \varphi_P \ : \ R_{\phi(P)} \to T_P \,) \longleftrightarrow (\, \phi_P^* \ : \ \mathcal{O}_{R,\phi(P)} \to \mathcal{O}_{T,P} \,) \\ (2.18) \quad &\Rightarrow \quad (\, h_P \ : \ R_{\psi(P)} \to T_P \,) \longleftrightarrow (\, \psi_P^* \ : \ \mathcal{O}_{R,\psi(P)} \to \mathcal{O}_{T,P} \,) \end{aligned}$$

But $\varphi_P = h_P$, hence $\phi_P^* = \psi_P^*$, $\forall\, P \in \operatorname{Spec} T$. Proposition 2.1.5 then implies that $\phi^* = \psi^*$. Finally we conclude that $(\psi, \psi^*) = (\phi, \phi^*) = \texttt{Spec}(\varphi)$, i.e. (ψ, ψ^*) has been induced by φ via the $\texttt{Spec}$–functor. □

2.6.3 Conclusion

Let us summarize the results of this chapter. Consider a commutative unital ring R.
To R we can associate the topological space $\operatorname{Spec} R$ together with its structure sheaf $\mathcal{O}_R$ and the spectrum $(\operatorname{Spec} R, \mathcal{O}_R)$ is a locally ringed space. Moreover we have the contravariant functor

$$\begin{aligned} \texttt{Spec} \ : \ \texttt{Ring} \to \texttt{LRS} \ : \ & R \longmapsto (\operatorname{Spec} R, \mathcal{O}_R) \\ & (\varphi \ : \ R \to T) \longmapsto \big(\texttt{Spec}(\varphi) \ : \ (\operatorname{Spec} T, \mathcal{O}_T) \to (\operatorname{Spec} R, \mathcal{O}_R)\big) \end{aligned}$$

This functor is in addition fully faithful, so that we have a bijection

$$\mathrm{Hom}_{\mathtt{Ring}}(R,T) \xrightarrow{\sim} \mathrm{Hom}_{\mathtt{LRS}}\big((\operatorname{Spec} T, \mathcal{O}_T), (\operatorname{Spec} R, \mathcal{O}_R)\big) \tag{2.19}$$

for any rings R, T. Hence equation (1.1) and proposition 1.2.4 imply that `Spec` is conservative and preserves isomorphisms "in both directions", i.e. if we are given a unital ring homomorphism $\varphi : R \to T$, then

$$\varphi \text{ is an isomorphism of rings } \Leftrightarrow \mathtt{Spec}(\varphi) \text{ is an isomorphism of locally ringed spaces}$$

As a consequence, the `Spec`–functor yields a contravariant equivalence between the category of commutative unital rings and a full subcategory of the category of locally ringed spaces. We will see in chapter 3 that this full subcategory is given by the category of affine schemes. Hence there is an equivalence between the opposite category of commutative unital rings (i.e. with reversed arrows) and the category of affine schemes ; each of these categories is often thought of as the opposite category of the other.

Chapter 3

Schemes

Schemes have been introduced by the pioneering mathematician **Alexander Grothendieck** (1928–) in the 1960s in order to generalize the notion of an algebraic variety. They connect the fields of algebraic geometry, commutative algebra and number theory and are considered to be the basic object of study of modern algebraic geometry. Roughly speaking, schemes arise from gluing together spectra of several commutative unital rings along some open subsets, augmented with a sheaf of rings that assigns to each open set a ring of "polynomial functions".
We are not going to study schemes in detail yet, but only give definitions, basic properties and some examples of affine, projective and general schemes that illustrate why they are useful. In particular, we will explain the very important concept of how schemes can be glued together. An important reason why we consider schemes is that they rigorously define the notion of the "multiplicity" of a point, for which an example will be discussed. Finally we also explain how schemes generalize the notion of an algebraic variety.
In this chapter, we collected ideas from several references, including e.g. [Ha], [Ga], [Ma], [Sch], [U1] and [U2].

3.1 Affine schemes

In the following, we usually denote locally ringed spaces by $X = (|X|, \mathcal{O}_X)$, where $|X|$ is the underlying topological space and $\mathcal{O}_X$ the sheaf of rings defined on $|X|$.

3.1.1 Definition

Let X be a locally ringed space. We say that X is an *affine scheme* if there exists a (commutative unital) ring R such that X is isomorphic to the spectrum of R, i.e.

$$X \text{ is an affine scheme} \quad \Leftrightarrow \quad (|X|, \mathcal{O}_X) \cong (\operatorname{Spec} R, \mathcal{O}_R)$$

where $\cong$ is an isomorphism of locally ringed spaces as defined in section 2.5.5. In particular, $|X|$ and $\operatorname{Spec} R$ are homeomorphic topological spaces. If X, Y are affine schemes, a *morphism of affine schemes* from X to Y is a morphism of locally ringed spaces $\Phi : X \to Y$.
Since affine schemes are only locally ringed spaces with additional properties, it follows that the category of affine schemes, denoted by $\mathtt{ASch}$, is a full subcategory of the category $\mathtt{LRS}$. Note that $\mathtt{ASch}$ is nothing but the image of the contravariant functor $\mathtt{Spec} : \mathtt{Ring} \to \mathtt{LRS}$ in the category of locally ringed spaces.
Hence, as already pointed out at the end of the previous chapter, it follows from theorem 2.6.2 that the category of affine schemes is equivalent to the opposite category of commutative unital rings :

$$\mathtt{Spec} \ : \ \mathtt{Ring}^{\mathrm{op}} \xrightarrow{\simeq} \mathtt{ASch} \subset \mathtt{LRS}$$

Remark :
Let X be an affine scheme with $|X| \cong \operatorname{Spec} R$ for some ring R. Then the points in $|X|$ correspond to the points in $\operatorname{Spec} R$, i.e. to the prime ideals in R. But $\mathcal{O}_X(|X|) \cong \mathcal{O}_R(\operatorname{Spec} R) \cong R$ as rings, hence points of $|X|$ are in 1-to-1 correspondence with prime ideals in the ring $\mathcal{O}_X(|X|)$.

3.1.2 Example

A ring R is called a *discrete valuation ring* if it is a principal ideal domain with exactly one non-zero maximal ideal $\mathfrak{M}$. This implies in particular that $\{0\}$ and $\mathfrak{M}$ are the only prime ideals in R since any non-zero prime ideal in a principal ideal domain is also maximal. Discrete valuation rings are a particular case of the more general notion of a *valuation ring* (we do not give a definition here). Equivalently, a discrete valuation ring is a principal ideal domain that is also a local ring, but not a field. The following example is taken from [Ha].

Let R be a discrete valuation ring and $T := (\operatorname{Spec} R, \mathcal{O}_R)$, where $\mathcal{O}_R$ is the structure sheaf of the spectrum of R. So T is an affine scheme and the underlying topological space $|T| = \operatorname{Spec} R$ consists of two points :

$$|T| = \big\{ \{0\} \, , \, \mathfrak{M} \big\}$$

– $\mathfrak{M}$ being maximal, we know that the point $\{\mathfrak{M}\}$ is closed in $|T|$ with associated local ring $\mathcal{O}_{R,\mathfrak{M}} \cong R_{\mathfrak{M}}$:

$$R_{\mathfrak{M}} = \left\{ \frac{a}{r} \;\middle|\; r \notin \mathfrak{M} \right\}$$

But $r \notin \mathfrak{M}$ means by lemma 2.5.2 that r is a unit. So $R_{\mathfrak{M}}$ consists of fractions with denominators being units and we get $R_{\mathfrak{M}} \cong R$ by the isomorphism

$$R_{\mathfrak{M}} \to R \;:\; \tfrac{a}{r} \mapsto a \cdot r^{-1} \qquad \text{with inverse} \quad R \to R_{\mathfrak{M}} \;:\; r \mapsto \tfrac{r}{1}$$

– Since R is an integral domain, the point $\{0\}$ is dense in $|T|$, i.e. it is a generic point, and it is also open because its complement is closed. Its corresponding local ring is $\mathcal{O}_{R,\{0\}} \cong R_{\{0\}} = \operatorname{Quot}(R) =: K$. Since K is a field, we know that $\operatorname{Spec} K$ consists of the single point $\{0\}$. We denote

$$|T| = \operatorname{Spec} R = \big\{ t_0 = \{0\} \, , \, t_1 = \mathfrak{M} \big\} \qquad , \qquad \operatorname{Spec} K = \big\{ k_0 = \{0\} \big\} = \{k_0\}$$

– The natural inclusion homomorphism $\varphi : R \hookrightarrow K$ (R contains no zero divisors) induces the continuous map $\operatorname{Spec} \varphi : \operatorname{Spec} K \to \operatorname{Spec} R$ with sends the unique point k_0 of $\operatorname{Spec} K$ to t_0 :

$$\operatorname{Spec} \varphi(k_0) = \varphi^{-1}(k_0) = \varphi^{-1}\big(\{0\}\big) = \ker \varphi = \{0\} = t_0$$

– Finally φ also induces the morphism of locally ringed spaces

$$\texttt{Spec}(\varphi) = (\phi, \phi^*) \;:\; (\operatorname{Spec} K, \mathcal{O}_K) \to (\operatorname{Spec} R, \mathcal{O}_R) = (|T|, \mathcal{O}_R)$$

where $\phi = \operatorname{Spec} \varphi$. The only open sets in $|T|$ are $\emptyset$, $\{t_0\}$ and $|T|$, so the only non-trivial pullbacks are

$$\phi^*_{\{t_0\}} \;:\; \mathcal{O}_R\big(\{t_0\}\big) \to \mathcal{O}_K\big(\{k_0\}\big) \cong K \qquad , \qquad \phi^*_{|T|} \;:\; \mathcal{O}_R\big(|T|\big) \cong R \to \mathcal{O}_K\big(\{k_0\}\big) \cong K$$

and the induced map on the stalk $\phi^*_{k_0} : \mathcal{O}_{R,t_0} \to \mathcal{O}_{K,k_0}$ is a local homomorphism, where $\mathcal{O}_{R,t_0} \cong K$ and $\mathcal{O}_{K,k_0} \cong \operatorname{Quot}(K) \cong K$. Using this, one can show that $\phi^*_{k_0}$ is even an isomorphism since all quotients we need to obtain surjectivity of the map have been added. Also note that $\phi^*_{|T|}$ is identified with φ.

Remark :
Of course, we can also define the map $\psi : \operatorname{Spec} K \to \operatorname{Spec} R : k_0 \mapsto t_1$ (continuous as well since $\operatorname{Spec} K$ is a discrete topological space). As above, we can associate the pullback morphisms

$$\psi^*_{\{t_0\}} \;:\; \mathcal{O}_R\big(\{t_0\}\big) \to \mathcal{O}_K(\emptyset) = \{0\} \qquad , \qquad \psi^*_{|T|} \;:\; \mathcal{O}_R\big(|T|\big) \cong R \to \mathcal{O}_K\big(\{k_0\}\big) \cong K$$

because $\psi^{-1}\big(\{t_0\}\big) = \emptyset$, together with the stalk homomorphism $\psi^*_{k_0} : \mathcal{O}_{R,t_1} \cong R \to \mathcal{O}_{K,k_0} \cong K$.

But $\psi^*_{k_0}$ is NOT a local homomorphism : since $|T|$ is the only open neighborhood of t_1, $\psi^*_{k_0}$ is by definition computed using $\psi^*_{|T|}$, which is identified with φ, i.e. $\psi^*_{k_0}$ corresponds to $\varphi : R \hookrightarrow K$ as well. But the unique maximal ideal in R is $\mathfrak{M}$ and the one in K is $\{0\}$ with $\varphi(\mathfrak{M}) \nsubseteq \{0\}$ since $\mathfrak{M} \neq \{0\}$ and φ is injective, so the stalk homomorphism does not preserve the maximal ideals.
In particular, the morphism of ringed spaces $(\psi, \psi^*) : (\operatorname{Spec} K, \mathcal{O}_K) \to (\operatorname{Spec} R, \mathcal{O}_R)$ is NOT induced by a ring homomorphism $R \to K$ as in (2.19) since it is not a morphism of <u>locally</u> ringed spaces.

3.1.3 Counter–example

Not every open subset of an affine scheme is again an affine scheme.
More precisely : if $X = (|X|, \mathcal{O}_X)$ is an affine scheme, $U \subset |X|$ an open subset and if we set $\mathcal{O}_U := \mathcal{O}_{X|U}$, then $(U, \mathcal{O}_U)$ is not necessarily an affine scheme as well. We show this by using an example from [Ja].

Let $\mathbb{K}$ be a field, $R = \mathbb{K}[X_1, X_2]$ the polynomial ring in two variables, $|X| = \mathbb{A}^2_{\mathbb{K}} = \operatorname{Spec} R$ the affine plane and $X = (\operatorname{Spec} R, \mathcal{O}_R) = (|X|, \mathcal{O}_X)$ the corresponding affine scheme. $M := \langle X_1, X_2 \rangle$ is a maximal ideal, so that the point $\{M\}$ is closed in $|X|$ and $U := |X| \setminus \{M\}$ is an open subset of $|X|$. We will show that $(U, \mathcal{O}_U)$ is not an affine scheme.
The open set U can be covered by the distinguished open subsets $U_1 = U(X_1)$ and $U_2 = U(X_2)$:

$$U = U_1 \cup U_2$$

$U_j \subset U$ since $M \notin U_j$ and if $P \notin U_1 \cup U_2$ is a prime ideal containing X_1 and X_2, then P also contains $\langle X_1, X_2 \rangle$ and it thus by maximality equal to M, i.e. $P \notin U$. Also note that $\mathcal{O}_R(U_j) = \mathcal{O}_R\big(U(X_j)\big) \cong R_{X_j}$ is the localized ring at X_j, $j = 1, 2$. More explicitly,

$$\mathcal{O}_R(U_j) \cong R_{X_j} = \Big\{ \frac{f}{X_j^n} \;\Big|\; f \in \mathbb{K}[X_1, X_2] \Big\} = \Big\{ \frac{f}{X_j^n} \;\Big|\; f = \sum_{ik} a_{ik}\, X_1^i X_2^k \Big\} = \mathbb{K}\big[X_1, X_2, \tfrac{1}{X_j}\big]$$

Obviously, $i : U \hookrightarrow |X|$ is not surjective, hence i cannot define an isomorphism of locally ringed spaces

$$(i, i^*) \;:\; (U, \mathcal{O}_U) \to (|X|, \mathcal{O}_X)$$

Nevertheless we can define a morphism of locally ringed spaces by specifying the pullback i^*. For any open subset $V \subseteq |X|$, we set

$$i^*_V \;:\; \mathcal{O}_X(V) \to \mathcal{O}_U\big(i^{-1}(V)\big) \quad \Leftrightarrow \quad i^*_V \;:\; \mathcal{O}_R(V) \to \mathcal{O}_R(U \cap V) \;:\; f \mapsto f \circ i_{|U \cap V}$$

Consider the global pullback $i^*_{|X|} : \mathcal{O}_R(\operatorname{Spec} R) \cong R \to \mathcal{O}_R(U)$. Lemma 2.1.6 gives the exact sequence

$$0 \longrightarrow \mathcal{O}_R(U) \xrightarrow{\alpha} \mathcal{O}_R(U_1) \oplus \mathcal{O}_R(U_2) \xrightarrow{\beta} \mathcal{O}_R(U_1 \cap U_2) \tag{3.1}$$

where $\mathcal{O}_R(U_1 \cap U_2) = \mathcal{O}_R\big(U(X_1) \cap U(X_2)\big) = \mathcal{O}_R\big(U(X_1X_2)\big) \cong R_{X_1X_2}$. Exactness of (3.1) yields that

$$\begin{aligned}\mathcal{O}_R(U) \cong \operatorname{im} \alpha = \ker \beta &= \big\{ (s_1, s_2) \;\big|\; s_i \in \mathcal{O}_R(U_i) \text{ and } s_{1|U_1 \cap U_2} = s_{2|U_1 \cap U_2} \big\} \\ &\cong \Big\{ \Big(\frac{f}{X_1^n} , \frac{g}{X_2^m} \Big) \;\Big|\; \frac{f}{X_1^n} \in R_{X_1},\ \frac{g}{X_2^m} \in R_{X_2} \text{ such that } \frac{f}{X_1^n} = \frac{g}{X_2^m} \text{ in } R_{X_1X_2} \Big\}\end{aligned}$$

where we used the identification explained at the end of section 2.4.4. But elements in the localized ring $R_{X_1X_2}$ must have fractions with denominators being powers of X_1X_2, hence we need that $n = m = 0$ and

$$\mathcal{O}_R(U) \cong \big\{ (f, g) \;\big|\; f, g \in R \text{ such that } f = g \big\} \cong R$$

So the global pullback $i^*_{|X|} : R \to \mathcal{O}_R(U) \cong R$ is actually an isomorphism of rings.

Now suppose that $(U, \mathcal{O}_U)$ is an affine scheme, i.e. there exists a ring T such that $(U, \mathcal{O}_U) \cong (\operatorname{Spec} T, \mathcal{O}_T)$. Then we have the morphism of locally ringed spaces

$$(i, i^*) \;:\; (U, \mathcal{O}_U) \to (|X|, \mathcal{O}_X) \quad \Leftrightarrow \quad (i, i^*) \;:\; (\operatorname{Spec} T, \mathcal{O}_T) \to (\operatorname{Spec} R, \mathcal{O}_R)$$

By full faithfulness of the `Spec`–functor, see equation (2.19), we know that $i : \operatorname{Spec} T \to \operatorname{Spec} R$ is induced by a ring homomorphism $\varphi : R \to T$, defined by the global pullback (as in the proof of theorem 2.6.2)

$$i^*_{\operatorname{Spec} R} \;:\; \mathcal{O}_R(\operatorname{Spec} R) \cong R \longrightarrow \mathcal{O}_T(\operatorname{Spec} T) \cong \mathcal{O}_U(U) = \mathcal{O}_X(U) \cong R$$

which is an isomorphism as shown above, hence φ is an isomorphism. But functors send isomorphisms to isomorphisms, i.e. `Spec`$(\varphi) = (i, i^*)$ would be an isomorphism as well. This is not possible since i is not surjective, hence U cannot be isomorphic to the spectrum of some ring, i.e. U is not an affine scheme.

Goal :
Our next goal is to show that the distinguished open set $U(r) \subseteq \operatorname{Spec} R$ with the restricted sheaf $\mathcal{O}_{R|U(r)}$ is an affine scheme, $\forall\, r \in R$. For this, we first need some preliminary results.

3.1.4 Proposition

Let $\varphi : R \to T$ be a surjective ring homomorphism with kernel $I = \ker \varphi$. Then $\operatorname{Spec}\varphi : \operatorname{Spec} T \to \operatorname{Spec} R$ is a homeomorphism onto $V(I) \subseteq \operatorname{Spec} R$ (Note : I is an ideal in R).

Proof. taken from [Ku]. Recall that $\operatorname{Spec}\varphi$ is defined by $\operatorname{Spec}\varphi(P) = \varphi^{-1}(P)$ for $P \in \operatorname{Spec} T$.
The image of $\operatorname{Spec}\varphi$ is contained in $V(I)$: $I \subseteq \varphi^{-1}(P)$, $\forall\, P \in \operatorname{Spec} T$ since $\varphi(I) = \{0\} \subseteq P$.

First we show that $\operatorname{Spec}\varphi$ is bijective : there is a 1-to-1 correspondence between prime ideals in T (i.e. elements in $\operatorname{Spec} T$) and prime ideals in R containing I (i.e. elements in $V(I)$). This bijection is

$$\operatorname{Spec} T \to V(I) \;:\; P \mapsto \varphi^{-1}(P) \qquad \text{with inverse} \;\; V(I) \to \operatorname{Spec} T \;:\; Q \mapsto \varphi(Q)$$

Note that $\varphi(Q)$ is an ideal since φ is surjective. Moreover $\varphi(Q)$ is prime because $1 \notin \varphi(Q)$, otherwise

$$1 \in \varphi(Q) \;\Rightarrow\; \exists\, r \in Q \text{ such that } \varphi(r) = 1 = \varphi(1) \;\Rightarrow\; 1 - r \in \ker\varphi = I \subseteq Q \;\Rightarrow\; 1 \in Q \;:\; \text{contradiction}$$

And if $a \cdot b \in \varphi(Q)$ for some $a, b \in T$, then by surjectivity $\exists\, r, s \in R$ such that $a = \varphi(r)$, $b = \varphi(s)$ and

$$a \cdot b = \varphi(r) \cdot \varphi(s) = \varphi(r \cdot s) \in \varphi(Q) \;\Rightarrow\; \exists\, r' \in Q \text{ such that } \varphi(r \cdot s) = \varphi(r') \;\Rightarrow\; r \cdot s - r' \in \ker\varphi = I \subseteq Q$$

and it follows that $r \cdot s \in Q \;\Rightarrow\; r \in Q$ or $s \in Q \;\Rightarrow\; \varphi(r) = a \in \varphi(Q)$ or $\varphi(s) = b \in \varphi(Q)$.
By surjectivity of φ, we have $\varphi\big(\varphi^{-1}(P)\big) = P$, $\forall\, P \in \operatorname{Spec} T$. In addition, $Q \subseteq \varphi^{-1}\big(\varphi(Q)\big)$, $\forall\, Q \in \operatorname{Spec} R$. For $Q \in V(I)$, the other inclusion holds true as well : if $a \in \varphi^{-1}\big(\varphi(Q)\big)$, then $\varphi(a) \in \varphi(Q)$, i.e. $\exists\, r \in Q$ such that $\varphi(a) = \varphi(r) \;\Rightarrow\; a - r \in \ker\varphi = I \subseteq Q \;\Rightarrow\; a \in Q$. Thus $\varphi^{-1}\big(\varphi(Q)\big) = Q$, $\forall\, Q \in V(I)$.

Hence $\operatorname{Spec}\varphi$ is continuous and bijective. To show that it is a homeomorphism, it is now sufficient to show that images of closed sets are again closed. Let $A = V(J)$ be closed in $\operatorname{Spec} T$ for some ideal $J \trianglelefteq T$. Then

$$\operatorname{Spec}\varphi(A) = \operatorname{Spec}\varphi\big(V(J)\big) = \varphi^{-1}\big(V(J)\big) = V(I) \cap V\big(\varphi^{-1}(J)\big) \;:\; \text{closed in } V(I) \subseteq \operatorname{Spec} R$$

$\supseteq$: $Q \in V\big(\varphi^{-1}(J)\big) \;\Leftrightarrow\; \varphi^{-1}(J) \subseteq Q \;\Rightarrow\; J = \varphi\big(\varphi^{-1}(J)\big) \subseteq \varphi(Q) \;\Leftrightarrow\; \varphi(Q) \in V(J) \;\Leftrightarrow\; Q \in \varphi^{-1}\big(V(J)\big)$
$\subseteq$: we know that $\operatorname{im}\operatorname{Spec}\varphi \subseteq V(I)$, hence $Q \in \varphi^{-1}\big(V(J)\big)$ immediately means that $J \subseteq \varphi(Q)$ and $I \subseteq Q$. Thus $\varphi^{-1}(J) \subseteq Q$ because for $a \in \varphi^{-1}(J)$, $\varphi(a) \in J \subseteq \varphi(Q) \;\Rightarrow\; \exists\, r \in Q$ such that $\varphi(a) = \varphi(r)$, so that $a - r \in \ker\varphi = I \subseteq Q \;\Rightarrow\; a \in Q$ as well and finally $Q \in V\big(\varphi^{-1}(J)\big)$. ☐

Corollary :
Let R be a ring and $I \trianglelefteq R$ an ideal. Then there is a homeomorphism between prime ideals in R/I and prime ideals in R containing I, i.e. $\operatorname{Spec}(R/I) \cong V(I)$.

Proof. This follows immediately from the previous proposition by taking the projection $\pi : R \to R/I$. Thus $\ker\pi = I$ and $\operatorname{Spec}\pi$ is a homeomorphism from $\operatorname{Spec}(R/I)$ onto $V(I)$. ☐

3.1.5 Theorem

Let R be a ring and $S \subseteq R$ a multiplicative subset. Then the natural map $i_S : R \to S^{-1}R$ (in general neither injective nor surjective since S may contain zero divisors) induces a homeomorphism

$$\operatorname{Spec} i_S \;:\; \operatorname{Spec}(S^{-1}R) \xrightarrow{\sim} \big\{\, P \in \operatorname{Spec} R \;\big|\; P \cap S = \emptyset \,\big\} =: D$$

where the topology on D is the induced Zariski topology of $\operatorname{Spec} R$ on D. The inverse map is given by the assignment $P \mapsto S^{-1}P$. Note that this is the restriction to all prime ideals of the map e in (2.5).

Proof. taken from [U1]. If $Q \in \operatorname{Spec}(S^{-1}R)$ and we denote $P = \operatorname{Spec} i_S(Q) = i_S^{-1}(Q)$, then $P \cap S = \emptyset$, otherwise if $\exists\, s \in S$ such that $s \in P = i_S^{-1}(Q)$, then $i_S(s) \in Q$ but $i_S(s) = \frac{s}{1}$ is a unit with inverse $\frac{1}{s}$, i.e. $1 \in Q$ which is a contradiction. Hence the image of $\operatorname{Spec} i_S$ is indeed contained in D.
Let $P \in D$, $\pi : R \to R/P$ and denote $S' = \pi(S)$. Then $P \cap S = \emptyset \Rightarrow 0 \notin S'$, hence $S'^{-1}(R/P)$ is not the zero ring. P being prime, we know that R/P and $S'^{-1}(R/P)$ are integral domains. By (2.4), we have

$$S^{-1}R/S^{-1}P \cong S'^{-1}(R/P)$$

via the isomorphism $\left[\frac{r}{s}\right] \mapsto \frac{\bar{r}}{\bar{s}}$ and it follows that $S^{-1}P$ is prime. Hence the inverse map is well-defined.

• $S^{-1}\big(i_S^{-1}(Q)\big) = Q,\ \forall\, Q \in \operatorname{Spec}(S^{-1}R)$:

$\subseteq$:

$$S^{-1}\big(i_S^{-1}(Q)\big) = \left\{ \frac{a}{s} \;\middle|\; a \in i_S^{-1}(Q),\ s \in S \right\} = \left\{ \frac{1}{s} \cdot \frac{a}{1} \;\middle|\; \tfrac{a}{1} \in Q,\ s \in S \right\} \subseteq Q$$

$\supseteq$: if $\frac{r}{s} \in Q$, then $\frac{r}{1} \cdot \frac{1}{s} \in Q$, but $\frac{1}{s} \notin Q$ since $\frac{1}{s}$ is a unit. Hence $\frac{r}{1} \in Q$ and therefore $\frac{r}{s} \in S^{-1}\big(i_S^{-1}(Q)\big)$

• $i_S^{-1}(S^{-1}P) = P,\ \forall\, P \in D$:

$\supseteq$: for $r \in P$, we have $i_S(r) = \frac{r}{1} \in S^{-1}P \Rightarrow P \subseteq i_S^{-1}(S^{-1}P)$

$\subseteq$: let $r \in i_S^{-1}(S^{-1}P)$, i.e. $\frac{r}{1} \in S^{-1}P \Rightarrow \left[\frac{r}{1}\right] = 0$. But we have the injective map

$$R/P \hookrightarrow S'^{-1}(R/P) \cong S^{-1}R/S^{-1}P \ :\ \bar{r} \longmapsto \tfrac{\bar{r}}{1} \longmapsto \left[\tfrac{r}{1}\right]$$

since R/P is a domain, hence $\left[\frac{r}{1}\right] = 0 \Rightarrow \bar{r} = 0 \Rightarrow r \in P$. This shows that $\operatorname{Spec} i_S$ is a bijection onto D.

We know that $\operatorname{Spec} i_S$ is continuous, hence it remains to show that it is an open map. It suffices to show that the image of a standard open set $U(g) \subseteq \operatorname{Spec}(S^{-1}R)$ is again open. Write $g = \frac{r}{s}$ for $r \in R$ and $s \in S$, so that $g = \frac{r}{1} \cdot \frac{1}{s}$ where $\frac{1}{s}$ is a unit and we have $U(g) = U\big(\frac{r}{1}\big)$ by 2.3.5. Then

$$\operatorname{Spec} i_S\big(U(g)\big) = i_S^{-1}\big(U\big(\tfrac{r}{1}\big)\big) = D \cap U(r) \ :\ \text{open in } D \subseteq \operatorname{Spec} R$$

$\subseteq$: we know that $\operatorname{im}\operatorname{Spec} i_S \subseteq D$ and if $P \in i_S^{-1}\big(U\big(\frac{r}{1}\big)\big)$, i.e. $\frac{r}{1} \notin i_S(P) = S^{-1}P$, then $r \notin P$

$\supseteq$: let $P \in D \cap U(r)$, i.e. $r \notin P$ and $P \cap S = \emptyset$. Then $\frac{r}{1} \notin i_S(P) = S^{-1}P$, otherwise

$$\exists\, a \in P,\ s \in S \text{ such that } \tfrac{r}{1} = \tfrac{a}{s} \ \Rightarrow\ \exists\, t \in S \text{ such that } t \cdot (rs - a) = 0 \ \Leftrightarrow\ r \cdot st = at \in P$$

which is impossible since $r \notin P$ and $st \notin P$ since $st \in S$ (S is multiplicative). □

3.1.6 Corollary

Let R be a ring and $r \in R$. Then the natural map $\varphi : R \to R_r$ induces a homeomorphism $\operatorname{Spec} R_r \cong U(r)$. Moreover, if $\mathcal{O}_R$ denotes the structure sheaf of $\operatorname{Spec} R$, then $\big(U(r), \mathcal{O}_{R|U(r)}\big)$ is an affine scheme.

Proof. taken from [U1] and [Ga]. Here we have $S = \{ r^n \mid n \in \mathbb{N}_0 \}$. If r is nilpotent, then $0 \in S$ and the statement holds true because $R_r = \{0\}$ and $\operatorname{Spec} R_r = U(r) = \emptyset$. Hence we may assume that r is not nilpotent and $0 \notin S$. It follows from theorem 3.1.5 that $\operatorname{Spec}\varphi$ is a homeomorphism of $\operatorname{Spec} R_r$ onto the set of all prime ideals in R that do not intersect S. But $S \cap P = \emptyset \Leftrightarrow r \notin P \Leftrightarrow P \in U(r)$ for any prime ideal $P \trianglelefteq R$, hence $\operatorname{Spec} R_r \cong U(r)$. Note that $\operatorname{Spec}\varphi : Q \mapsto \varphi^{-1}(Q)$ is the inverse of the map $P \mapsto S^{-1}P$ in (2.9).

To show that $\big(U(r), \mathcal{O}_{R|U(r)}\big)$ is isomorphic to $\big(\operatorname{Spec} R_r, \mathcal{O}_{R_r}\big)$ as locally ringed spaces, it remains to show that the corresponding structure sheaves on $U(r)$ and on $\operatorname{Spec} R_r$ agree. We first check this property on the standard open sets : let $U(s) \subseteq U(r)$, which means by 2.3.5 that $U(s) = U(r \cdot s)$. It corresponds to

$$U(r) \supseteq U(r \cdot s) \cong \operatorname{Spec} R_{rs} \cong \operatorname{Spec}\big((R_r)_s\big) \cong U_r\big(\tfrac{s}{1}\big) \subseteq \operatorname{Spec} R_r \tag{3.2}$$

where $U_r\big(\frac{s}{1}\big)$ denotes the topological subspace of $\operatorname{Spec} R_r$ consisting of all prime ideals in R_r not containing $\frac{s}{1}$. By theorem 2.4.4, we have

$$\mathcal{O}_{R|U(r)}\big(U(r \cdot s)\big) = \mathcal{O}_R\big(U(rs)\big) \cong R_{rs} \qquad , \qquad \mathcal{O}_{R_r}\big(U_r\big(\tfrac{s}{1}\big)\big) \cong (R_r)_s \cong R_{rs}$$

so the sheaves $\mathcal{O}_{R|U(r)}$ and $\mathcal{O}_{R_r}$ agree on the standard open sets.

For a general open set $V \subseteq U(r)$, we write V as a union of distinguished open sets $V = \bigcup_i U(rs_i)$ and the result follows using (2.1) :

$$\begin{aligned}\mathcal{O}_{R|U(r)}(V) = \mathcal{O}_R\Big(\bigcup\nolimits_i U(rs_i)\Big) &\cong \prod_i \Big(\mathcal{O}_R(U(rs_i))\,\big|\,\text{sections coincide on intersections}\Big)\\ &\cong \prod_i \Big(\mathcal{O}_{R_r}(U_r(\tfrac{s_i}{1}))\,\big|\,\text{sections coincide on intersections}\Big)\\ &\cong \mathcal{O}_{R_r}\Big(\bigcup\nolimits_i U_r(\tfrac{s_i}{1})\Big) = \mathcal{O}_{R_r}(V') \quad \text{where } V \cong V' \text{ via (3.2)}\end{aligned}$$

Both structure sheaves being isomorphic everywhere, we obtain that $\big(U(r), \mathcal{O}_{R|U(r)}\big)$ is an affine scheme. □

3.2 General schemes

3.2.1 Definitions

A *scheme* is a locally ringed space $X = (|X|, \mathcal{O}_X)$, i.e. a pair consisting of a topological space $|X|$ and a sheaf of rings $\mathcal{O}_X$ on X such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in |X|$, that is locally isomorphic to an affine scheme. More precisely : X is a scheme if every point $x \in |X|$ has an open neighborhood $U \subseteq |X|$ such that the locally ringed space $(U, \mathcal{O}_{X|U})$ is isomorphic, in the sense of locally ringed spaces, to the spectrum $(\operatorname{Spec} R, \mathcal{O}_R)$ of some commutative unital ring R (the ring may depend on x). Recall that this means that there is a homeomorphism $\psi : U \to \operatorname{Spec} R$ and an isomorphism of sheaves $\psi^* : \mathcal{O}_R \to \psi_*(\mathcal{O}_{X|U})$; by proposition 2.5.6, we do not need to require that the stalk homomorphisms are local.
Equivalently, X is a scheme if there exists an open covering $\{U_i\}_{i\in J}$ of $|X|$ such that $(U_i, \mathcal{O}_{X|U_i})$ is isomorphic to an affine scheme $(\operatorname{Spec} R_i, \mathcal{O}_{R_i})$ for some commutative unital rings R_i, $\forall\, i \in J$.

Obviously, the restriction of a scheme $X = (|X|, \mathcal{O}_X)$ to an open subset $V \subset |X|$ is again a scheme since if every point $x \in |X|$ has a neighborhood that is isomorphic to an affine scheme, then the same of course also holds for all $x \in V \;\Rightarrow\; (V, \mathcal{O}_{X|V})$ is a scheme as well. It is called an *open subscheme* of X.
Affine schemes are in particular schemes since they are even globally isomorphic to the spectrum of a ring.

Morphisms of schemes
Let $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ be two schemes. A *morphism of schemes* $\Phi : X \to Y$ is a morphism of locally ringed spaces, i.e. $\Phi = (\phi, \phi^*)$ consists of a continuous map $\phi : |X| \to |Y|$ and a morphism of sheaves $\phi^* : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ such that $\phi^*_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism on the stalks, $\forall\, x \in |X|$. This definition implies that the category of schemes, denoted by `Sch`, is a full subcategory of `LRS` since schemes are just locally ringed spaces with additional properties.

3.2.2 Motivation

We may ask why it is necessary to "complicate" the task by considering this generalization. Why do we need these local properties and don't just consider affine schemes ? This has actually several reasons :
1) As shown in 3.1.3, affine schemes have the problem that their restriction to open subsets may no longer be an affine scheme. The notion of a general scheme is therefore much more flexible.
2) We will see in the following that there are a lot of interesting examples of schemes which are not affine schemes, such as projective schemes (see section 3.5) or spaces with "double points" (see example 3.3.4).
3) Schemes being obtained by "gluing" affine schemes, they have the useful property that several schemes can again be glued together (see section 3.3.3), giving rise to a new scheme.
4) The most important reason however is that we actually don't gain anything if we only consider affine schemes since the category `ASch` is equivalent to the opposite category of rings. Anything we could do with affine schemes, we could do equally well with just commutative rings (with reversed arrows).

Schemes and morphisms of schemes are actually motivated by the following example, which can also be found in [Nel] :
A smooth real manifold consists of a topological space M together with a sheaf of differentiable real-valued functions on M, denoted by C^∞_M, such that the locally ringed space (M, C^∞_M) is locally isomorphic to some model $\mathbb{R}^n$ with its standard sheaf of differentiable functions. A continuous map $\psi : M \to N$ between smooth manifolds M, N is called smooth if $\forall\, U \subseteq N$ open and $\forall\, f \in C^\infty_N(U)$, we have $f \circ \psi \in C^\infty_M(V)$, where $V = \psi^{-1}(U)$. This can be translated into the language of morphisms of schemes.
Denote the sheaf of continuous real-valued functions on a manifold M by C^0_M. First note that any continuous function $\psi : M \to N$ between smooth manifolds defines a morphism of sheaves $\psi^* : C^0_N \to \psi_* C^0_M$ by mapping a function to its pullback, i.e. $\forall\, U \subseteq N$

$$\psi^*_U \;:\; C^0_N(U) \longrightarrow \psi_* C^0_M(U) = C^0_M\big(\psi^{-1}(U)\big) \;:\; f \longmapsto \psi^*_U(f) = f \circ \psi_{|\psi^{-1}(U)}$$

$\psi^*_U(f)$ is called the pullback of f under ψ. Obviously we have the inclusions $C^\infty_\cdot(U) \subset C^0_\cdot(U)$. Hence ψ is smooth if ψ^* maps $C^\infty_N(U)$ into $\psi_* C^\infty_M(U)$ for all open sets $U \subseteq N$, i.e. if we have the commutative diagram

$$\begin{tikzcd}
C^\infty_N(U) \arrow[r, "\psi^*_U"] \arrow[d] & \psi_* C^\infty_M(U) \arrow[d] \\
C^0_N(U) \arrow[r, "\psi^*_U"] & \psi_* C^0_M(U)
\end{tikzcd}$$

where the vertical arrows are the inclusion maps.

The problem with adapting this definition to schemes is that the structure sheaf $\mathcal{O}_X$ of a general scheme X is not necessarily a subscheme of a sheaf of functions that already exists. Hence we need to specify a continuous base map $\phi : |X| \to |Y|$ **and** a pullback morphism $\phi^* : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$, these data satisfying some compatibility conditions as in the above diagram.

3.2.3 Proposition

Any scheme has a basis of affine open subsets.
(We say that an open subset $U \subset |X|$ of a scheme $X = (|X|, \mathcal{O}_X)$ is *affine* if $(U, \mathcal{O}_{X|U})$ is an affine scheme.)

Proof. taken from [Ma]. Let X be a scheme. By definition, there exists an open covering $\{U_i\}_{i \in J}$ of $|X|$ such that $(U_i, \mathcal{O}_{X|U_i})$ is an affine scheme, i.e. $\forall\, i \in J$, there is a ring R_i, a homeomorphism $\psi_i : U_i \to \operatorname{Spec} R_i$ and an isomorphism

$$\psi^*_i \;:\; \mathcal{O}_{R_i} \to \psi_{i*}(\mathcal{O}_{X|U_i})$$

where $\mathcal{O}_{R_i}$ is the structure sheaf of $\operatorname{Spec} R_i$. For each i, we know that $\{\, U(r_i) \subseteq \operatorname{Spec} R_i \;\big|\; r_i \in R_i \,\}$ is a basis for the topology of $\operatorname{Spec} R_i$. Moreover these $U(r_i)$ again define affine schemes by corollary 3.1.6 :

$$\big(U(r_i), \mathcal{O}_{R_i|U(r_i)}\big) \cong \big(\operatorname{Spec}(R_i)_{r_i}, \mathcal{O}_{(R_i)_{r_i}}\big)$$

Define $W(r_i) := \psi_i^{-1}\big(U(r_i)\big) \subseteq U_i$, so that $\big(W(r_i), \mathcal{O}_{X|W(r_i)}\big) \cong \big(U(r_i), \mathcal{O}_{R_i|U(r_i)}\big)$ is an affine scheme and

$$\mathcal{U}_i := \big\{\, W(r_i) \subseteq U_i \;\big|\; r_i \in R_i \,\big\}$$

is a basis for the topology on $U_i \subseteq |X|$. Then $\mathcal{U} = \bigcup_{i \in J} \mathcal{U}_i$ is a basis for the topology of $|X|$ consisting of affine open subsets. Indeed if $U \subseteq |X|$ is open, we can write $U = U \cap |X| = U \cap \bigcup_i U_i = \bigcup_i (U \cap U_i)$, where $U \cap U_i$ is open in U_i and can hence be written as a union $\bigcup_j W(r_{ij})$ with $W(r_{ij}) \in \mathcal{U}$, $r_{ij} \in R_i$, $\forall\, j$. Thus

$$U = \textstyle\bigcup_i \big(U \cap U_i\big) = \bigcup_i \bigcup_j W(r_{ij}) = \displaystyle\bigcup_{i,j} W(r_{ij})$$

$\square$

3.2.4 Proposition

If $X = (|X|, \mathcal{O}_X)$ is a scheme, then the underlying topological space $|X|$ is a Kolmogorov space.

Proof. Let $x, y \in |X|$ with $x \neq y$ and take an affine open neighborhood U of x. If $y \notin U$, we are done since we found an open set in $|X|$ that contains x but not y. So assume that $x, y \in U$. U being affine, we know that there exists a ring R and a homeomorphism $\phi : U \to \operatorname{Spec} R$. In particular, U as topological subspace of $|X|$ has the same properties as the topological space $\operatorname{Spec} R$, which is a Kolmogorov space by theorem 2.3.7. ϕ being bijective, we have $\phi(x) \neq \phi(y)$, so there either exists an open neighborhood V of $\phi(x)$ such that $\phi(y) \notin V$ or an open neighborhood W of $\phi(y)$ such that $\phi(x) \notin W$. The preimage $\phi^{-1}(V)$, resp. $\phi^{-1}(W)$, is therefore an open neighborhood of one of x or y that does not contain the other point. $\square$

3.3 Gluing schemes

3.3.1 Lemma : Gluing sheaves

Before being able to explain how to glue schemes, we first need a method for gluing sheaves. The following statements are taken from [S], where the proof has only been sketched. Here we develop the proof and add some more computations (however, not every detail is indeed instructive).

Let X be a topological space and $\{U_i\}_{i\in J}$ an open covering of X. We set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_{ij} \cap U_k$ and consider a sheaf (of modules, of rings, etc.) $\mathcal{F}$ on X. If we define $\mathcal{F}_i := \mathcal{F}_{|U_i}$, then obviously

$$\theta_i \ : \ \mathcal{F}_{|U_i} \xrightarrow{\sim} \mathcal{F}_i$$

is an isomorphism of sheaves (actually the identity) and if we set $\theta_{ji} := \theta_j \circ \theta_i^{-1} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$, then

$$\theta_{ii} = \mathrm{id} \text{ on } U_i \qquad \text{and} \qquad \theta_{ij} \circ \theta_{jk} = \theta_{ik} \text{ on } U_{ijk} \tag{3.3}$$

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (3.3) is called a *cocycle* and it describes the sheaf $\mathcal{F}$ locally. The goal is to show that, given the data of a cocycle, we can reconstruct the sheaf $\mathcal{F}$.

Theorem :
Let $\{U_i\}_{i\in J}$ be an open covering of a topological space X and $\mathcal{F}_i$ be a sheaf on U_i, $\forall\, i \in J$. Assume we are given an isomorphism of sheaves $\theta_{ji} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$ for each pair $(i,j) \in J \times J$ such that the θ_{ji} satisfy the cocycle conditions (3.3). Then there exists a sheaf $\mathcal{F}$ on X and for each $i \in J$, we have isomorphisms of sheaves $\theta_i : \mathcal{F}_{|U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that $\theta_j = \theta_{ji} \circ \theta_i$ on U_{ij}. Moreover $\mathcal{F}$ is unique up to isomorphism.

Proof. Uniqueness (up to isomorphism) is admitted. For existence, let $V \subseteq X$ be open. We define

$$\mathcal{F}(V) := \left\{ s = \{s_i\}_i \;\middle|\; s_i \in \mathcal{F}_i(V \cap U_i),\ \theta_{ji}(s_{i|V\cap U_{ij}}) = s_{j|V\cap U_{ij}},\ \forall\, i,j \in J \right\} \subseteq \prod_{i\in J} \mathcal{F}_i(V \cap U_i)$$

with restriction morphisms $\rho^V_W : \mathcal{F}(V) \to \mathcal{F}(W) : \{s_i\}_i \mapsto \{s_{i|W\cap U_i}\}_i$ for $W \subseteq V$ open. This is a sheaf :
– S1 : let $V = \bigcup_k V_k$ be an open covering and $s = \{s_i\}_i \in \mathcal{F}(V)$ such that $\rho^V_{V_k}(s) = 0$, $\forall\, k$. This means

$$\forall\, k,\ 0 = \rho^V_{V_k}(s) = \rho^V_{V_k}\big(\{s_i\}_i\big) = \{s_{i|V_k\cap U_i}\}_i \quad \Rightarrow \quad \forall\, i : s_{i|V_k\cap U_i} = 0,\ \forall\, k \ \Rightarrow \ s_i = s_{i|V\cap U_i} = 0$$

since $\{V_k \cap U_i\}_k$ is an open covering of $V \cap U_i$ and each $\mathcal{F}_i$ satisfies S1. Hence $s = \{s_i\}_i = 0$ as well.
– S2 : let $V = \bigcup_k V_k$ be an open covering and $s^k = \{s^k_i\}_i \in \mathcal{F}(V_k)$ for each k such that $\forall\, k, l$:

$$\rho^{V_k}_{V_k\cap V_l}(s^k) = \rho^{V_l}_{V_k\cap V_l}(s^l) \ \Leftrightarrow \ \{s^k_{i|V_k\cap V_l\cap U_i}\}_i = \{s^l_{i|V_k\cap V_l\cap U_i}\}_i \ \Leftrightarrow \ \forall\, i,\ s^k_{i|(V_k\cap U_i)\cap(V_l\cap U_i)} = s^l_{i|(V_k\cap U_i)\cap(V_l\cap U_i)}$$

$\{V_k \cap U_i\}_k$ is an open covering of $V \cap U_i$ and $\mathcal{F}_i$ satisfies S2, so for each i we can glue the $s^k_i \in \mathcal{F}(V_k \cap U_i)$ to an element $s_i \in \mathcal{F}_i(V \cap U_i)$ such that $s_{i|V_k\cap U_i} = s^k_i$, $\forall\, k$. Set $s := \{s_i\}_i$, then $\rho^V_{V_k}(s) = s^k$, $\forall\, k$ because

$$\rho^V_{V_k}(s) = \rho^V_{V_k}\big(\{s_i\}_i\big) = \{s_{i|V_k\cap U_i}\}_i = \{s^k_i\}_i = s^k$$

It remains to show that s satisfies the defining condition for belonging to $\mathcal{F}(V)$. Since $s^k \in \mathcal{F}(V_k)$, we have

$$\forall\, k,i,j \;:\; \theta_{ji}(s^k_{i|V_k\cap U_{ij}}) = s^k_{j|V_k\cap U_{ij}}$$

θ_{ji} is an isomorphism of sheaves, so we know that it commutes with restrictions and $\forall\, k,i,j$:

$$\begin{aligned}\theta_{ji}\big(s_{i|V\cap U_{ij}}\big)_{|V_k\cap U_{ij}} &= \theta_{ji}\big((s_{i|V\cap U_{ij}})_{|V_k\cap U_{ij}}\big) = \theta_{ji}\big(s_{i|V_k\cap U_{ij}}\big) = \theta_{ji}\big((s_{i|V_k\cap U_i})_{|V_k\cap U_{ij}}\big)\\ &= \theta_{ji}\big(s^k_{i|V_k\cap U_{ij}}\big) = s^k_{j|V_k\cap U_{ij}} = (s_{j|V_k\cap U_j})_{|V_k\cap U_{ij}} = s_{j|V_k\cap U_{ij}} = \big(s_{j|V\cap U_{ij}}\big)_{|V_k\cap U_{ij}}\end{aligned}$$

Now $\{V_k\cap U_{ij}\}_k$ is an open covering of $V\cap U_{ij}$, hence by S1 we get $\theta_{ji}(s_{i|V\cap U_{ij}}) = s_{j|V\cap U_{ij}}$, $\forall\, i,j$.

Next we shall construct the isomorphisms $\theta_i : \mathcal{F}_{|U_i} \xrightarrow{\sim} \mathcal{F}_i$. For this, we take the canonical projections

$$\forall\, W\subseteq U_i \text{ open},\; \theta_i : \mathcal{F}(W)\to\mathcal{F}_i(W) \;:\; s=\{s_j\}_j \mapsto s_i$$

θ_i commutes with restrictions since $(\theta_i\circ\rho^W_{W'})(s) = \theta_i\big(\{s_{j|W'}\}_j\big) = s_{i|W'} = \theta_i(s)_{|W'}$, $\forall\, W'\subseteq W\subseteq U_i$ open.
Moreover $\theta_j = \theta_{ji}\circ\theta_i$ on U_{ij} follows by definition of $\mathcal{F}$: $\forall\, W\subseteq U_{ij}$ open and $s=\{s_i\}_i\in\mathcal{F}(W)$,

$$s_i\in\mathcal{F}_i(W) \;\;\Rightarrow\;\; \theta_{ji}\big(\theta_i(s)\big) = \theta_{ji}(s_i) = \theta_{ji}(s_{i|W\cap U_{ij}}) = s_{j|W\cap U_{ij}} = s_j = \theta_j(s)$$

Hence it remains to show that θ_i maps $\mathcal{F}_{|U_i}$ isomorphically to $\mathcal{F}_i$ on any open subset $W\subseteq U_i$.
$-$ θ_i is injective : let $s=\{s_j\}_j\in\mathcal{F}(W)$ and $t=\{t_j\}_j\in\mathcal{F}(W)$ be such that $s_i=t_i$. Then $\forall\, j\neq i$,

$$s_j = s_{j|W\cap U_j} = s_{j|W\cap U_{ij}} = \theta_{ji}(s_{i|W\cap U_{ij}}) = \theta_{ji}(t_{i|W\cap U_{ij}}) = t_{j|W\cap U_{ij}} = t_{j|W\cap U_j} = t_j$$

since $s_j,t_j\in\mathcal{F}_j(W\cap U_j)$ by definition, hence $s_j=t_j$, $\forall\, j$ and $s=t$.
$-$ θ_i is surjective : let $t\in\mathcal{F}_i(W)$.
We shall construct $s=\{s_j\}_j\in\mathcal{F}(W)$ with $s_j\in\mathcal{F}_j(W\cap U_j)$, $\forall\, j$, such that $\theta_i(s)=t$. We set

$$s_j := \theta_{ji}(t_{|W\cap U_j}),\;\forall\, j$$

Thus $s_i = \theta_{ii}(t_{|W\cap U_i}) = \mathrm{id}(t) = t$ since $W\subseteq U_i$ and $s=\{s_j\}_j\in\mathcal{F}(W)$ because of the cocycle condition :

$$\begin{aligned}\theta_{kj}(s_{j|W\cap U_{kj}}) &= \theta_{kj}\big(\theta_{ji}(t_{|W\cap U_j})_{|W\cap U_{kj}}\big) = \theta_{kj}\big(\theta_{ji}(t_{|W\cap U_{kj}})\big) = (\theta_{kj}\circ\theta_{ji})(t_{|W\cap U_{kj}})\\ &= \theta_{ki}(t_{|W\cap U_{kj}}) = \theta_{ki}(t_{|W\cap U_k})_{|W\cap U_{kj}} = s_{k|W\cap U_{kj}}\end{aligned}$$

Hence $s\in\mathcal{F}(W)$ and we showed that $\theta_i : \mathcal{F}_{|U_i}\xrightarrow{\sim}\mathcal{F}_i$ is an isomorphism of sheaves, $\forall\, i$. □

3.3.2 Introductory example

Now we give a first example of how general schemes can be glued in order to define a new scheme. Here we do not describe the procedure in detail since it will be a particular case of a more general theorem which we will prove in section 3.3.3. The example is taken from [Ha].
Let $X_1 = (|X_1|,\mathcal{O}_{X_1})$ and $X_2 = (|X_2|,\mathcal{O}_{X_2})$ be schemes, $U_1\subseteq|X_1|$, $U_2\subseteq|X_2|$ open subsets (not necessarily affine) and let $\Phi = (\phi,\phi^*) : (U_1,\mathcal{O}_{X_1|U_1}) \to (U_2,\mathcal{O}_{X_2|U_2})$ be an isomorphism of locally ringed spaces. We will construct a new scheme X, obtained by *gluing* X_1 and X_2 along U_1 and U_2 via the isomorphism Φ.

The topological space $|X|$ is defined by the disjoint union $|X_1|\sqcup|X_2|$, modulo a relation that glues U_1 to U_2 via the homeomorphism $\phi : U_1\to U_2$. For $x_1\in|X_1|$, $x_2\in|X_2|$, we say that

$$x_1\sim x_2 \;\Leftrightarrow\; x_1\in U_1,\; x_2\in U_2 \;\text{ and }\; x_2=\phi(x_1)$$

So we identify $x_1\sim\phi(x_1)$, $\forall\, x_1\in U_1$, i.e. we identify the open subsets U_1 and U_2 inside $|X_1|\sqcup|X_2|$. Note that this gluing may "twist" the spaces $|X_1|$ and $|X_2|$ onto each other since ϕ is not necessarily the identity.

Hence we have $|X| := \big(|X_1| \sqcup |X_2|\big)/\sim$ as a set. To define a topology on $|X|$, consider the natural maps

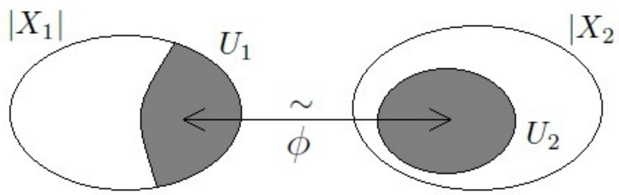
$$i_1 \;:\; |X_1| \to |X_1| \sqcup |X_2| \to |X| \qquad , \qquad i_2 \;:\; |X_2| \to |X_1| \sqcup |X_2| \to |X|$$

Denote $V_1 := \operatorname{im} i_1$ and $V_2 := \operatorname{im} i_2$. We endow $|X|$ with the quotient topology relative to i_1 and i_2, i.e.

$$V \subseteq |X| \text{ is open } \;\Leftrightarrow\; i_1^{-1}(V) \text{ is open in } |X_1| \text{ and } i_2^{-1}(V) \text{ is open in } |X_2|$$

which turns $|X|$ into a topological space. Since i_1 and i_2 are moreover injective, we have that $|X_i| \cong V_i$.

Figure 3.1: gluing $|X_1|$ and $|X_2|$ via ϕ



Now we shall define the structure sheaf $\mathcal{O}_X$ on $|X|$.
Note that $\{V_1, V_2\}$ is an open covering of $|X|$ and that we have the sheaves $i_{1*}\mathcal{O}_{X_1}$ and $i_{2*}\mathcal{O}_{X_2}$ on $|X|$. One can show that these 2 sheaves agree on $V_1 \cap V_2$ (proof in 3.3.3), i.e. there is an isomorphism of sheaves

$$\theta \;:\; (i_{1*}\mathcal{O}_{X_1})_{|V_1 \cap V_2} \xrightarrow{\sim} (i_{2*}\mathcal{O}_{X_2})_{|V_1 \cap V_2}$$

Hence we can glue both sheaves as in 3.3.1 to obtain a new sheaf $|X|$, denoted by $\mathcal{O}_X$. Explicitely,

$$\begin{aligned} V \subseteq |X| \text{ open } : \; \mathcal{O}_X(V) &= \left\{ (s_1, s_2) \;\middle|\; s_j \in i_{j*}\mathcal{O}_{X_j}(V \cap V_j),\; \big(\theta \circ \rho^{V\cap V_1}_{*V\cap V_1 \cap V_2}\big)(s_1) = \rho^{V \cap V_2}_{*V \cap V_1 \cap V_2}(s_2) \right\} \\ &= \left\{ (s_1, s_2) \;\middle|\; s_j \in \mathcal{O}_{X_j}\big(i_j^{-1}(V \cap V_j)\big),\; \theta\big(s_{1|i_1^{-1}(V\cap V_1 \cap V_2)}\big) = s_{2|i_2^{-1}(V \cap V_1 \cap V_2)} \right\} \\ &= \left\{ (s_1, s_2) \;\middle|\; s_j \in \mathcal{O}_{X_j}\big(i_j^{-1}(V)\big),\; \theta\big(s_{1|i_1^{-1}(V)\cap U_1}\big) = s_{2|i_2^{-1}(V)\cap U_2} \right\} \end{aligned} \tag{3.4}$$

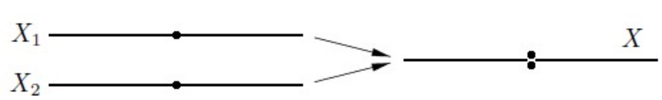
where $i_j^{-1}(V \cap V_j) = i_j^{-1}(V)$, $i_j^{-1}(V_1 \cap V_2) = U_j$ and we used (2.2) for the restrictions. $(|X|, \mathcal{O}_X)$ is moreover a locally ringed space because $\mathcal{O}_{X|V_j} \cong i_{j*}\mathcal{O}_{X_j}$ and the $i_j : |X_j| \to V_j$ are even homeomorphisms, thus the natural map in (2.3) is an isomorphism and $\forall\, x \in |X|$ with, say, $x \in V_1$,

$$\mathcal{O}_{X,x} = (\mathcal{O}_{X|V_1})_x \cong (i_{1*}\mathcal{O}_{X_1})_x = (i_{1*}\mathcal{O}_{X_1})_{i_1(i_1^{-1}(x))} \cong \mathcal{O}_{X_1, i_1^{-1}(x)} \;:\; \text{local ring}$$

by proposition 2.1.4 and (2.3). Finally $X = (|X|, \mathcal{O}_X)$ is also a scheme because any point $x \in |X|$ either writes as $x = i_1(x_1)$ for $x_1 \in |X_1|$ or as $x = i_2(x_2)$ for $x_2 \in |X_2|$, or both, and x_j has a neighborhood which is affine since X_j is a scheme. The image of this neighborhood is then an affine neighborhood of x in $|X|$.

Example :
A good example visualizing this gluing is the following : let $\mathbb{K}$ be a field, $|X_1| = |X_2| = \mathbb{A}^1_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[X]$ the affine line and $U_1 = U_2 = \mathbb{A}^1_{\mathbb{K}} \setminus \{M\}$ where M is the maximal ideal $\langle X \rangle$, so U_1 and U_2 are open. We get the (affine) schemes $X_1 = (\mathbb{A}^1_{\mathbb{K}}, \mathcal{O}_{\mathbb{K}[X]}) = X_2$ with open subsets $U_1 \subset |X_1|$ and $U_2 \subset |X_2|$. As homeomorphism, we simply take the identity map $\phi : U_1 \to U_2$.

Figure 3.2: gluing 2 affine lines to get the affine line with a double point



Hence we glue 2 affine lines everywhere except on that one maximal ideal, which is a closed point in $\mathbb{A}^1_{\mathbb{K}}$. Let X denote the scheme obtained by gluing X_1 and X_2 along U_1 and U_2 via ϕ. Then $|X|$ can be visualized by a line with a "double point". X is another example of a scheme that is not an affine scheme (a proof of this fact will be given in 3.3.4).

3.3.3 The Gluing Lemma

Now we generalize and prove the preceding results. Here we base ourselves on [U2], which we expanded at some points for a more explicit presentation.
Let I be a (possibly infinite) set and consider a family of schemes $\{X_i\}_{i\in I}$ where $X_i = (|X_i|, \mathcal{O}_{X_i})$, $\forall\, i \in I$. For each pair $(i,j) \in I \times I$, suppose we are given an open subset $U_{ij} \subseteq |X_i|$ and an isomorphism of schemes

$$\Phi_{ij} = (\phi_{ij}, \phi_{ij}^*) \ :\ (U_{ij}, \mathcal{O}_{X_i|U_{ij}}) \xrightarrow{\ \sim\ } (U_{ji}, \mathcal{O}_{X_j|U_{ji}})$$

For convenience, we set $U_{ii} = |X_i|$ and $\Phi_{ii} = \mathrm{id}$. Assume in addition that $\forall\, i,j,k \in I$:
a) $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$
b) we have the commutative diagram

$$\begin{tikzcd} U_{ij}\cap U_{ik} \arrow[rr, "\phi_{ik}"] \arrow[dr, "\phi_{ij}"'] & & U_{ki}\cap U_{kj} \\ & U_{ji}\cap U_{jk} \arrow[ur, "\phi_{jk}"'] & \end{tikzcd}$$

i.e. $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. This implies in particular that $\phi_{ji} = \phi_{ij}^{-1}$. Then there exists a scheme $X = (|X|, \mathcal{O}_X)$, called the scheme obtained by *gluing* the schemes X_i, and maps $\psi_i : |X_i| \to |X|$ such that
1) ψ_i is a homeomorphism of $|X_i|$ onto an open subset $V_i \subseteq |X|$.
2) $\{V_i\}_{i\in I}$ is an open covering of $|X|$.
3) $\psi_i(U_{ij}) = V_i \cap V_j$.
4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.
These conditions can be summarized by the following commutative diagram :

$$\begin{tikzcd} V_i & V_i\cap V_j \arrow[l] \arrow[r, "="] & V_j\cap V_i \arrow[r] & V_j \\ {|X_i|} \arrow[u, "\psi_i", "\cong"'] & U_{ij} \arrow[l] \arrow[u, "\psi_i", "\cong"'] \arrow[r, "\phi_{ij}"] & U_{ji} \arrow[u, "\cong", "\psi_j"'] \arrow[r] & {|X_j|} \arrow[u, "\cong", "\psi_j"'] \end{tikzcd}$$

In particular, $\psi_i : |X_i| \to V_i$ then induces an isomorphism of schemes $(|X_i|, \mathcal{O}_{X_i}) \cong (V_i, \mathcal{O}_{X|V_i})$, $\forall\, i \in I$.

Proof. The proof is a generalization of the idea introduced in section 3.3.2. As a set, we define

$$|X| := \Big(\coprod_{i\in I} |X_i|\Big) / \sim$$

where the equivalence relation $\sim$ on $\coprod_i |X_i|$ is defined as follows : for $x \in |X_i|$ and $y \in |X_j|$,

$$x \sim y \ \Leftrightarrow\ x \in U_{ij},\ y \in U_{ji} \ \text{ and } \ y = \phi_{ij}(x)$$

$\sim$ is reflexive : for $x \in |X_i|$, $x \sim x$ since $x \in U_{ii}$ and $x = \phi_{ii}(x)$.
$\sim$ is symmetric : let $x \in U_{ij} \subseteq |X_i|$ and $y \in U_{ji} \subseteq |X_j|$ such that $y = \phi_{ij}(x)$. Then $x = \phi_{ij}^{-1}(y) = \phi_{ji}(y)$.
$\sim$ is transitive : let $x \in |X_i|$, $y \in |X_j|$ and $z \in |X_k|$ such that $x \sim y$ and $y \sim z$. Then

$$x \in U_{ij},\ y \in U_{ji},\ y = \phi_{ij}(x) \ \text{ and } \ y \in U_{jk},\ z \in U_{kj},\ z = \phi_{jk}(y)$$

Thus $y \in U_{ji} \cap U_{jk}$ and assumption a) implies that $x \in U_{ij} \cap U_{ik}$ and $z \in U_{kj} \cap U_{ki}$. So b) holds and we get

$$z = \phi_{jk}(y) = \phi_{jk}\big(\phi_{ij}(x)\big) = \phi_{ik}(x) \quad \Rightarrow \ x \sim z$$

For all $i \in I$, we have the natural maps $|X_i| \to \coprod_i |X_i| \to |X|$. We denote them by $\psi_i : |X_i| \to |X|$ and set $V_i := \operatorname{im} \psi_i \subseteq |X|$. The ψ_i are injective by definition (2 different points from the same space $|X_i|$ are never identified), so that $\psi_i : |X_i| \to V_i$ is a bijection. Note however that the map $\coprod_i |X_i| \to |X|$ is not injective!

The formula $\psi_i(U_{ij}) = V_i \cap V_j$ is now immediate and $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij} follows because

$$\forall\, x \in U_{ij},\ x \sim \phi_{ij}(x) \in U_{ji} \quad \Rightarrow \quad \psi_i(x) = \psi_j\big(\phi_{ij}(x)\big) \text{ in } |X|$$

The topology on $|X|$ is given by the quotient topology relative to all the ψ_i, i.e. we say that

$$V \subseteq |X| \text{ is open} \quad \Leftrightarrow \quad \psi_i^{-1}(V) \text{ is open in } |X_i|,\ \forall\, i \in I$$

This indeed defines a topology because preimages commute with unions and intersections, so that $|X|$ is a topological space. In particular the ψ_i are immediately continuous with respect to this definition. Moreover the V_i are open because $\psi_j^{-1}(V_i) = \psi_j^{-1}(V_i \cap V_j) = U_{ji}$, which is open in $|X_j|$ by assumption. It follows that $\{V_i\}_{i\in I}$ is an open covering since $|X| = \bigcup_i \operatorname{im}\psi_i = \bigcup_i V_i$. Finally, if $W \subseteq |X_i|$ is open, then

$$\begin{aligned}\psi_j^{-1}\big(\psi_i(W)\big) = \psi_j^{-1}\big(\psi_i(W) \cap V_i \cap V_j\big) = \psi_j^{-1}\big(\psi_i(W) \cap \psi_i(U_{ij})\big) = \psi_j^{-1}\big(\psi_i(W \cap U_{ij})\big) \\ = (\psi_j^{-1} \circ \psi_i)(W \cap U_{ij}) = \phi_{ij}(W \cap U_{ij}) \ :\ \text{open in } |X_j|\end{aligned}$$

So $\psi_i(W)$ is open in $|X|$ and ψ_i is also an open map, i.e. $\psi_i : |X_i| \to V_i$ is a homeomorphism.

Now we shall define a sheaf of rings $\mathcal{O}_X$ on $|X|$. If we denote $\mathcal{O}_{V_i} := \psi_{i*}\mathcal{O}_{X_i}$, then $\mathcal{O}_{V_i}$ is a sheaf of rings on V_i, $\forall\, i \in I$ with the open covering $\{V_i\}_{i\in I}$ of $|X|$. In order to glue these sheaves as in 3.3.1, we have to show that $\mathcal{O}_{V_i|V_i\cap V_j} \cong \mathcal{O}_{V_j|V_i\cap V_j}$, $\forall\, i,j \in I$. Consider the commutative diagram

$$\begin{tikzcd} U_{ij} \arrow[rr, "\phi_{ij}"] \arrow[dr, "\psi_{i|U_{ij}}"'] & & U_{ji} \arrow[dl, "\psi_{j|U_{ji}}"] \\ & V_i \cap V_j & \end{tikzcd}$$

where each map is a homeomorphism and induces a corresponding isomorphism of locally ringed spaces :

$$\begin{aligned}\psi_i \ &:\ U_{ij} \xrightarrow{\sim} V_i \cap V_j \ \Rightarrow\ (U_{ij}, \mathcal{O}_{X_i|U_{ij}}) \cong (V_i \cap V_j, \mathcal{O}_{V_i|V_i\cap V_j}) \\ \psi_j \ &:\ U_{ji} \xrightarrow{\sim} V_j \cap V_i \ \Rightarrow\ (U_{ji}, \mathcal{O}_{X_j|U_{ji}}) \cong (V_i \cap V_j, \mathcal{O}_{V_j|V_i\cap V_j}) \\ \phi_{ij} \ &:\ U_{ij} \xrightarrow{\sim} U_{ji} \ \Rightarrow\ (U_{ij}, \mathcal{O}_{X_i|U_{ij}}) \cong (U_{ji}, \mathcal{O}_{X_j|U_{ji}})\end{aligned}$$

For example, $\forall\, W \subseteq V_i \cap V_j$, $\psi_i^* : \mathcal{O}_{V_i|V_i\cap V_j}(W) \to (\psi_{i*}\mathcal{O}_{X_i|U_{ij}})(W) \ \Leftrightarrow\ (\psi_{i*}\mathcal{O}_{X_i})(W) \xrightarrow{\sim} (\psi_{i*}\mathcal{O}_{X_i})(W)$.

Thus $\mathcal{O}_{V_i|V_i\cap V_j} \cong \mathcal{O}_{V_j|V_i\cap V_j}$, $\forall\, i,j \in I$, and we can glue the $\mathcal{O}_{V_i}$ to a unique sheaf $\mathcal{O}_X$ on $|X|$ such that

$$\mathcal{O}_{X|V_i} \cong \mathcal{O}_{V_i},\ \forall\, i \in I$$

Using proposition 2.1.4 and equation (2.3), this also allows to prove that $(|X|, \mathcal{O}_X)$ is a locally ringed space. Indeed, the $\psi_i : |X_i| \to V_i$ are homeomorphisms, so if $x \in |X|$ with, say, $x \in V_i$, then

$$\mathcal{O}_{X,x} = (\mathcal{O}_{X|V_i})_x \cong \mathcal{O}_{V_i,x} = (\psi_{i*}\mathcal{O}_{X_i})_x = (\psi_{i*}\mathcal{O}_{X_i})_{\psi_i(\psi_i^{-1}(x))} \cong \mathcal{O}_{X_i,\psi_i^{-1}(x)}$$

where $\mathcal{O}_{X_i,\psi_i^{-1}(x)}$ is a local ring since X_i is a scheme (in particular a locally ringed space).
Finally $X = (|X|, \mathcal{O}_X)$ is in addition a scheme because any point $x \in |X| = \bigcup_i \operatorname{im}\psi_i$ can be written as $x = \psi_i(x_i)$ for some $x_i \in |X_i|$ and this x_i has an open neighborhood U which is affine since X_i is a scheme. ψ_i being an open map, the image $\psi_i(U)$ is then an affine open neighborhood of x in $|X|$. $\square$

3.3.4 Example : The affine space with doubled zero

We can again generalize the example of the affine line with a double point. This can also be found in [U2]. Let $\mathbb{K}$ be a field, $n \geq 1$, $R_n = \mathbb{K}[X_1, \ldots, X_n]$, $R'_n = \mathbb{K}[Y_1, \ldots, Y_n]$ be polynomial rings and $|X_1| = \operatorname{Spec} R_n$, $|X_2| = \operatorname{Spec} R'_n$ the corresponding affine spaces. So we have the (affine) schemes

$$X_1 = (|X_1|, \mathcal{O}_{X_1}) = (\operatorname{Spec} R_n, \mathcal{O}_{R_n}) \qquad , \qquad X_2 = (|X_2|, \mathcal{O}_{X_2}) = (\operatorname{Spec} R'_n, \mathcal{O}_{R'_n})$$

Consider the maximal ideals $M_1 = \langle X_1, \dots, X_n \rangle \trianglelefteq R_n$, $M_2 = \langle Y_1, \dots, Y_n \rangle \trianglelefteq R'_n$ and denote their corresponding points in the spectrum by $0_1 \in |X_1|$ and $0_2 \in |X_2|$, i.e. we consider M_1 and M_2 as the "origins" of the affine spaces. We have the isomorphism of rings

$$\phi \ : \ \mathbb{K}[X_1, \dots, X_n] \xrightarrow{\sim} \mathbb{K}[Y_1, \dots, Y_n] \ : \ X_i \mapsto Y_i$$

which preserves these maximal ideals : $\phi(M_1) = M_2$. Via Spec, it induces a homeomorphism on the spectra

$$\operatorname{Spec}\phi \ : \ \operatorname{Spec} R'_n \xrightarrow{\sim} \operatorname{Spec} R_n \ \Leftrightarrow \ \operatorname{Spec}\phi \ : \ |X_2| \xrightarrow{\sim} |X_1|$$

with $\operatorname{Spec}\phi(0_2) = 0_1$. Now let $U_{12} = |X_1| \setminus \{0_1\}$ and $U_{21} = |X_2| \setminus \{0_2\}$, which are open since 0_1 and 0_2 are closed points. Then ϕ also induces homeomorphisms

$$\phi_{21} = (\operatorname{Spec}\phi)_{|U_{21}} \ : \ U_{21} \xrightarrow{\sim} U_{12} \qquad , \qquad \phi_{12} = \phi_{21}^{-1} \ : \ U_{12} \xrightarrow{\sim} U_{21}$$

(Actually, ϕ_{12} and ϕ_{21} are nothing but the identities since ϕ just represents an identity.) Now we glue X_1 and X_2 along the open sets U_{12} and U_{21} via the isomorphism of locally ringed spaces

$$\Phi_{12} = (\phi_{12}, \phi_{12}^*) \ : \ (U_{12}, \mathcal{O}_{X_1|U_{12}}) \xrightarrow{\sim} (U_{21}, \mathcal{O}_{X_2|U_{21}})$$

We denote by $X = (|X|, \mathcal{O}_X)$ the resulting scheme : it is an affine n-space with a "double origin". As shown in the proof of 3.3.3 ($|X_i| \cong V_i$), we may see $|X_1|$ and $|X_2|$ as topological subspaces of $|X|$, so that X_1 and X_2 become open subschemes of X.

X is not an affine scheme. To see this, let $(f, f^*) \ : \ (|X|, \mathcal{O}_X) \to (\operatorname{Spec} R, \mathcal{O}_R)$ be an arbitrary morphism of locally ringed spaces for some ring R. Restricting the continuous map $f \ : \ |X| \to \operatorname{Spec} R$ to the open subspaces $|X_1|$ and $|X_2|$, we get continuous maps arising from ring homomorphisms by theorem 2.6.2 :

$$\begin{aligned} f_1 \ &: \ |X_1| \to \operatorname{Spec} R \ \Leftrightarrow \ f_1 \ : \ \operatorname{Spec} R_n \to \operatorname{Spec} R \quad \Rightarrow \ \exists\, \varphi_1 \ : \ R \to R_n \\ f_2 \ &: \ |X_2| \to \operatorname{Spec} R \ \Leftrightarrow \ f_2 \ : \ \operatorname{Spec} R'_n \to \operatorname{Spec} R \quad \Rightarrow \ \exists\, \varphi_2 \ : \ R \to R'_n \end{aligned}$$

i.e. $f_1 = \operatorname{Spec}\varphi_1$, $f_2 = \operatorname{Spec}\varphi_2$ and they have to coincide on $|X_1| \cap |X_2|$ (as restrictions of the same map). Since ϕ is "nothing but identity", we get $\phi \circ \varphi_1 = \varphi_2$ and obtain the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R'_n \arrow[rr, "\operatorname{Spec}\phi"] \arrow[dr, "f_2 = \operatorname{Spec}\varphi_2"'] & & \operatorname{Spec} R_n \arrow[dl, "f_1 = \operatorname{Spec}\varphi_1"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

hence $f(0_1) = f_1(0_1) = \operatorname{Spec}\varphi_1(0_1) = \operatorname{Spec}\varphi_1\big(\operatorname{Spec}\phi(0_2)\big) = \operatorname{Spec}\varphi_2(0_2) = f_2(0_2) = f(0_2)$ with $0_1 \neq 0_2$. So f cannot be a homeomorphism and there does not exist an isomorphism $(|X|, \mathcal{O}_X) \cong (\operatorname{Spec} R, \mathcal{O}_R)$.

3.3.5 Example : The projective line

We go back to the case of the affine line ($n = 1$). The following example is again taken from [U2].
Let $|X_1| = \operatorname{Spec}\mathbb{K}[X]$, $|X_2| = \operatorname{Spec}\mathbb{K}[Y]$ (defining affine schemes X_1, X_2) and consider the maximal ideals $M_1 = \langle X \rangle$, $M_2 = \langle Y \rangle$. We set again $U_{12} = |X_1| \setminus \{M_1\}$ and $U_{21} = |X_2| \setminus \{M_2\}$. Now observe that

$$|X_1| \setminus \{M_1\} = U(X) \qquad , \qquad |X_2| \setminus \{M_2\} = U(Y)$$

For example, a prime ideal in $\mathbb{K}[X]$ not containing X must be different from M_1 (since M_1 contains X) and any prime ideal $P \trianglelefteq \mathbb{K}[X]$ different from M_1 cannot contain X, otherwise $M_1 = \langle X \rangle \subseteq P \ \Rightarrow \ P = M_1$ by maximality. By corollary 3.1.6, we know that distinguished open sets are again affine. More precisely :

$$U(X) \cong \operatorname{Spec}\big(\mathbb{K}[X]_X\big) = \operatorname{Spec}\mathbb{K}\big[X, \tfrac{1}{X}\big] \qquad , \qquad U(Y) \cong \operatorname{Spec}\big(\mathbb{K}[Y]_Y\big) = \operatorname{Spec}\mathbb{K}\big[Y, \tfrac{1}{Y}\big]$$

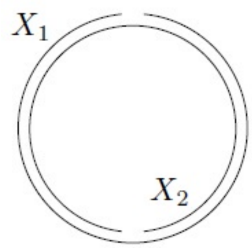
In order to glue X_1 and X_2 along the open subsets $U_{12} = U(X)$ and $U_{21} = U(Y)$, we need an isomorphism

$$(U_{12}, \mathcal{O}_{X_1|U_{12}}) \cong (U_{21}, \mathcal{O}_{X_2|U_{21}}) \quad \Leftrightarrow \quad \big(\operatorname{Spec} \mathbb{K}\big[X, \tfrac{1}{X}\big], \mathcal{O}_{\mathbb{K}[X, \frac{1}{X}]}\big) \cong \big(\operatorname{Spec} \mathbb{K}\big[Y, \tfrac{1}{Y}\big], \mathcal{O}_{\mathbb{K}[Y, \frac{1}{Y}]}\big)$$

By (2.19), such an isomorphism necessarily arises from a ring isomorphism $\mathbb{K}\big[Y, \frac{1}{Y}\big] \xrightarrow{\sim} \mathbb{K}\big[X, \frac{1}{X}\big]$. There are 2 obvious choices, namely $\phi : Y \mapsto X$ and $\psi : Y \mapsto \frac{1}{X}$.
If we glue U_{12} and U_{21} via $\operatorname{Spec} \phi$, we obtain the affine line with a double point.

If we however glue them via $\operatorname{Spec} \psi$, we glue U_{12} and U_{21} "along opposite edges" (see figure below) and obtain another scheme, called the *projective line over* $\mathbb{K}$ and denoted by $X = \mathbb{P}^1_{\mathbb{K}}$. Again we may see X_1 and X_2 as open subschemes of $\mathbb{P}^1_{\mathbb{K}} = (|X|, \mathcal{O}_X)$. Using formula (3.4), one can show that $\mathcal{O}_X\big(|X|\big) \cong \mathbb{K}$ since the only polynomials $g(X)$ such that $g\big(\frac{1}{Y}\big)$ is also a polynomial in Y are constant polynomials (details omitted). It follows that $\mathbb{P}^1_{\mathbb{K}}$ is not an affine scheme, otherwise $\mathbb{P}^1_{\mathbb{K}} \cong (\operatorname{Spec} R, \mathcal{O}_R)$ implies that $\mathbb{K} \cong \mathcal{O}_X\big(|X|\big) \cong \mathcal{O}_R(\operatorname{Spec} R) \cong R$. This is impossible since $\operatorname{Spec} \mathbb{K}$ is finite, but $|X|$ is not.

Figure 3.3: gluing $|X_1|$ and $|X_2|$ to obtain the projective line ; picture taken from [Va]



As in the geometric analogue of the projective line, $\mathbb{P}^1_{\mathbb{K}}$ contains the points $\mathbf{0}$ and ∞, where $\mathbf{0} \in |X|$ corresponds to the maximal ideal M_1 in $|X_1|$ and $\infty \in |X|$ corresponds to M_2. There even exists an open subset $U \subset |X|$ which contains both of them, namely $U = |X| \setminus \{\mathbf{1}\}$, where $\mathbf{1} \in |X_1| \cap |X_2|$ is the point corresponding to the maximal ideals $\langle\, X - 1 \,\rangle$ in $|X_1|$ and $\langle\, 1 - Y \,\rangle$ in $|X_2|$. Indeed $\mathbf{0}, \infty \in U$ and

$$\langle\, X - 1 \,\rangle \neq M_1 \;\Rightarrow\; \langle\, X - 1 \,\rangle \in U_{12} \cong \operatorname{Spec} \mathbb{K}\big[X, \tfrac{1}{X}\big] \quad , \quad \langle\, 1 - Y \,\rangle \neq M_2 \;\Rightarrow\; \langle\, 1 - Y \,\rangle \in U_{21} \cong \operatorname{Spec} \mathbb{K}\big[Y, \tfrac{1}{Y}\big]$$

In U_{21}, we have $\langle\, 1 - Y \,\rangle = \big\langle\, \frac{1-Y}{Y} \,\big\rangle$. If we want that $\langle\, X - 1 \,\rangle$ and $\langle\, 1 - Y \,\rangle$ define the same point in $\mathbb{P}^1_{\mathbb{K}}$, we need that they can be identified via $\operatorname{Spec} \psi$. But

$$\psi\big(\tfrac{1-Y}{Y}\big) = \psi\big(\tfrac{1}{Y}\big) \cdot \big(1 - \psi(Y)\big) = X \cdot \big(1 - \tfrac{1}{X}\big) = X - 1$$

$$\Rightarrow\; \langle\, 1 - Y \,\rangle = \big\langle\, \tfrac{1-Y}{Y} \,\big\rangle = \big\langle\, \psi^{-1}(X - 1) \,\big\rangle = \psi^{-1}\big(\langle\, X - 1 \,\rangle\big) = \operatorname{Spec} \psi\big(\langle\, X - 1 \,\rangle\big)$$

Hence $\langle\, X - 1 \,\rangle \sim \langle\, 1 - Y \,\rangle$ and they represesent the same point $\mathbf{1} \in |X|$.

3.4 Applications and examples

As a first application, we can give the following generalization of theorem 2.6.2 and formula (2.19), which can also be found in [Ga].

3.4.1 Theorem

Let $X = (|X|, \mathcal{O}_X)$ be any scheme and $Y = (|Y|, \mathcal{O}_Y)$ an affine scheme, i.e. $|Y| \cong \operatorname{Spec} R$ for some ring R. Then we have a 1-to-1 correspondence between morphisms of schemes $\Phi : X \to Y$ and ring homomorphisms $\varphi : \mathcal{O}_Y\big(|Y|\big) \to \mathcal{O}_X\big(|X|\big)$, i.e.

$$\operatorname{Hom}_{\mathtt{Sch}}(X, Y) = \operatorname{Hom}_{\mathtt{Sch}}\big((|X|, \mathcal{O}_X), (|Y|, \mathcal{O}_Y)\big) \cong \operatorname{Hom}_{\mathtt{Ring}}\big(\mathcal{O}_Y\big(|Y|\big), \mathcal{O}_X\big(|X|\big)\big) \tag{3.5}$$

In particular,

$$\operatorname{Hom}_{\mathtt{Sch}}\big((|X|, \mathcal{O}_X), (\operatorname{Spec} R, \mathcal{O}_R)\big) \cong \operatorname{Hom}_{\mathtt{Ring}}\big(R, \mathcal{O}_X\big(|X|\big)\big)$$

Proof. The particular case follows from the fact that $\mathcal{O}_Y\big(|Y|\big) \cong \mathcal{O}_R(\operatorname{Spec} R) \cong R$ for any affine scheme Y.

Given a morphism of schemes $\Phi = (\phi, \phi^*) : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$, the global pullback induces a homomorphism of rings

$$\phi^*_{|Y|} \ : \ \mathcal{O}_Y\big(|Y|\big) \to (\phi_*\mathcal{O}_X)\big(|Y|\big) = \mathcal{O}_X\big(|X|\big)$$

To show that Φ is also induced by this homomorphism, let $\{U_i\}_i$ be a cover of $|X|$ consisting of affine open sets (this exists by definition of a scheme). Denote the restriction of Φ to these open subsets by

$$\Phi_i := \Phi_{|U_i} = (\phi_i, \phi_i^*) \ : \ (U_i, \mathcal{O}_{X|U_i}) \to (|Y|, \mathcal{O}_Y)$$

where $\phi_i = \phi_{|U_i}$ and for any open subset $U \subseteq |Y|$, the new pullback is

$$\phi^*_{i,U} \ : \ \mathcal{O}_Y(U) \to (\phi_*\mathcal{O}_X)(U) = \mathcal{O}_X\big(\phi^{-1}(U)\big) \to \mathcal{O}_X\big(\phi^{-1}(U) \cap U_i\big) = \mathcal{O}_{X|U_i}\big(\phi_i^{-1}(U)\big) = (\phi_{i*}\mathcal{O}_{X|U_i})(U)$$

Moreover $(U_i, \mathcal{O}_{X|U_i}) \cong (\operatorname{Spec} R_i, \mathcal{O}_{R_i})$ for some rings R_i. By (2.19), these morphisms correspond exactly to ring homomorphisms given by the global pullbacks $\phi^*_{i,|Y|} : \mathcal{O}_Y\big(|Y|\big) \cong R \to \mathcal{O}_{X|U_i}(U_i) \cong R_i$.

As restrictions, these Φ_i need to coincide on intersections, i.e. $\Phi_{i|U_i\cap U_j} = \Phi_{j|U_i \cap U_j}, \forall\, i, j$. Since the $U_i \cap U_j$ define open subschemes of X, we know again that there exists a cover $\{U_{ijk}\}_k$ of $U_i \cap U_j$ consisting of affine open subsets. Hence we have $\Phi_{i|U_{ijk}} = \Phi_{j|U_{ijk}}, \forall\, i, j, k$, and the $\Phi_{i|U_{ijk}} : (U_{ijk}, \mathcal{O}_{X|U_{ijk}}) \to (|Y|, \mathcal{O}_Y)$ are again morphisms between affine schemes, so they correspond exactly to ring homomorphisms given by their global pullbacks $\rho^{U_i}_{U_{ijk}} \circ \phi^*_{i,|Y|} : \mathcal{O}_Y\big(|Y|\big) \to \mathcal{O}_{X|U_{ijk}}(U_{ijk})$. Due to the compatibility condition, we need that

$$\forall\, i, j, k\,,\ \rho^{U_i}_{U_{ijk}} \circ \phi^*_{i,|Y|} = \rho^{U_j}_{U_{ijk}} \circ \phi^*_{j,|Y|} \ : \ \mathcal{O}_Y\big(|Y|\big) \to \mathcal{O}_{X|U_{ijk}}(U_{ijk})$$

i.e. these pullbacks also have to coincide on intersections. But $\mathcal{O}_X$ is a sheaf, so they can be glued "pointwise" to a (unique) global ring homomorphism $\phi^*_{|Y|} : \mathcal{O}_Y\big(|Y|\big) \to \mathcal{O}_X\big(|X|\big)$. Each $\phi^*_{i,|Y|}$ inducing the restricted morphism $\Phi_i = \Phi_{|U_i}$, it follows that the global morphism Φ is indeed induced by $\phi^*_{|Y|}$. $\square$

Remark :
One can show, see e.g. [Ma], that the same formula also holds true for any locally ringed space, i.e.
For any locally ringed space $X = (|X|, \mathcal{O}_X)$ and any affine scheme $Y = (|Y|, \mathcal{O}_Y)$, we have a bijection

$$\operatorname{Hom}_{\texttt{LRS}}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\texttt{Ring}}\big(\mathcal{O}_Y\big(|Y|\big), \mathcal{O}_X\big(|X|\big)\big)$$

The proof in the case of X being a scheme is easier since we can recover the results locally.

3.4.2 Residue fields

Let R be a local ring with unique maximal ideal $M \trianglelefteq R$. We define the *residue field* of R as the quotient R/M. This is indeed a field since M is maximal.
We already encountered the notion of a residue field in section 2.4.1. The relation is the following :

Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space and $x \in |X|$. The stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{M}_x \trianglelefteq \mathcal{O}_{X,x}$. We denote the corresponding residue field by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{M}_x$.
Now assume that X is a scheme. Then any $x \in |X|$ has an affine neighborhood $U \subseteq |X|$, i.e. if $\mathcal{O}_U = \mathcal{O}_{X|U}$, then $(U, \mathcal{O}_U) \cong (\operatorname{Spec} R, \mathcal{O}_R)$ for some ring R. The point $x \in U \cong \operatorname{Spec} R$ therefore corresponds to a prime ideal $P \trianglelefteq R$, implying that $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} \cong \mathcal{O}_{R,P} \cong R_P$, the localization of R at P. Then

$$\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{M}_x \cong R_P/\mathfrak{M}_P \cong \operatorname{Quot}(R/P) = \kappa(P) \tag{3.6}$$

where $\kappa(P)$ is the residue field of R at P as defined in 2.4.1.
Note that this construction is independent of the chosen neighborhood of $x \in |X|$: if $V \subseteq |X|$ is another affine neighborhood of x with $(V, \mathcal{O}_V) \cong (\operatorname{Spec} T, \mathcal{O}_T)$, then x corresponds to some prime ideal $Q \trianglelefteq T$ and

$$R_P \cong \mathcal{O}_{U,x} = \mathcal{O}_{X,x} = \mathcal{O}_{V,x} \cong T_Q$$

Hence $\kappa(x) \cong R_P/\mathfrak{M}_P \cong T_Q/\mathfrak{M}_Q$, so that $\kappa(P) \cong \kappa(Q)$. It remains to prove (3.6).

Proof. We know that the maximal ideal $\mathfrak{M}_P \trianglelefteq R_P$ is described by $\mathfrak{M}_P = \{ \frac{r}{s} \mid r \in P,\ s \notin P \}$. Define

$$\varphi \ : \ R_P/\mathfrak{M}_P \longrightarrow \mathrm{Quot}(R/P) \ : \ \left[\frac{r}{s}\right] \longmapsto \frac{\bar{r}}{\bar{s}}$$

φ is well-defined : $\bar{s} \neq 0$ since $s \notin P$. Moreover if $\left[\frac{r}{s}\right] = \left[\frac{a}{b}\right]$, then $\frac{r}{s} - \frac{a}{b} = \frac{rb-as}{sb} \in \mathfrak{M}_P$, which means that

$$rb - as \in P \ \Rightarrow \ \bar{r} \cdot \bar{b} - \bar{a} \cdot \bar{s} = 0 \ \Rightarrow \ \frac{\bar{r}}{\bar{s}} = \frac{\bar{a}}{\bar{b}}$$

As homomorphism between fields with $\varphi(1) = 1$, φ is immediately injective. Its inverse is obviously $\frac{\bar{r}}{\bar{s}} \mapsto \left[\frac{r}{s}\right]$, which is also well-defined since $\bar{s} \neq 0 \Rightarrow s \notin P$ and $\frac{\bar{r}}{\bar{s}} = \frac{\bar{a}}{\bar{b}}$ means that $\bar{r}\bar{b} - \bar{a}\bar{s} = 0$ (R/P is an integral domain), so $rb - as \in P$ and $sb \notin P$ (P is prime). Finally $\frac{r}{s} - \frac{a}{b} = \frac{rb-as}{sb} \in \mathfrak{M}_P$, i.e. $\left[\frac{r}{s}\right] = \left[\frac{a}{b}\right]$. □

Examples :
The following applications are taken from [Sch] and illustrate some residue field constructions.

1) Consider a field $\mathbb{K}$ and the affine line $\mathbb{A}^1_{\mathbb{K}} = \mathrm{Spec}\,\mathbb{K}[X]$. $\mathbb{K}[X]$ being a principal ideal domain, we know that

$$\mathbb{A}^1_{\mathbb{K}} = \left\{ \{0\}, \langle f \rangle \ \middle| \ f \text{ is irreducible with } \deg f \geq 1 \right\}$$

$\mathbb{A}^1_{\mathbb{K}}$ is an affine scheme by definition. The residue field at the zero ideal is the function field over $\mathbb{K}$:

$$\kappa\big(\{0\}\big) = \mathrm{Quot}\left(\mathbb{K}[X]/\{0\}\right) = \mathrm{Quot}\,\mathbb{K}[X] = \mathbb{K}(X) = \left\{ \frac{f}{g} \ \middle| \ f, g \in \mathbb{K}[X],\ g \neq \mathbf{0} \right\}$$

Let f be an irreducible polynomial of the type $f(X) = X - \alpha$ for some $\alpha \in \mathbb{K}$. Then $\langle f \rangle$ is a maximal ideal, so that $\mathbb{K}[X]/\langle f \rangle$ is a field and we don't need to take the quotient field. The residue field at $\langle f \rangle$ is therefore

$$\kappa\big(\langle f \rangle\big) = \mathrm{Quot}\left(\mathbb{K}[X]/\langle f \rangle\right) \cong \mathbb{K}[X]/\langle X - \alpha \rangle \cong \mathbb{K}(\alpha) = \mathbb{K}$$

If $\mathbb{K}$ is algebraically closed, then all irreducible polynomials are of this form, so all residue fields (except the one at $\{0\}$) will be $\mathbb{K}$. Consider e.g. $\mathbb{K} = \mathbb{R}$ and $f(X) = X^2 + 1$. Then

$$\kappa\big(\langle f \rangle\big) \cong \mathbb{R}[X]/\langle X^2 + 1 \rangle \cong \mathbb{R}(i) = \mathbb{C}$$

In general, if $\mathbb{K}$ is not algebraically closed and f is a non-linear irreducible polynomial (e.g. for $\mathbb{K} = \mathbb{Q}$ and $f(X) = X^2 - 2$), we obtain as residue field $\kappa\big(\langle f \rangle\big) \cong \mathbb{K}[X]/\langle f \rangle$, which is a finite field extension of $\mathbb{K}$.

2) Now let $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[X,Y]$ be the affine complex plane. Here we have 3 types of prime ideals :

− the maximal ideals, which are all of the form $M = \langle X - \alpha, Y - \beta \rangle$ for $\alpha, \beta \in \mathbb{C}$ since $\mathbb{C}$ is algebraically closed. M being maximal, we again know that the residue field at M is

$$\kappa(M) = \mathrm{Quot}\left(\mathbb{C}[X,Y]/M\right) \cong \mathbb{C}[X,Y]/\langle X - \alpha, Y - \beta \rangle \cong \mathbb{C}(\alpha, \beta) = \mathbb{C}$$

Let us also compute the value of $f \in \mathbb{C}[X,Y]$ at M (as defined in 2.4.1). Due to maximality of M, we get $M + \mathbb{C} \cdot r = \mathbb{C}[X,Y]$ for any non-zero $r \in \mathbb{C}$, hence any polynomial f can be written as

$$f(X,Y) = f_0 + (X - \alpha) \cdot g(X,Y) + (Y - \beta) \cdot h(X,Y)$$

for some $g, h \in \mathbb{C}[X,Y]$ and where $f_0 = f(\alpha, \beta)$. Hence $f^f(M) = f \mod M = f_0 = f(\alpha, \beta) \in \mathbb{C} \cong \kappa(M)$. The value of f at M indeed coincides with the value (in the usual sense) obtained by plugging the point $(\alpha, \beta) \in \mathbb{C}^2$ into the polynomial f.

− the zero ideal $\{0\}$ since $\mathbb{C}[X,Y]$ is an integral domain. As in the case of the affine line, the corresponding residue field is given by the function field over $\mathbb{C}$ in two variables : $\kappa\big(\{0\}\big) = \mathrm{Quot}\left(\mathbb{C}[X,Y]\right) = \mathbb{C}(X,Y)$.

– the prime ideals of the form $P = \langle\, f \,\rangle$ where f is a non-constant irreducible polynomial in X and Y. In this case, the quotient $\mathbb{C}[X,Y]/\langle\, f \,\rangle$ is not a field and we obtain as residue field

$$\kappa(P) = \mathrm{Quot}\,\big(\mathbb{C}[X,Y]/\langle\, f \,\rangle\big) = \mathbb{C}(X,Y)/\langle\, f \,\rangle$$

This field consists of all rational expressions in the variables X and Y with the relation $f(X,Y) = 0$.
– One can show that there are no other types of prime ideals in $\mathbb{C}[X,Y]$.

3.4.3 The scheme of integers

Now we want to analyze the very important affine scheme $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_\mathbb{Z})$, called the *scheme of integers*. This is also presented in [Sch]. For this, recall that $\mathbb{Z}$ is a principal ideal domain (so all non-zero prime ideals are maximal) and the prime ideals in $\mathbb{Z}$ are $\{0\} = \langle\, 0 \,\rangle$ and $\langle\, p \,\rangle$ where $p \in \mathbb{Z}$ is a prime number (we denote $p \in \mathbb{P}$). So as a set :

$$\mathrm{Spec}\,\mathbb{Z} = \big\{ \{0\}\,,\, \langle\, p \,\rangle \;\big|\; p \in \mathbb{P} \big\}$$

$\{0\}$ is a generic point since $\mathbb{Z}$ does not contain zero divisors. In particular, $\overline{\{0\}} = \mathrm{Spec}\,\mathbb{Z}$ implies that the point is not closed. Its residue field is $\kappa\big(\{0\}\big) = \mathrm{Quot}\,\mathbb{Z} = \mathbb{Q}$. All other points in $\mathrm{Spec}\,\mathbb{Z}$ are closed since $\langle\, p \,\rangle$ is a maximal ideal, $\forall\, p \in \mathbb{P}$. Hence $\mathbb{Z}/\langle\, p \,\rangle$ is a field and we obtain

$$\kappa\big(\langle\, p \,\rangle\big) = \mathrm{Quot}\,\big(\mathbb{Z}/\langle\, p \,\rangle\big) \cong \mathbb{Z}/\langle\, p \,\rangle = \mathbb{F}_p$$

where $\mathbb{F}_p$ is the finite field of characteristic p. We conclude that, even for maximal ideals, the corresponding residue field can vary in an essential way. This can also be seen by the following computations :
Consider $7 \in \mathbb{Z}$ and compute the value of 7 at several points $P \in \mathrm{Spec}\,\mathbb{Z}$.

$$f^7\big(\langle\, 2 \,\rangle\big) = 7 \bmod 2 = 1 \in \mathbb{F}_2 \quad , \quad f^7\big(\langle\, 5 \,\rangle\big) = 7 \bmod 5 = 2 \in \mathbb{F}_5 \quad , \quad f^7\big(\langle\, 7 \,\rangle\big) = 7 \bmod 7 = 0 \in \mathbb{F}_7$$

In particular, we observe that the values of 7 lie in different fields.

Let us determine all closed subsets in $\mathrm{Spec}\,\mathbb{Z}$. By proposition 2.3.4, we know that there is a 1-to-1 correspondence between closed subsets of $\mathrm{Spec}\,\mathbb{Z}$ and radical ideals in $\mathbb{Z}$. For this, we recall that

$$\mathrm{Rad}\big(\langle\, n \,\rangle\big) = \langle\, p_1 \cdot p_2 \cdot \ldots \cdot p_r \,\rangle$$

for $n \in \mathbb{Z}$, $n \geq 2$ and where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$ is the prime factor decomposition of n with $p_i \in \mathbb{P}$ and $\alpha_i \geq 1$, $\forall\, i$. Hence radical ideals in $\mathbb{Z}$ are of the following form :

$$\langle\, n \,\rangle \trianglelefteq \mathbb{Z} \text{ is radical} \quad \Leftrightarrow \quad n = 0,\ n = 1 \text{ or } n \text{ has no repeated factors in its prime factor decomposition}$$

The above 1-to-1 correspondence is given by mapping a radical ideal I to the closed subset $V(I)$. Obviously

$$V\big(\langle\, 0 \,\rangle\big) = \mathrm{Spec}\,\mathbb{Z} \qquad \text{and} \qquad V\big(\langle\, 1 \,\rangle\big) = \emptyset$$

are closed. And if $I = \langle\, p_1 \cdot p_2 \cdot \ldots \cdot p_r \,\rangle \trianglelefteq \mathbb{Z}$ is radical with $p_i \in \mathbb{P}$, we get

$$V(I) = \big\{ P \in \mathrm{Spec}\,\mathbb{Z} \;\big|\; I \subseteq P \big\} = \big\{ \langle\, p \,\rangle \;\big|\; \langle\, p_1 \cdot p_2 \cdot \ldots \cdot p_r \,\rangle \subseteq \langle\, p \,\rangle,\, p \in \mathbb{P} \big\} = \big\{ \langle\, p_1 \,\rangle, \langle\, p_2 \,\rangle, \ldots, \langle\, p_r \,\rangle \big\}$$

Thus closed sets in $\mathrm{Spec}\,\mathbb{Z}$ are, beside the whole space and the empty set, just sets of finitely many points.

The reason why $\mathrm{Spec}\,\mathbb{Z}$ is an important example is because it is a terminal object in the category `Sch` of schemes : for any scheme $X = (|X|, \mathcal{O}_X)$, there exists a unique morphism of schemes, i.e. a morphism of locally ringed spaces, from X to $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_\mathbb{Z})$.
We first show this in the case where X is an affine scheme : let $|X| \cong \mathrm{Spec}\,R$ for some commutative unital ring R. By theorem 2.6.2, we know that any morphism $(\mathrm{Spec}\,R, \mathcal{O}_R) \to (\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_\mathbb{Z})$ is induced by a ring homomorphism $\mathbb{Z} \to R$. But $\mathbb{Z}$ is an initial object in the category of rings because $\mathbb{Z} \to R : n \mapsto n \cdot 1_R$ is the only choice for such a ring homomorphism. Hence there is also only one such morphism of schemes :

$$\{\mathrm{pt}\} = \mathrm{Hom}_{\mathtt{Ring}}(\mathbb{Z}, R) \cong \mathrm{Hom}_{\mathtt{ASch}}\big((\mathrm{Spec}\,R, \mathcal{O}_R), (\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_\mathbb{Z})\big)$$

The case of a general scheme X follows from theorem 3.4.1, which claims that

$$\mathrm{Hom}_{\mathtt{Sch}}\big((|X|,\mathcal{O}_X),(\operatorname{Spec}\mathbb{Z},\mathcal{O}_{\mathbb{Z}})\big)\cong\mathrm{Hom}_{\mathtt{Ring}}\big(\mathbb{Z},\mathcal{O}_X\big(|X|\big)\big)=\{\mathrm{pt}\}$$

In particular, a map $\psi:|X|\to\operatorname{Spec}\mathbb{Z}$ is uniquely determined by the ring homomorphism $\varphi:\mathbb{Z}\to\mathcal{O}_X\big(|X|\big)$. More precisely, for $x\in|X|$, $\psi(x)=\langle\, p\,\rangle$ where p is the characteristic of the residue field $\kappa(x)$.

Proof. We first consider the case of a morphism of affine schemes $\psi:\operatorname{Spec}R\to\operatorname{Spec}\mathbb{Z}$, where $\psi=\operatorname{Spec}\varphi$ for $\varphi:\mathbb{Z}\to R:\,n\mapsto n\cdot 1$. Let $P\in\operatorname{Spec}R$ and $p=\operatorname{char}\kappa(P)$. Then by definition

$$\psi(P)=\varphi^{-1}(P)=\big\{\,n\in\mathbb{Z}\;\big|\;n\cdot 1=1+\ldots+1\in P\,\big\}=\langle\,p\,\rangle$$

because $1+\ldots+1\in P\;\Leftrightarrow\;\bar{1}+\ldots+\bar{1}=0$ in $R/P\;\Leftrightarrow\;\bar{1}+\ldots+\bar{1}=0$ in $\kappa(P)\;\Leftrightarrow\;n$ is a multiple of p.
The result for a general scheme X then follows by locality : every point $x\in|X|$ has an affine neighborhood $U\cong\operatorname{Spec}T$ and corresponds to a prime ideal $Q\trianglelefteq T$, hence $\kappa(x)\cong\kappa(Q)$. $\square$

Note that the remark after theorem 3.4.1 even implies that $(\operatorname{Spec}\mathbb{Z},\mathcal{O}_{\mathbb{Z}})$ is a terminal object in the category `LRS` of all locally ringed spaces.

3.4.4 Multiplicities

Now we give an example of how schemes can be used in order to define the "multiplicity" of a point. Here we will follow the ideas of [Ga]. In particular, we postulate a few definitions and use some notions that we did not yet define rigorously. The goal is to explain the idea behind the constructions.

Let $|X|=\operatorname{Spec}R$ define an affine scheme. Given an ideal $I\trianglelefteq R$, we can construct another affine scheme by $|Y|=\operatorname{Spec}(R/I)$ and the projection map $\pi:R\to R/I$ induces a morphism of schemes $\operatorname{Spec}\pi:|Y|\to|X|$. By the corollary of proposition 3.1.4, we know that $\operatorname{Spec}\pi$ defines a homeomorphism between prime ideals in R/I and prime ideals in R containing I, i.e. $\operatorname{Spec}\pi:\operatorname{Spec}(R/I)\xrightarrow{\sim}V(I)$. Thus we can view Y as a "closed subscheme" of X.
If $|Y_1|=\operatorname{Spec}(R/I_1)$ and $|Y_2|=\operatorname{Spec}(R/I_2)$ define 2 closed subschemes of X, we define their intersection as

$$|Y_1|\cap|Y_2|:=\operatorname{Spec}\big(R/(I_1+I_2)\big)$$

This definition is justified by the following example.

Example :
Let $|X|=\operatorname{Spec}\mathbb{C}[X_1,X_2]$ be the affine complex plane and consider the closed subschemes

$$|Y_1|=\operatorname{Spec}\big(\mathbb{C}[X_1,X_2]/\langle\,X_2\,\rangle\big)\qquad\text{and}\qquad|Y_2|=\operatorname{Spec}\big(\mathbb{C}[X_1,X_2]/\langle\,X_2-X_1^2+a^2\,\rangle\big)$$

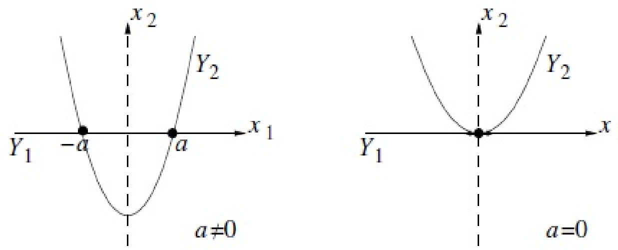
where $a\in\mathbb{C}$. Note that $\mathbb{C}[X_1,X_2]/\langle\,X_2\,\rangle\cong\mathbb{C}[X_1]$, so that $|Y_1|=\operatorname{Spec}\big(\mathbb{C}[X_1]\big)$ is actually an affine complex line in the affine plane. We try to understand what they represent.

Consider $|X|$ as a 2-dimensional plane (in the usual sense) with a coordinate system (x_1,x_2). Then we may see $|Y_1|$ as the complex line $\{\,X_2=0\,\}$ and $|Y_2|$ represents the parabola $\{\,X_2=X_1^2-a^2\,\}$. We of course think of $|Y_1|\cap|Y_2|$ as the intersection of the line and the parabola. Computing explicitly :

$$\langle\,X_2\,\rangle+\langle\,X_2-X_1^2+a^2\,\rangle=\langle\,X_2\,,\,X_1^2-a^2\,\rangle=\langle\,X_2\,,\,(X_1-a)(X_1+a)\,\rangle$$
$$\Rightarrow\;|Y_1|\cap|Y_2|=\operatorname{Spec}\Big(\mathbb{C}[X_1,X_2]/\big(\langle\,X_2\,,\,(X_1-a)(X_1+a)\,\rangle\big)\Big)\cong\operatorname{Spec}\Big(\mathbb{C}[X_1]/\langle\,(X_1-a)(X_1+a)\,\rangle\Big)$$

and solving $(X_1-a)(X_1+a)=0\Leftrightarrow X_1=\pm\,a$ gives the first coordinate of the intersection points. Hence we see that they intersect in exactly 2 points if $a\neq 0$, but only in 1 point if $a=0$. Intuitively we would say that the intersection point for $a=0$ is a point of multiplicity 2. Now we can make this precise.

Figure 3.4: intersections of multiplicity 1, resp. 2 ; the picture is taken from [Ga]



First assume that $a \neq 0$. Then we have $\mathbb{C}[X_1]/\langle (X_1-a)(X_1+a) \rangle \cong \mathbb{C} \oplus \mathbb{C}\bar{X}_1$ as vector spaces and

$$\operatorname{Spec}\Big(\mathbb{C}[X_1]/\langle (X_1-a)(X_1+a) \rangle\Big) = \{ \langle \bar{X}_1 - a \rangle , \langle \bar{X}_1 + a \rangle \}$$

with the relation $\bar{X}_1^2 = a^2$. This is because $\langle X-a \rangle$ and $\langle X-a \rangle$ are the only prime ideals in $\mathbb{C}[X]$ lying above $\langle X^2 - a^2 \rangle$ (similar proof as in 2.4.1). Hence the topological space $|Y_1| \cap |Y_2|$ consists of two points, corresponding to the points $(a,0)$ and $(-a,0)$ in $\mathbb{C}^2$.

For $a = 0$, we however have $|Y_1| \cap |Y_2| = \operatorname{Spec}\big(\mathbb{C}[X_1]/\langle X_1^2 \rangle\big)$, which has only one point as shown in 2.4.1, this point corresponding to $(0,0) \in \mathbb{C}^2$. But as a vector space $\mathbb{C}[X_1]/\langle X_1^2 \rangle$ still has dimension 2 over $\mathbb{C}$. This represents the "multiplicity" of that point. We say that $Y_1 \cap Y_2$ is a scheme of "dimension" 2 and its underlying topological space either consists of two points of multiplicity 1 or one point of multiplicity 2.

Another way to "detect" this multiplicity is by seeing that there is always a unique line in $\mathbb{A}^2_{\mathbb{C}}$ that passes through $|Y_1| \cap |Y_2|$, even in the case where it just consists of a single point. In coordinates, a line is represented by an equation of the form $c_1X_1 + c_2X_2 = 0$ for some $c_1, c_2 \in \mathbb{C}$. Saying that this lines "passes through" $|Y_1| \cap |Y_2|$ means that $Y_1 \cap Y_2$ should be a subscheme of the scheme corresponding to that line, i.e.

$$|Y_1| \cap |Y_2| = \operatorname{Spec}\Big(\mathbb{C}[X_1,X_2]/\langle X_2 \, , \, (X_1-a)(X_1+a) \rangle\Big) \subset \operatorname{Spec}\Big(\mathbb{C}[X_1,X_2]/\langle c_1X_1 + c_2X_2 \rangle\Big)$$

This is true if and only if $\langle c_1X_1 + c_2X_2 \rangle \subset \langle X_2 \, , \, (X_1-a)(X_1+a) \rangle$, which is the case if and only if $c_1 = 0$, i.e. the only line passing through $|Y_1| \cap |Y_2|$ must be the "horizontal" line $\{ X_2 = 0 \}$. Hence in the case where $a = 0$, the X_1-axis is the only line in $\mathbb{A}^2_{\mathbb{C}}$ that contains $\operatorname{Spec}\big(\mathbb{C}[X_1,X_2]/\langle X_2, X_1^2 \rangle\big)$, which we can therefore interpret as "the origin together with a tangent direction along the X_1-axis".

3.5 Projective schemes

Next we will define the most important class of schemes which are examples of non-affine schemes. They are constructed by means of graded rings. Here we again follow the text of [Ha], but also add of few ideas from [Si] and [Va].

3.5.1 Graded rings and homogeneous ideals

We say that a ring S is a *graded ring* if has a decomposition into a direct sum

$$S = \bigoplus_{d \geq 0} S_d = \bigoplus_{d \in \mathbb{N}_0} S_d$$

where each S_d is an abelian group and such that $S_d \cdot S_e \subseteq S_{d+e}$, $\forall\, d, e \in \mathbb{N}_0$. The S_d are the *homogeneous components* of S and elements of S_d are called *homogeneous elements* of *degree d*. Thus any element of the graded ring S can be written uniquely as a finite sum of homogeneous elements and $S_d \cdot S_e \subseteq S_{d+e}$ means that multiplying homogeneous elements of degrees d and e yields a homogeneous element of degree $d+e$. This implies in particular that $1 \in S_0$ and $0 \in S_d$, $\forall\, d \geq 0$ (i.e. 0 "has all degrees").
In the following, we will indicate that an element is homogenous by adding its degree as a lower index.

An ideal $I \trianglelefteq S$ is a *homogeneous ideal* if it can be decomposed into its homogeneous components, i.e. if

$$I = \bigoplus_{d \geq 0} \big(I \cap S_d \big) \tag{3.7}$$

The fundamental property of a homogeneous ideal is that it contains an element if any only if it contains all homogeneous components of that element : $r = \sum_d r_d \in I \;\Leftrightarrow\; r_d \in I,\ \forall\, d \geq 0$. Now we have :
1) An ideal I is homogeneous if and only if it can be generated by homogeneous elements.
2) The sum, product and intersection of homogeneous ideals are again homogeneous.
3) A homogeneous ideal is prime if and only if for any two homogeneous elements $r_d, s_e \in S$, we have

$$r_d \cdot s_e \in I \;\Rightarrow\; r_d \in I \text{ or } s_e \in I$$

Proof. 1) $\underline{\Rightarrow}$: (3.7) implies that every element in I writes as a finite sum of homogeneous elements that are also in I, so I is generated by the homogeneous elements that it contains.
$\underline{\Leftarrow}$: let $r \in I$ and write it as a (finite) sum $r = \sum_i r_i a_i$ where $r_i \in S$ and each a_i is a homogeneous generator of I. We can write again $r_i = \sum_j r_{ij}$ where each r_{ij} is homogeneous and rearrange the sum $r = \sum_i r_i a_i$ in terms of the degrees of the components, i.e.

$$r = \sum_i r_i a_i = \sum_{ij} r_{ij} a_i = (\text{ degree } 0\,) + (\text{ degree } 1\,) + \ldots$$

where each $r_{ij} a_i$ is homogeneous and belongs to I since $a_i \in I$. This shows formula (3.7).

2) Let $\{I_\lambda\}_\lambda$ be a family of homogeneous ideals and $\{a_{\lambda i}\}_i$ a generating system for I_λ of homogenous elements. Since the sum $\sum_\lambda I_\lambda$ is the ideal generated by the union $\bigcup_\lambda I_\lambda$, it suffices to take as generators the union of all generators $a_{\lambda i}$: $\sum_\lambda I_\lambda = \big\langle \{a_{\lambda i}\}_{\lambda, i} \big\rangle$, so the sum is again homogeneous.
If we have a finite family $\{I_i\}_{i=1,\ldots,m}$ of homogeneous ideals with (homogeneous) generators $\{a_{ij}\}_j$, the product ideal is generated by all product of these generators, which are homogeneous as well :

$$I_1 \cdot \ldots \cdot I_m = \big\langle \{a_{1j_1} \cdot a_{2j_2} \cdot \ldots \cdot a_{mj_m}\} \big\rangle$$

Finally, for the intersection $\bigcap_\lambda I_\lambda$ of a family $\{I_\lambda\}_\lambda$ of homogeneous ideals, we shall show that

$$\bigcap_\lambda I_\lambda = \bigoplus_{d \geq 0} \Big(\textstyle\bigcap_\lambda I_\lambda \cap S_d \Big)$$

Inclusion $\supseteq$ is obvious. Conversely, let $r \in \bigcap_\lambda I_\lambda$ and write $r = \sum_d r_d$ for some $r_d \in S_d$. Then $r \in I_\lambda$ for all λ, which means that $r_d \in I_\lambda,\ \forall\, \lambda,\ \forall\, d$, hence $r_d \in \bigcap_\lambda I_\lambda \cap S_d,\ \forall\, d \geq 0$.

3) Necessity is clear by definition, so it remains to prove sufficiency. We will do this in a simple case ; the proof in the general case follows by iterating this process. Let $r, s \in I$ such that $r = r_0 + r_1$, $s = s_0 + s_1$ and assume that $r \cdot s \in I$. We rearrange $r \cdot s$ into a sum of homogeneous elements with increasing degrees :

$$r \cdot s = (r_0 + r_1) \cdot (s_0 + s_1) = \underbrace{r_0 s_0}_{\in\, I \cap S_0} + \underbrace{(r_0 s_1 + r_1 s_0)}_{\in\, I \cap S_1} + \underbrace{r_1 s_1}_{\in\, I \cap S_2} \in I$$

By assumption, $r_0 s_0 \in I$ implies that $r_0 \in I$ or $s_0 \in I$. Now we have to proceed by cases :
a) $r_0 \in I \;\Rightarrow\; r_0 s_1 \in I \;\Rightarrow\; r_1 s_0 \in I \;\Rightarrow\; r_1 \in I$ or $s_0 \in I$. If $r_1 \in I$, then $r = r_0 + r_1 \in I$ and we are done.
b) $s_0 \in I \;\Rightarrow\; r_1 s_0 \in I \;\Rightarrow\; r_0 s_1 \in I \;\Rightarrow\; r_0 \in I$ or $s_1 \in I$. If $s_1 \in I$, then $s = s_0 + s_1 \in I$.
c) We are left with the case where $r_0 \in I$ and $s_0 \in I$. Then $r_1 s_1 \in I$ implies that $r_1 \in I$ or $s_1 \in I$, so at least one of r or s will surely belong to I.

For the general case, let $r = \sum\limits_{d=0}^{n} r_d$, $s = \sum\limits_{e=0}^{m} s_e$ and assume that $r \cdot s \in I$. Rearranging the terms, we get

$$r \cdot s = \sum_{k=0}^{n+m} \Big(\sum_{d+e=k} r_i \cdot s_e \Big) \in I \qquad \Rightarrow \quad \sum_{d+e=k} r_i \cdot s_e \in I,\ \forall\, k$$

By iterating the above procedure (analyze all possible cases), one finds that $r \in I$ or $s \in I$ in the end. $\square$

Example :
Let R be a ring and $S = R[X_1, \ldots, X_n]$ the polynomial ring in n variables. Then we can make S into a graded ring by taking S_d to be the set of all (finite) linear combinations of monomials of weight d, i.e.

$$S_d = \Big\{ \sum{}' r_i \cdot X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n} \ \Big| \ r_i \in R, \ \alpha_1 + \ldots + \alpha_n = d \Big\}$$

Then $S = \bigoplus_{d \geq 0} S_d$ since any polynomial can uniquely be written as a linear combination of such monomials.

3.5.2 The Proj–functor

Let S be a graded ring and denote $S_+ := \bigoplus_{d>0} S_d$. This is an ideal because of the condition $S_d \cdot S_e \subseteq S_{d+e}$. S_+ is moreover homogeneous since

$$\bigoplus_{d \geq 0} (S_+ \cap S_d) = \bigoplus_{d \geq 0} \Big(\bigoplus_{e>0} S_e \cap S_d \Big) = \varnothing \oplus \bigoplus_{d>0} S_d \ = \ S_+$$

We define the set $\operatorname{Proj} S$ as the set of all homogeneous prime ideals in S which do not contain all of S_+ :

$$\operatorname{Proj} S := \big\{ \, P \trianglelefteq S \ \big| \ P \text{ prime, homogeneous}, \ S_+ \not\subseteq P \, \big\}$$

In section 3.5.4 we will explain by means of an example why one should ignore this ideal ; S_+ is therefore also called the *irrelevant ideal*. For a homogeneous ideal $I \trianglelefteq S$, we moreover define the subset

$$V_+(I) := \big\{ \, P \in \operatorname{Proj} S \ \big| \ I \subseteq P \, \big\}$$

One shows exactly as in lemma 2.3.2 that the following properties hold true :

– If I, J are homogeneous ideals in S, then $V_+(I \cdot J) = V_+(I) \cup V_+(J)$.
– If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of homogeneous ideals in S, then $V_+\big(\sum_{\lambda \in \Lambda} I_\lambda \big) = \bigcap_{\lambda \in \Lambda} V_+(I_\lambda)$.

Note that both formulas make sense since products and sums of homogeneous ideals are again homogeneous. In addition they allow to define a topology on the set $\operatorname{Proj} S$ by taking as closed sets the subsets of the form $V_+(\ )$, i.e. by saying that $A \subseteq \operatorname{Proj} S$ is *closed* if there exists a homogeneous ideal $I \trianglelefteq S$ such that $A = V_+(I)$. Then $\emptyset = V_+(S)$, $\operatorname{Proj} S = V_+\big(\{0\}\big)$, so $\emptyset$ and $\operatorname{Proj} S$ are closed since S and $\{0\}$ are homogeneous ideals. Moreover finite unions and arbitrary intersections of closed sets are again closed, which turns $\operatorname{Proj} S$ into a topological space. Next we want to define a sheaf of rings on this topological space. For this, we have to introduce the following notions.

Let $P \in \operatorname{Proj} S$ and define T as the set consisting of all homogeneous elements of S which are not in P :

$$T = \Big(\bigcup_{d \geq 0} S_d \Big) \setminus P$$

T is a multiplicative set since $1 \in S_0$ and $1 \notin P$, so $1 \in T$ and if $r, s \in T$ are homogeneous of degrees d, e, then $r \cdot s \in S_{d+e}$ with $r \cdot s \notin P$ (since P is prime), hence $r \cdot s \in T$ again. Now consider the localization

$$T^{-1}S = \Big\{ \, \frac{s}{t} \ \Big| \ s \in S, \ t \in T \, \Big\} = \Big\{ \, \frac{s}{t} \ \Big| \ s \in S, \ t \notin P \text{ and } t \text{ is homogeneous} \Big\}$$

This is a graded ring with respect to the following grading. If s and t are homogeneous of degree d and e respectively, we define $\frac{s}{t}$ to be of degree $d - e$. This is well-defined : if $\frac{s}{t} = \frac{a}{b}$, then $\exists \, c \in T$ such that

$$c \cdot (sb - at) = 0 \ \Leftrightarrow \ c \cdot s \cdot b - c \cdot a \cdot t = 0$$

which implies that csb and cat must be of the same degree (otherwise their difference cannot vanish). Then

$$\begin{aligned} \deg(c \cdot s \cdot b) = \deg(c \cdot a \cdot t) \ &\Leftrightarrow \ \deg c + \deg s + \deg b = \deg c + \deg a + \deg t \\ &\Leftrightarrow \ \deg s - \deg t = \deg a - \deg b \end{aligned}$$

Here it is however possible that we can have negative degrees as well, so that $T^{-1}S$ is $\mathbb{Z}$-graded :

$$T^{-1}S = \bigoplus_{d\in\mathbb{Z}} T_d$$

We also denote the subring T_0 consisting of homogeneous elements of degree 0 by $S_{(P)}$. Hence $S_{(P)}$ consists of fractions whose numerator and denominator are homogeneous elements of the same degree in S.

$S_{(P)}$ is even a local ring. For this, it suffices to show that $S_{(P)} \setminus S_{(P)}^\times$ is an ideal in $S_{(P)}$. We have

$$\begin{aligned} S_{(P)}^\times &= \left\{ \frac{s}{t} \;\middle|\; s,t \text{ homogeneous of the same degree, } s,t \notin P \right\} \\ \Rightarrow\; S_{(P)} \setminus S_{(P)}^\times &= \left\{ \frac{s}{t} \;\middle|\; s,t \text{ homogeneous of the same degree, } s\in P,\ t \notin P \right\} =: M \end{aligned}$$

where $\frac{0}{1} \in M$ (since 0 has every degree) and M is closed under addition and multiplication with elements of the whole ring. Hence $S_{(P)}$ is a local ring with unique maximal ideal M.

Now let $U \subseteq \operatorname{Proj} S$ be open. We define $\mathcal{P}_S(U)$ to be the set of functions $f : U \to \coprod_{P\in U} S_{(P)}$ satisfying

1) $\forall\, P \in U,\ f(P) \in S_{(P)}$.
2) f is locally a quotient of elements of S : $\forall\, P \in U$, there is a neighborhood V of P in U and there are homogeneous elements $s,t \in S$ of the same degree such that $\forall\, Q \in V$, $t \notin Q$ and $f(Q) = \frac{s}{t} \in S_{(Q)}$.

This construction is very similar to the definition of the structure sheaf on the spectrum of a ring. By the same arguments, it follows that $\mathcal{P}_S(U)$ is a commutative unital ring and, due to the local nature of the definition, $U \mapsto \mathcal{P}_S(U)$ is a sheaf with respect to the usual restriction of functions $\rho_V^U : \mathcal{P}_S(U) \to \mathcal{P}_S(V)$ for any inclusion of open sets $V \subseteq U \subseteq \operatorname{Proj} S$.

Conclusion :
Hence we can associate a ringed space $(\operatorname{Proj} S, \mathcal{P}_S)$ to any graded ring S. And this is even a locally ringed space since one can show that the stalk $\mathcal{P}_{S,P}$ at a point $P \in \operatorname{Proj} S$ is isomorphic to the local ring $S_{(P)}$. The proof is completely analogous to the one in proposition 2.4.3.

Now Proj even defines a functor from the category of graded rings to the category of locally ringed spaces. We only explain this very briefly because this is not the goal of the section and, secondly, the construction is again similar to the one of the `Spec`–functor in section 2.6.1.
Let $\varphi : S \to T$ be a morphism of graded rings, i.e. a ring homomorphism that preserves the degrees, which means that $\varphi(S_d) \subseteq T_d$, $\forall\, d \geq 0$. Then we can construct a morphism of locally ringed spaces

$$\texttt{Proj}(\varphi) \;:\; (\operatorname{Proj} T, \mathcal{P}_T) \to (\operatorname{Proj} S, \mathcal{P}_S)$$

where $\operatorname{Proj}\varphi : \operatorname{Proj} T \to \operatorname{Proj} S$ is a continuous map defined by $\operatorname{Proj}\varphi(P) = \varphi^{-1}(P)$. The fact that φ preserves degrees ensures that $\varphi^{-1}(P)$ is again a homogeneous ideal.

3.5.3 Theorem

The importance of $\operatorname{Proj} S$ will become clear now. The proof of the following theorem is inspired from [Ha], but we expanded it by adding some more verifications which have been omitted in the textbook.

Let $r \in S_+$ be homogeneous and define the set $U_+(r) = \{ P \in \operatorname{Proj} S \mid r \notin P \}$. Then $U_+(r)$ is open in $\operatorname{Proj} S$ and open sets of this type form a basis for the topology on $\operatorname{Proj} S$. In particular they cover $\operatorname{Proj} S$. Moreover for any homogeneous $r \in S_+$, we have an isomorphism of locally ringed spaces

$$\big(U_+(r), \mathcal{P}_{S|U_+(r)}\big) \cong \big(\operatorname{Spec} S_{(r)}, \mathcal{O}_{S_{(r)}}\big)$$

where $S_{(r)}$ is the subring of elements of degree 0 in the localized ring S_r (not a local ring in general). This implies in particular that $(\operatorname{Proj} S, \mathcal{P}_S)$ is a scheme. Schemes of this type are called *projective schemes*.

Proof. We have $U_+(r) = \operatorname{Proj} S \setminus V_+\big(\langle\, r\,\rangle\big)$ because $r \notin P \Leftrightarrow \langle\, r\,\rangle \nsubseteq P$, $\forall\, P \in \operatorname{Proj} S$. Also note that $\langle\, r\,\rangle$ is a homogeneous ideal (being generated by r), thus $U_+(r)$ is open. As in the affine case, we have

$$U_+(r) \cap U_+(s) = U_+(r \cdot s),\ \forall\, r, s \in S_+ \text{ homogeneous}$$

Hence for proving that these sets form a basis of topology, it remains to show that

$$\operatorname{Proj} S = \big\{\, P \trianglelefteq S \;\big|\; P \text{ prime, homogeneous},\ S_+ \nsubseteq P \,\big\} = \bigcup_{r \in S_+ \text{ homogeneous}} U_+(r)$$

Inclusion $\supseteq$ is clear by definition. Conversely, let $P \in \operatorname{Proj} S$. Since $S_+ \nsubseteq P$, $\exists\, r' \in S_+$ such that $r' \notin P$. Hence there must be at least one homogeneous component r of r' which does not belong to P (otherwise $r' \in P$). Therefore $P \in U_+(r)$.

Now fix a homogeneous element $r \in S_+$. We shall define an isomorphism of locally ringed spaces

$$\Phi = (\phi, \phi^*)\ :\ \big(U_+(r), \mathcal{P}_{S|U_+(r)}\big) \xrightarrow{\ \sim\ } \big(\operatorname{Spec} S_{(r)}, \mathcal{O}_{S_{(r)}}\big)$$

Recall that

$$S_{(r)} = \left\{\, \frac{s}{r^n} \;\middle|\; n \in \mathbb{N}_0,\ s \in S \text{ homogeneous},\ \deg s = \deg(r^n) = n \cdot \deg r \,\right\}$$

and this is a subring of $S_r = T_r^{-1} S$, where $T_r = \{\, r^n \mid n \in \mathbb{N}_0 \,\}$. For a homogeneous ideal $I \trianglelefteq S$, we define

$$\phi(I) := T_r^{-1} I \cap S_{(r)} \quad \Rightarrow \quad \phi(I) \trianglelefteq S_{(r)}$$

And for $P \in \operatorname{Proj} S$ such that $r \notin P$, we know that $T_r^{-1} P$ is a prime ideal in S_r. In particular it does not contain 1, so that $S_{(r)} \subsetneq T_r^{-1} P$ and $\phi(P)$ is a prime ideal in $S_{(r)}$. Hence we have $\phi : U_+(r) \to \operatorname{Spec} S_{(r)}$. We prove that ϕ is a bijection in a separate lemma after the proof.
It is even a homeomorphism : if $J \trianglelefteq S_{(r)}$ is an ideal and $I \trianglelefteq S$ a homogeneous ideal, then

$$\phi^{-1}\big(V(J)\big) = V_+\big(\phi^{-1}(J)\big) \cap U_+(r) \qquad , \qquad \phi\big(V_+(I) \cap U_+(r)\big) = V\big(\phi(I)\big)$$

because $I \subseteq P \ \Leftrightarrow\ \phi(I) \subseteq \phi(P)$, $\forall\, P \in \operatorname{Proj} S$ (here we use that ϕ is bijective, so $\phi(I)$ is again an ideal).

Next we show that $\forall\, P \in U_+(r)$, $S_{(P)} \cong (S_{(r)})_{\phi(P)}$, the localization of $S_{(r)}$ at the prime ideal $\phi(P)$.

$$S_{(P)} = \left\{\, \frac{s}{t} \;\middle|\; s \in S,\ t \notin P \text{ both homogeneous such that } \deg s = \deg t \,\right\}$$

For $(S_{(r)})_{\phi(P)}$, note that $\frac{b}{r^m} \in S_{(r)} \setminus \phi(P) \ \Leftrightarrow\ b \notin P$. Hence we get

$$(S_{(r)})_{\phi(P)} = \left\{\, \frac{a/r^n}{b/r^m} \;\middle|\; \deg a = n \cdot \deg r,\ \deg b = m \cdot \deg r,\ b \notin P \,\right\} = \left\{\, \frac{a}{b} \cdot r^{\frac{\deg b - \deg a}{\deg r}} \;\middle|\; b \notin P \,\right\}$$

so the bijection follows since $\deg t - \deg s = 0$ and $\deg\left(a \cdot r^{\frac{\deg b - \deg a}{\deg r}}\right) = \deg a + \frac{\deg b - \deg a}{\deg r} \cdot \deg r = \deg b$.

This allows to define an isomorphism of sheaves $\phi^* : \mathcal{O}_{S_{(r)}} \xrightarrow{\ \sim\ } \phi_*\big(\mathcal{P}_{S|U_+(r)}\big)$. Let $U \subseteq \operatorname{Spec} S_{(r)}$ be open. By definition :

$$\mathcal{O}_{S_{(r)}}(U) = \left\{\, f : U \to \coprod_{Q \in U} (S_{(r)})_Q \;\middle|\; f \text{ is locally a quotient} \,\right\}$$

where bijectivity of ϕ implies that $(S_{(r)})_Q = (S_{(r)})_{\phi(\phi^{-1}(Q))} \cong S_{(\phi^{-1}(Q))}$ and

$$\coprod_{\phi^{-1}(Q) \in U} (S_{(r)})_{\phi^{-1}(Q)} \quad \cong \quad \coprod_{P \in \phi^{-1}(U)} (S_{(r)})_{(P)}$$

Hence it suffices to set $\phi_U^* : \mathcal{O}_{S_{(r)}}(U) \to \mathcal{P}_{S|U_+(r)}\big(\phi^{-1}(U)\big) : f \mapsto f \circ \phi$ because

$$\phi_U^*(f) \;:\; \phi^{-1}(U) \xrightarrow{\phi} U \xrightarrow{f} \coprod_{Q \in U} (S_{(r)})_Q \quad \cong \coprod_{P \in \phi^{-1}(U)} (S_{(r)})_{(P)}$$

i.e. $\phi_U^*(f) \in \mathcal{P}_S\big(\phi^{-1}(U)\big)$ since it is still locally a quotient. ϕ being bijective, we obtain that ϕ_U^* is an isomorphism, $\forall\, U \subseteq \operatorname{Spec} S_{(r)}$ open, and ϕ^* also commutes with restrictions (usual restriction of functions). It follows that (ϕ, ϕ^*) is an isomorphism of locally ringed spaces. □

As a corollary, we also obtain that $\mathcal{P}_S\big(U_+(r)\big) \cong \mathcal{O}_{S_{(r)}}\big(\operatorname{Spec} S_{(r)}\big) \cong S_{(r)}$, $\forall\, r \in S_+$ homogeneous.

Lemma. $\phi : U_+(r) \to \operatorname{Spec} S_{(r)}$ is a bijection.

Proof. For this proof, we solved an exercise from [Va]. Note that ϕ is defined in 2 steps :

$$U_+(r) \ni P \overset{\phi_1}{\longmapsto} T_r^{-1} P \overset{\phi_2}{\longmapsto} T_r^{-1} P \cap S_{(r)} \in \operatorname{Spec} S_{(r)}$$

As in corollary 3.1.6, ϕ_1 defines a 1-to-1 correspondence between prime ideals in S_r and prime ideals in S not containing r. This correspondence preserves homogeneity, i.e. there is also a bijection between homogeneous prime ideals in S_r and homogeneous prime ideals in S not containing r.
So it remains to show that $\phi_2 : P \mapsto P \cap S_{(r)}$ defines a bijection between homogeneous prime ideals in S_r and prime ideals in $S_{(r)}$. Constructing an inverse is however a bit tricky. For $Q \in \operatorname{Spec} S_{(r)}$, we define Q^* as the ideal generated by homogeneous elements $\frac{a}{r^n} \in S_r$ satisfying the condition

$$\frac{a^{\deg r}}{r^{\deg a}} \in Q \tag{3.8}$$

This is well-defined : $\deg\big(a^{\deg r}\big) = \deg\big(r^{\deg a}\big)$ and if $\frac{a}{r^n} = \frac{b}{r^m}$, $\exists\, c \in T_r$ such that $c \cdot (ar^m - br^n) = 0$, so

$$\deg a + m \cdot \deg r = \deg b + n \cdot \deg r$$

$$\Rightarrow \quad \frac{a^{\deg r}}{r^{\deg a}} = \frac{(c \cdot a \cdot r^m)^{\deg r}}{r^{\deg a} \cdot (c \cdot r^m)^{\deg r}} = \frac{(c \cdot b \cdot r^n)^{\deg r}}{r^{\deg a + m \cdot \deg r} \cdot c^{\deg r}} = \frac{b^{\deg r} \cdot r^{n \cdot \deg r}}{r^{\deg b + n \cdot \deg r}} = \frac{b^{\deg r}}{r^{\deg b}}$$

However we still need to check that Q^* is indeed an ideal (if so, it is homogeneous since it is generated by homogeneous elements), i.e. sums and products of elements satisfying (3.8) must again satisfy (3.8). First note that if a satisfies (3.8), then due to independence of the representative, $a \cdot r^k$ also satisfies (3.8) because $\frac{a}{r^n} = \frac{ar^k}{r^{n+k}}$, $\forall\, k \in \mathbb{N}_0$. Moreover we have $\frac{a}{r^n} \in Q^* \;\Leftrightarrow\; \big(\frac{a}{r^n}\big)^2 = \frac{a^2}{r^{2n}} \in Q^*$ because

$$\frac{(a^2)^{\deg r}}{r^{\deg(a^2)}} = \frac{a^{2 \cdot \deg r}}{r^{2 \cdot \deg a}} = \Big(\frac{a^{\deg r}}{r^{\deg a}}\Big)^2$$

and Q is prime (i.e. it contains an element if and only if it contains the square of that element). Q^* is closed under multiplication with elements of the whole ring : $\forall\, \frac{a}{r^n} \in Q^*$, $\frac{s}{r^m} \in S_r$,

$$\frac{(sa)^{\deg r}}{r^{\deg(sa)}} = \frac{s^{\deg r}}{r^{\deg s}} \cdot \frac{a^{\deg r}}{r^{\deg a}} \in Q \tag{3.9}$$

and under addition : $\forall\, \frac{a}{r^n}, \frac{b}{r^m} \in Q^*$, we have $\big(\frac{a}{r^n} + \frac{b}{r^m}\big)^2 = \frac{a^2 r^{2m} + 2abr^{n+m} + b^2 r^{2n}}{r^{2(m+n)}} \in Q^*$ since each term satisfies (3.8), so $\frac{a}{r^n} + \frac{b}{r^m} \in Q^*$ as well. In addition, (3.9) shows that Q^* is prime. Now one checks that

$$\forall\, Q \in \operatorname{Spec} S_{(r)} \;:\; Q^* \cap S_{(r)} = Q$$

$\supseteq$: if $\frac{a}{r^n} \in Q$, $\frac{a^{\deg r}}{r^{\deg a}} = \frac{a^{(deg r)-1}}{r^{(\deg a)-n}} \cdot \frac{a}{r^n} \in Q$ since $\deg r > 0$, so $\frac{a}{r^n} \in Q^*$
$\subseteq$: if $\frac{a}{r^n} \in Q^*$ with $\deg a = n \cdot \deg r$, then $\frac{a^{\deg r}}{r^{\deg a}} = \big(\frac{a}{r^n}\big)^{\deg r} \in Q \;\Rightarrow\; \frac{a}{r^n} \in Q$

$$\forall\, P \in \operatorname{Spec} S_r \text{ homogeneous } : \; P = (P \cap S_{(r)})^*$$

$\subseteq$: if $\frac{a}{r^n} \in P$, $\frac{a^{\deg r}}{r^{\deg a}} = \frac{a^{(\deg r)-1}}{r^{(\deg a)-n}} \cdot \frac{a}{r^n} \in P$ and $\deg\left(a^{\deg r}\right) = \deg r + \deg a = \deg\left(r^{\deg a}\right)$
$\supseteq$: if $\frac{a}{r^n} \in S_r$ is such that $\frac{a^{\deg r}}{r^{\deg a}} \in P \cap S_{(r)}$, then $\frac{a}{r^n} \in P$ since P is prime and cannot contain powers of r
Hence the assignment $Q \mapsto Q^*$ is the inverse of the above map ϕ_2 and we finally get that ϕ is bijective. □

3.5.4 Example : The projective space

Let R be a ring and consider the graded ring $S = R[X_0, X_1, \ldots, X_n]$. We define the *projective n-space* over R, denoted by $\mathbb{P}^n_R$, to be the projective scheme $\operatorname{Proj} S = \operatorname{Proj} R[X_0, \ldots, X_n]$.
We want $\operatorname{Proj} S$ to be related to the usual projective space $\mathbb{P}^n$ (see also section 3.6.1). And indeed one can show that if $R = \mathbb{K}$ is an algebraically closed field, then the subspace consisting of all closed points in $\mathbb{P}^n_{\mathbb{K}}$ (i.e. the homogeneous prime ideals that are maximal among those not containing the irrelevant ideal) is homeomorphic to $\mathbb{P}^n$.
This example also allows to illustrate why the ideal S_+ is irrelevant. In fact, $S_+ = \langle\, X_0, \ldots, X_n \,\rangle$ since the only polynomials of degree 0 are the constants. However, when considering the vanishing set associated to this ideal, we have to require that all the homogeneous coordinates shall vanish. But this is not a point in the projective space, so we shall ignore this ideal in general.
Let us finally point out that for $n = 1$, $\operatorname{Proj} \mathbb{K}[X_0, X_1] \cong \mathbb{P}^1_{\mathbb{K}}$, where $\mathbb{P}^1_{\mathbb{K}}$ is the projective line from section 3.3.5 obtained by gluing two affine lines. We do not prove this result. So we see that the Proj–construction of schemes allows to "avoid messy gluings".

3.6 Relation between schemes and varieties

Now we explain that the notion of a scheme does in fact generalize the notion of an algebraic variety. For this we first need to broaden the notion of a variety since affine varieties as defined in 2.2.1 are not sufficient. Moreover it is not quite true that such varieties are schemes because there are examples where the underlying topological space of a scheme has more points than the corresponding variety. But there is a way of adding generic points to the variety so that it becomes a scheme. We again follow the ideas developed in [Ha].
The main theorem of this section is however a very deep result and we will only give a sketch of its proof.

3.6.1 The category of varieties

Let $\mathbb{K}$ be a field, denote $R_n = \mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring and recall that an *affine variety* is an irreducible closed subset of $\mathbb{K}^n$ with respect to the Zariski topology :

$$A \subseteq \mathbb{K}^n \text{ is closed } \;\Leftrightarrow\; \exists\, I \trianglelefteq R_n \text{ such that } A = \mathcal{V}(I)$$

An open subset of an affine variety is called a *quasi-affine* variety.

Recall that the *projective space* $\mathbb{P}^n$ is defined as $\left(\mathbb{K}^{n+1} \setminus \{(0, \ldots, 0)\}\right)/\sim$ where the equivalence relation $\sim$ identifies points lying on the same line through the origin : $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$, $\forall\, \lambda \in \mathbb{K}$, $\lambda \neq 0$. We denote

$$\mathbb{P}^n = \left\{\, (a_0 : \ldots : a_n) \;\middle|\; \exists\, i \in \{0, \ldots, n\} \text{ such that } a_i \neq 0 \,\right\}$$

where $(a_0 : \ldots : a_n)$ is the equivalence class of $(a_0, \ldots, a_n) \in \mathbb{K}^{n+1} \setminus \{(0, \ldots, 0)\}$. We say that $(a_0, \ldots, a_n)$ is an $(n+1)$–tuple of *homogeneous coordinates* for $(a_0 : \ldots : a_n)$.
A polynomial $f \in S = \mathbb{K}[X_0, \ldots, X_n]$ is called *homogeneous of degree d* if $f \in S_d$. Hence f is of the form

$$f = \sum_{|\alpha| = d} a_{\alpha_0 \ldots \alpha_n} \cdot X_0^{\alpha_0} \cdot \ldots \cdot X_n^{\alpha_n} \qquad \text{where } |\alpha| = \alpha_0 + \ldots + \alpha_n$$

and since $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d \cdot f(a_0, \ldots, a_n)$, $\forall\, \lambda \in \mathbb{K}$, it makes sense to define the zero set of f.

As in the affine case, we again define a topology on the space $\mathbb{P}^n$. If $I \trianglelefteq S$ is a homogeneous ideal, we define

$$\mathcal{Z}(I) := \{ (a_0 : \ldots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0, \ \forall f \in I \text{ homogeneous} \}$$

and say that a subset $B \subseteq \mathbb{P}^n$ is closed if there exists a homogeneous ideal $I \trianglelefteq S$ such that $B = \mathcal{Z}(I)$. Then we define a *projective variety* to be an irreducible closed subset of $\mathbb{P}^n$. Similarly, an open subset of a projective variety is called a *quasi-projective variety*.

The goal is to define the category of varieties over $\mathbb{K}$, denoted by $\mathtt{Var}(\mathbb{K})$. As objects we take any affine, quasi-affine, projective or quasi-projective variety as defined above. We simply call these objects *varieties over* $\mathbb{K}$. In order to define morphisms between varieties, we need the following definition :
Let $Y \subseteq \mathbb{K}^n$ be a quasi-affine variety. A function $f : Y \to \mathbb{K}$ is called *regular at a point* $p \in Y$ if there exist an open neighborhood U of p in Y and polynomials $g, h \in R_n$ such that $h(x) \neq 0$, $\forall x \in U$ and $f = \frac{g}{h}$ on U. In the case where $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, we say that a function $f : Y \to \mathbb{K}$ is regular at $p \in Y$ if there exist an open neighborhood U of p in Y and homogeneous polynomials $g, h \in S$ of the same degree such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. Note that the quotient is well-defined since g and h are homogeneous of the same degree. In both cases, we say that f is *regular on* Y if it is regular at every point of Y, i.e. if f is locally a quotient of (homogeneous) polynomial functions.
We denote by $\mathcal{R}_Y(Y)$ the ring of all regular functions on Y. Similarly, we can define $\mathcal{R}_Y(U)$ for every open subset $U \subseteq Y$ and thus obtain a presheaf $U \mapsto \mathcal{R}_Y(U)$ with respect to the usual restriction of functions. Due to the local nature of the definition of regular functions, this presheaf is even a sheaf. $\mathcal{R}_Y$ is called the *sheaf of regular functions on* Y.
Now let X and Y be two varieties. A *morphism of varieties* is a continuous map $\varphi : X \to Y$ such that for every open set $V \subseteq Y$ and every regular function $f : V \to \mathbb{K}$, the pullback $f \circ \varphi_{|\varphi^{-1}(V)} : \varphi^{-1}(V) \to \mathbb{K}$ is regular on the open set $\varphi^{-1}(V)$, i.e. if φ pulls regular functions back to regular functions.
In particular we again have the notion of an isomorphism, which is necessarily a homeomorphism between the topological spaces. However, not every homeomorphism $X \to Y$ is necessarily an isomorphism.

Remark :
Regular functions $f : Y \to \mathbb{K}$ are continuous (where we take the Zariski topology on $\mathbb{K}^1$).

Proof. Let Y for example be a quasi-affine variety. In order to show that f is continuous, it suffices to prove that f is continuous in a neighborhood of every point in Y. Let $\{U_i\}_i$ be an open covering of Y such that f is given by a quotient $f_{|U_i} = \frac{g_i}{h_i}$ on each U_i where $g_i, h_i \in R_n$ such that $h_i(x) \neq 0$, $\forall x \in U_i$. We have to show that preimages under $f_{|U_i}$ of closed sets in $\mathbb{K}$ are closed in U_i. Since $\mathbb{K}$ is 1-dimensional, a closed subset of $\mathbb{K}$ is a finite set of points, so it suffices to prove that $f_{|U_i}^{-1}(\{a\})$ is closed in U_i, $\forall a \in \mathbb{K}$ and $\forall i$.

$$\forall x \in U_i, \ f_{|U_i}(x) = a \ \Leftrightarrow \ \frac{g_i(x)}{h_i(x)} = a \ \Leftrightarrow \ (g_i - a \cdot h_i)(x) = 0 \ \Leftrightarrow \ x \in Z(g_i - a \cdot h_i) \cap U_i$$

Hence $f_{|U_i}^{-1}(\{a\}) = f^{-1}(\{a\}) \cap U_i = Z(g_i - a \cdot h_i) \cap U_i$ which is closed in U_i, so $f_{|U_i}$ is continuous, $\forall i$. $\square$

3.6.2 Schemes over schemes

Grothendieck has also introduced the relative viewpoint, whose idea is to study morphisms of schemes and how they behave instead of studying a scheme by itself.
Let S be a fixed scheme. A *schemes over* S, or an S*-scheme*, is a scheme X together with a morphism of schemes $X \to S$. We call S the *base* of X. If X and Y are two schemes over S, a *morphism of schemes over* S (also called an S*-morphism*), is a morphism of schemes $f : X \to Y$ that is compatible with the morphisms $X \to S$ and $Y \to S$, i.e. such that we have the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We denote the category of schemes over S (made up by S-schemes and S-morphisms) by $\texttt{Sch}(S)$. If R is a ring, we also denote $\texttt{Sch}(R)$ for the category of schemes over the affine scheme $\operatorname{Spec} R$.

Examples :
1) Every affine scheme is a scheme over $\operatorname{Spec}\mathbb{Z}$:
For any ring R, we have the natural map $\mathbb{Z} \to R : n \mapsto n \cdot 1_R$ which induces a morphism $\operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}$.

2) If A is an R-algebra (where R is a commutative unital ring), the canonical ring homomorphism $R \to A$ induces a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} R$, so that $\operatorname{Spec} A \in \texttt{Var}(R)$.
Conversely, assume that $S = \operatorname{Spec} R$ for some ring R and let X be a scheme over S. By theorem 3.4.1 and formula (3.5), the morphism $X \to S$ is induced by a ring homomorphism $R \to \mathcal{O}_X\big(|X|\big)$, which therefore induces an R-algebra structure on $\mathcal{O}_X\big(|X|\big)$.

3) If X is a scheme over S, then every open subset $U \subset |X|$ also defines a scheme over S by the chain of morphisms $U \to X \to S$. In the particular case where $S = \operatorname{Spec} R$, this implies that we obtain an R-algebra structure on $\mathcal{O}_X(U)$ for any $U \subset |X|$ open as well. Summarizing :
Given a scheme X over an affine scheme $\operatorname{Spec} R$, the sheaf $\mathcal{O}_X$ is a sheaf of R-algebras.

4) If $\mathbb{K}$ is a field, we know that $\operatorname{Spec}\mathbb{K}$ only consists of the point $\{0\}$ and its structure sheaf $\mathcal{O}_{\mathbb{K}}$ is the constant sheaf $\mathcal{O}_{\mathbb{K}}\big(\{0\}\big) = \mathcal{O}_{\mathbb{K}}(\operatorname{Spec}\mathbb{K}) \cong \mathbb{K}$. By the above, giving a scheme X over $\operatorname{Spec}\mathbb{K}$ is the same as endowing its structure sheaf $\mathcal{O}_X$ with the structure of a sheaf of $\mathbb{K}$-vector spaces.

5) If X is any scheme, then morphisms $\operatorname{Spec}\mathbb{K} \to X$ can be described explicitly. This is e.g. done in [Ma].
Assume we are given a morphism of schemes $(\phi, \phi^*) : \operatorname{Spec}\mathbb{K} \to X$. Since $\operatorname{Spec}\mathbb{K}$ only consists of one point, it is mapped to some $x \in |X|$, i.e. $\phi\big(\{0\}\big) = x$. Moreover we have the local homomorphism on the stalks

$$\phi^*_{\{0\}} \ : \ \mathcal{O}_{X,x} \to \mathcal{O}_{\mathbb{K},\{0\}} \cong \mathbb{K}$$

Being local, it preserves the maximal ideals, which means that the maximal ideal $\mathfrak{M}_x \trianglelefteq \mathcal{O}_{X,x}$ is completely mapped to 0. Hence it induces a well-defined homomorphism of fields

$$\mathcal{O}_{X,x}/\mathfrak{M}_x = \kappa(x) \longrightarrow \mathbb{K}$$

Conversely, let a point $x \in |X|$ and a field homomorphism $\varphi : \kappa(x) \to \mathbb{K}$ be given. This induces a map

$$\mathcal{O}_{X,x} \xrightarrow{\pi} \mathcal{O}_{X,x}/\mathfrak{M}_x = \kappa(x) \xrightarrow{\varphi} \mathbb{K}$$

which is also a local homomorphism since $\pi(\mathfrak{M}_x) = \{0\}$. Now let $U \cong \operatorname{Spec} R$ be an affine open neighborhood of x. Then $x \in |X|$ corresponds to a prime ideal $P \in \operatorname{Spec} R$ and we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{R,P} \cong R_P$. Hence we get a ring homomorphism

$$R \longrightarrow R_P \cong \mathcal{O}_{X,x} \longrightarrow \mathbb{K}$$

which finally induces a morphism of schemes $\operatorname{Spec}\mathbb{K} \to \operatorname{Spec} R \cong U \hookrightarrow X$, i.e. $\operatorname{Spec}\mathbb{K}$ is a scheme over X.

Conclusion :
Giving a morphism of schemes $\operatorname{Spec}\mathbb{K} \to X$ is equivalent to picking a point $x \in |X|$ and a field homomorphism $\kappa(x) \to \mathbb{K}$. In particular, $\forall\, x \in |X|$, there is a canonical morphism $\operatorname{Spec}\kappa(x) \to X$.

3.6.3 Recalls on irreducible spaces

Before stating the main theorem, we need some last preliminaries about irreducible topological spaces. The following statements should only be reminders and we will not prove them (they can be found in any book about algebra and topology, as for example in [Ha] or [Sch]).

A topological space X is called *irreducible* if for any decomposition of the type $X = A_1 \cup A_2$ where A_1 and A_2 are closed, we have $A_1 = X$ or $A_2 = X$. Some equivalent conditions are :
− Every non-empty open set is dense in X.
− Every proper closed subset in X has empty interior.
− Any 2 non-empty open subsets of X have a non-empty intersection.

The last property also shows that irreducible topological spaces are never Hausdorff, unless $X = \{\mathrm{pt}\}$.
A subset $Y \subseteq X$ is called irreducible if it irreducible with respect to the induced topology. If X is irreducible, then any open subset $U \subseteq X$ and every image $f(X)$ under a continuous map f are irreducible as well. Moreover a subset $Y \subseteq X$ is irreducible $\Leftrightarrow$ its topological closure $\bar{Y}$ is irreducible.

Let X be a topological space and denote by $t(X)$ the set of non-empty irreducible closed subsets of X. Hence if $Y \subseteq X$ is closed, then $t(Y) \subseteq t(X)$. Moreover $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ if $Y_1, Y_2 \subseteq X$ are closed and for a family of closed subsets $\{Y_i\}_i$, we have $t\big(\bigcap_i Y_i\big) = \bigcap_i t(Y_i)$. This defines a topology on the set $t(X)$ by saying that $A \subseteq t(X)$ is closed if and only if $A = t(Y)$ for some closed subset $Y \subseteq X$.
In addition, a continuous map $f : X_1 \to X_2$ induces a continuous map $t(f) : t(X_1) \to t(X_2)$ given by

$$t(f) \ : \ Z \longmapsto \overline{f(Z)}$$

This is well-defined since Z irreducible $\Rightarrow$ $f(Z)$ irreducible $\Rightarrow$ $\overline{f(Z)}$ irreducible. Thus t defines a functor $\texttt{Top} \to \texttt{Top}$. Furthermore we have a continuous map $\alpha : X \to t(X) : x \mapsto \overline{\{x\}}$.

This map α is the tool we have to use to add generic points in order to construct a scheme from a variety. We will only sketch the proof of the following theorem. A more detailed proof can e.g. be found in [Ha].

3.6.4 Theorem

Let $\mathbb{K}$ be an algebraically closed field. Then there exists a fully faithful functor $\mathcal{T} : \texttt{Var}(\mathbb{K}) \to \texttt{Sch}(\mathbb{K})$ from the category of varieties over $\mathbb{K}$ to the category of schemes over $\operatorname{Spec} \mathbb{K}$.

Sketch of proof. Let V be a variety over $\mathbb{K}$ and denote by $\mathcal{O}_V$ its sheaf of regular functions. We set

$$\mathcal{T}(V) := \big(t(V), \alpha_* \mathcal{O}_V\big)$$

One has to show that this is indeed a scheme over $\operatorname{Spec} \mathbb{K}$ (which requires more hard work). One first proves that $\big(t(V), \alpha_* \mathcal{O}_V\big)$ is a scheme if V is an affine variety. Then, by section 3.6.2, we know that giving a morphism of schemes $\mathcal{T}(V) \to \operatorname{Spec} \mathbb{K}$ is equivalent to endowing the sheaf $\alpha_* \mathcal{O}_V$ with the structure of a vector space over $\mathbb{K}$. This is done by using theorem 3.4.1 : since $\alpha^{-1}\big(t(V)\big) = V$, we have

$$\operatorname{Hom}_{\texttt{Sch}}\Big(\big(t(V), \alpha_* \mathcal{O}_V\big), \big(\operatorname{Spec} \mathbb{K}, \mathcal{O}_{\mathbb{K}}\big)\Big) \cong \operatorname{Hom}_{\texttt{Ring}}\Big(\mathbb{K}\,,\, (\alpha_* \mathcal{O}_V)\big(t(V)\big)\Big) = \operatorname{Hom}_{\texttt{Ring}}\big(\mathbb{K}, \mathcal{O}_V(V)\big)$$

We define this ring homomorphism $\mathbb{K} \to \mathcal{O}_V(V)$ by mapping $\lambda \in \mathbb{K}$ to the constant function λ on V. It follows that $\mathcal{T}(V)$ is a scheme over $\operatorname{Spec} \mathbb{K}$.
Now if V and W are two varieties, one also needs to check that the natural map induced by $\mathcal{T}$

$$\operatorname{Hom}_{\texttt{Var}(\mathbb{K})}(V, W) \longrightarrow \operatorname{Hom}_{\texttt{Sch}(\mathbb{K})}\big(\mathcal{T}(V), \mathcal{T}(W)\big)$$

is a bijection, which implies that the functor $\mathcal{T} : \texttt{Var}(\mathbb{K}) \to \texttt{Sch}(\mathbb{K})$ is fully faithful.

Conclusion :
The functor $\mathcal{T}$ being fully faithful, it follows again that we may identify the category of varieties over $\mathbb{K}$ with a full subcategory of the category of schemes over $\operatorname{Spec} \mathbb{K}$ in the case of an algebraically closed field. Thus we may see varieties as being "embedded" into the category of schemes. In particular, proposition 1.2.4 then implies that $\mathcal{T}(V) \cong \mathcal{T}(W)$ as schemes if and only if $V \cong W$ as varieties.

Chapter 4

Moduli problems and moduli spaces

Moduli problems are a quite general class of problems and it is highly complicated, say impossible, to formalize the concept in whole generality. Simplifying, a moduli problem consists of the following data :
First we have to specify a class of algebro-geometric objects (e.g. schemes, sheaves, manifolds, morphisms or combinations of these), together with an appropriate notion of a "family" of such objects over a given object B. Next we define an equivalence relation $\sim$ on the set $F(B)$ of all such families over B. The moduli space is then a geometric space (for example a scheme) whose points represent these equivalence classes of objects of a fixed kind. Moduli spaces frequently arise as solutions of classification problems. The goal is to show that a collection of interesting objects can be given the structure of a geometric space and to parameterize their equivalence classes by introducing "coordinates" on the resulting space.
The possibilities one has in mind for a moduli problem are wide-ranging (thus the vague term "object") and the properties we want to study may vary with the considered problem. Thus we first focus on particular examples which shall help to understand the ideas behind and to get used to the language of modern Algebraic Geometry. In particular, we will not yet be able to establish deep results. The main references that we used for this chapter are [HM], [Ho] and [Sch].

4.1 Families of objects

We start by giving several examples of what it means to have a family of "objects" of a certain "type" over a certain "space". The general idea is the following :
A *family* over a *base* B is a surjective map $\pi : X \to B$ such that the fibers $X_b := \pi^{-1}(b)$ are spaces of the same type, $\forall\, b \in B$. b is called a *parameter* of the family. Depending on the structures of X and B one may require additional properties for the map π (e.g. smooth, linear, ...). Later on, we will also introduce equivalence relations on the set of all fibers; the most interesting case occurs if these equivalence classes are given by isomorphism classes. The goal is to classify our families of interest e.g. up to isomorphism.

4.1.1 Families of vector spaces

A *family of vector spaces* is given by a surjective map $\pi : X \to B$ as above such that the fibers X_b are vector spaces for all $b \in B$. Intuitively, one would think of X as a vector bundle over B. But this is only a very particular case. Indeed the above definition is still very vague and it may e.g. happen that
− the vector spaces X_b have different dimensions or are vector spaces over different fields
− X and B are not real differentiable or complex manifolds
− the map π is not smooth, resp. holomorphic
− the "bundle" X is not locally trivial
If one explicitly wants to talk about vector bundles, it has to be specified in the description of the moduli problem. This is what is meant by fixing the "type" of the objects one wants to study. Of course it is still possible to study a completely abstract family of vector spaces, but this usually complicates the problem and it is rarely possible to obtain any interesting results.

Let's for example analyze the case of vector bundles. For this, we need that our base B is a smooth (or complex) manifold. A vector bundle of rank r is then a family of vector spaces $\pi : X \to B$ over the base manifold B where X is another smooth (or complex) manifold and π is a smooth (or holomorphic) map such that each fiber $X_b = \pi^{-1}(b)$ is an r-dimensional vector space over $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and X is locally trivial, i.e. $\forall\, b \in B$, there is an open neighborhood U of b in B such that $\pi^{-1}(U) \cong U \times \mathbb{K}^r$. In this case

$$F(B) := \big\{ \text{ families over } B \big\} = \big\{ \text{ vector bundles of rank } r \text{ over } B \big\}$$

Now consider the particular case where $B = \{\mathrm{pt}\}$ is just a point. A vector bundle of rank r over 1 point is nothing but a vector space of dimension r. Hence if we are e.g. interested in isomorphism classes, the set $F(\mathrm{pt})/\!\sim$ (where $\sim$ means that 2 vector bundles are equivalent if they are isomorphic) only consists of one point because every vector space V over $\mathbb{K}$ of dimension r satisfies $V \cong \mathbb{K}^r$ after some choice of a basis. A rigorous definition of what it means to say that two families are isomorphic will be given in section 4.1.6.
Also note that the condition of local triviality is always satisfied over 1 point, so that we can even analyze the case of arbitrary families of $\mathbb{K}$-vector spaces $\pi : X \to \{\mathrm{pt}\}$. Since the whole space X is a fiber, we need that X is a vector space itself over the fixed base field $\mathbb{K}$. Hence if we consider isomorphism classes of families of $\mathbb{K}$-vector spaces over 1 point, we obtain exactly one point for each dimension. The set $F(\mathrm{pt})/\!\sim$ is therefore parameterized by $\mathbb{N}_0$.

4.1.2 Families of vector bundles

After vector spaces, the next step is to consider *families of vector bundles* over a base B, i.e. a surjective map $\pi : X \to B$ where the fibers X_b are vector bundles for all $b \in B$. Again for each $b \in B$ one needs to specify the base manifold M_b of X_b, the rank of bundle X_b and its projection map $X_b \to M_b$. Note that this does not necessarily require X or B to be manifolds; the only restriction is that X_b must be a manifold, $\forall\, b \in B$. A trivial example is obtained by taking $\pi : X_b \to \{b\}$, which implies that each X_b is again a vector space. Moreover $\pi_{|X_b}$ is immediately smooth (or holomorphic) since it is constant.
Another possibility is to choose $M_b = B$, $\forall\, b \in B$, but with maybe different projection maps $\pi_b : X_b \to B$. Hence if B is a manifold, we may see X as a big "collection of points" over B which is a disjoint union of vector bundles over B. In this case we can write

$$F(B) = \big\{ \text{ families over } B \big\} = \big\{ X \mid X \text{ is a disjoint union of vector bundles over } B \big\}$$

If we consider again the particular case where $B = \{\mathrm{pt}\}$ is a point, the condition $\pi^{-1}(\mathrm{pt}) = X$ implies that X is a vector bundle itself (and if it is a bundle over $\{\mathrm{pt}\}$, then it is again a vector space).

4.1.3 Families of smooth manifolds

We move on to *families of smooth manifolds* over B, which means that each fiber X_b of the map $\pi : X \to B$ must be a smooth manifold. Here one may think of fiber bundles over B. But this again just a particular case since the general definition of a family of smooth manifolds does e.g. not require that X and B are manifolds themselves, that the "bundle" is locally trivial or that the dimension of the fibers must be locally constant.
However a family of smooth manifolds over B is a fiber bundle if $\pi : X \to B$ is in addition a smooth map between smooth manifolds and $\forall\, b \in B$, there exists an open neighborhood U of b in B and a smooth manifold M such that $\pi^{-1}(U) \cong U \times M$. If we do not require the base B to be connected, it is even possible that the dimension of the fibers may vary (but locally it has to constant). So we get

$$F(B) = \big\{ \text{ families over } B \big\} = \big\{ \text{ fiber bundles over } B \big\}$$

4.1.4 Families of n points

We can even define very "simple" moduli problems, as e.g. *families of n points* over B, which requires that the fibers X_b of the surjective map $\pi : X \to B$ consist of n points for each $b \in B$. We do not always need some predefined structures on X or B; everything here is only about sets and cardinalities (but it is allowed to require additional properties). In particular, if $B = \{\mathrm{pt}\}$ is a point, then X itself should consist of n points.

But even in such a simple situation, major problems may arise. Consider e.g. the following examples :

1) Let $\mathbb{K}$ be an algebraically closed field and take the map $\pi : \mathbb{K} \to \mathbb{K} : x \mapsto x^n$. Hence for $x, b \in \mathbb{K}$,

$$\pi(x) = b \;\Leftrightarrow\; x^n = b \;\Leftrightarrow\; x^n - b = 0$$

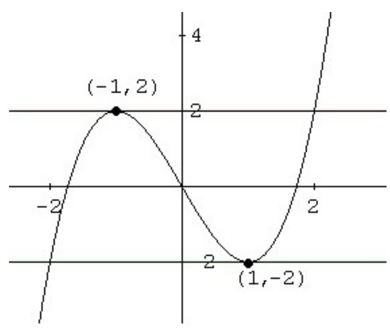
π is surjective since $\mathbb{K}$ is algebraically closed, so that every element in $\mathbb{K}$ has at least one n^{th} root, $\forall\, n \in \mathbb{N}$. The fibers are obviously

$$X_b = \left\{ x \in \mathbb{K} \;\middle|\; x^n = b \right\} = Z(X^n - b) \subset \mathbb{K}$$

i.e. $\forall\, b \in B$, X_b is the zero set of the polynomial $X^n - b$. If $b \neq 0$, then X_b consists of n distinct points. However for $b = 0$, we only obtain a single point : $X_0 = \{0\}$. But we may see 0 as a root of multiplicity n of the polynomial X^n (this can again be formalized by using schemes). Hence it still makes sense to speak about n points, now counted with multiplicities.

2) Multiplicities may also appear in more than just one point. Consider the map $\pi : \mathbb{C} \to \mathbb{C} : x \mapsto x^3 - 3x$.

Figure 4.1: the function $\mathbb{R} \to \mathbb{R} : x \mapsto x^3 - 3x$



π is surjective since the equation $x^3 - 3x = b$ has a solution, $\forall\, b \in \mathbb{C}$. The fibers are therefore

$$X_b = \left\{ x \in \mathbb{C} \;\middle|\; x^3 - 3x - b = 0 \right\} = Z(X^3 - 3X - b) \subset \mathbb{C}$$

and they consist of 3 points for all $b \in B$, except for $b \in \{-2, 2\}$ where the equation has only 2 solutions :

$$X^3 - 3X - 2 = (X-2)(X+1)^2 \qquad , \qquad X^3 - 3X + 2 = (X+2)(X-1)^2 \tag{4.1}$$

But (4.1) shows that this drop of cardinality is again due to multiplicities. So it still makes sense to speak of a family of 3 points over $\mathbb{C}$: $\forall\, b \notin \{-2, 2\}$, X_b contains 3 points of multiplicity 1, whereas X_2 and X_{-2} only contain 2 points, one of multiplicity 1 and one of multiplicity 2.

3) In $\mathbb{R}$ it may however happen that the above equation has only 1 solution if $b \notin [-2, 2]$ (see figure 4.1). This does not happen because of multiplicities, but because $\mathbb{R}$ is not algebraically closed : the equation $X^3 - 3X - b$ factorizes into a product of a linear and a quadratic polynomial, where the quadratic term has no solutions in $\mathbb{R}$. Since the cardinality of the fibers drops, the surjective map $\pi : \mathbb{R} \to \mathbb{R} : x \mapsto x^3 - 3x$ does NOT define a family of 3 points over $\mathbb{R}$.

The reason for this failure is that there is no 1-to-1 correspondence between geometric points and maximal ideals over a field which is not algebraically closed, for example no point in $\mathbb{R}$ corresponds to the maximal ideal $\langle X^2 + 1 \rangle$. We can solve the problem by working with the affine scheme $\operatorname{Spec} \mathbb{R}[X]$ instead of the affine space $\mathbb{R}$. This example shows that it is "easier" to study families over fields which are algebraically closed.

4.1.5 Families of lines

The following is another example to show how multiplicities affect the definition of a family. We want to consider *families of 2 lines* over an algebraically closed field $\mathbb{K}$. Define

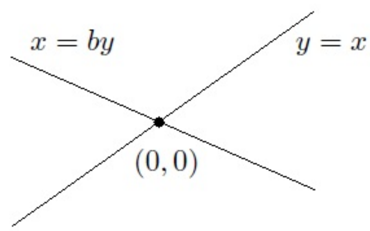
$$M := \bigcup_{b \in \mathbb{K}} \mathcal{V}\big(\langle\, b - Z \,,\, (X-Y)(X-ZY)\,\rangle\big) \subset \mathbb{K}^3 \tag{4.2}$$

If we see M as a subset of $\mathbb{K}^3$ with a coordinate system (x,y,z), then M is given by the zero set of the polynomial $(X-Y)(X-bY)$ at the level $z=b$.
The union in (4.2) is disjoint, so we have a well-defined projection map $\pi : M \to \mathbb{K} : (x,y,z) \mapsto z$. The fiber over $b \in \mathbb{K}$ is given exactly by "shape" of M at the level $z=b$, i.e.

$$M_b = \mathcal{V}\big(\langle\,(X-Y)(X-bY)\,\rangle\big) \times \{b\} \;\cong\; Z\big((X-Y)(X-bY)\big)$$

For $b \neq 1$, the fiber M_b is a union of the lines $y=x$ and $x=by$ in $\mathbb{K}^2$ intersecting at the origin $(0,0)$.

Figure 4.2: two lines intersecting at the origin



More precisely, M_b defines an algebraic subset of $\mathbb{K}^2$ which has 2 irreducible components. However for $b=1$, these two lines coincide and the fiber X_1 only has of 1 component, given by the line $y=x$. But again one should count this line twice because it is defined by the zero set of the polynomial $(X-Y)^2$.

Another reason why it is useful to consider multiplicities is given by considering the coordinate ring of the algebraic subsets of $\mathbb{K}^2$ defined by these polynomials. If $A_b := \mathcal{V}\big(\langle\,(X-Y)(X-bY)\,\rangle\big) \subset \mathbb{K}^2$ denotes the corresponding algebraic subset, its coordinate ring is

$$\mathbb{K}[A_b] = \mathbb{K}[X,Y]/\langle\,(X-Y)(X-bY)\,\rangle$$

But for $b=1$, we get $\mathbb{K}[A_1] = \mathbb{K}[X,Y]/\langle\, X-Y\,\rangle$ since $\mathcal{V}\big(\langle\,(X-Y)^2\,\rangle\big) = \mathcal{V}\big(\langle\, X-Y\,\rangle\big)$. We see that the structure of the coordinate ring "jumps" at $b=1$. This happens because the polynomial assigned to an algebraic set is not unique. It is for example possible to increase the vanishing order of the polynomial without changing the vanishing set. Hence it is necessary also to introduce multiplicities of algebraic sets; this would give a closer correspondence to their defining polynomials.

4.1.6 Isomorphic families

Let $\pi : X \to B$ and $\pi' : X' \to B$ be two families over the same base B. We want to define what it means that these families are isomorphic. Unfortunately it is not possible to give a completely general definition because this always depends on the properties of our family. A necessary condition at least is the following :
If the families are isomorphic, then the sets X and X' are objects of the same category and there exists an isomorphism $f : X \to X'$ in this category such that $\pi' \circ f = \pi$. This implies in particular that

$$\forall\, b \in B \;\;:\;\; X_b = \pi^{-1}(b) = (\pi' \circ f)^{-1}(b) = f^{-1}\big(\pi'^{-1}(b)\big) = f^{-1}(X'_b)$$

Since f is an isomorphism, this means that $f(X_b) = X'_b$, $\forall\, b \in B$, so f respects the fibers of π and π'. Moreover we see that fibers over the same basepoint $b \in B$ are in 1-to-1 correspondence.

Depending on our family, one may need to add some more properties that the morphism f has to satisfy. In particular, we want that the fibers X_b and X'_b are isomorphic (this is more than just a set bijection) for all $b \in B$. If we consider e.g. vector bundles over B as in 4.1.1, we have to add the condition of $f_{|X_b} : X_b \to X'_b$ being a linear map for all $b \in B$ in order to make it an isomorphism of vector bundles. However, in the case where B is a smooth manifold and we consider fiber bundles over B (section 4.1.3), the above definition already coincides with the notion of an isomorphism of fiber bundles. Summarizing :
An isomorphism of families is an isomorphism that is compatible with the properties of our family.

4.2 Fine moduli spaces

Roughly speaking we want a moduli space to be a space which parameterizes all our families. In order to obtain some interesting results, we however need to restrict the possibilities of defining families. For this, we always require in the following that the base B is a locally ringed space, as e.g. a scheme $(X, \mathcal{O}_X)$ or a manifold (M, C^∞_M). Thus if we speak of a family over B, we always mean a family over the underlying topological space.

4.2.1 The moduli functor

Let B be a locally ringed space and consider the set $F(B)$ of families over B. We introduce an equivalence relation $\sim$ on $F(B)$ which reflects the classes of families we want to study (e.g. isomorphism classes). Of course the relation $\sim$ can also be trivial, so that we consider all possible families over B. Then we define

$$\mathcal{F} \;:\; \texttt{LRS} \longrightarrow \texttt{Set} \;:\; B \longmapsto \mathcal{F}(B) := F(B)/\!\sim \; = \big\{\text{families over } B \text{ up to equivalence}\big\}$$

We want this assignment to be functorial.
Let $f : B \to B'$ be a morphism of locally ringed spaces. If $\pi : X \to B$ is a family over B, the composition

$$X \xrightarrow{\pi} B \xrightarrow{f} B'$$

is in general not a family over B'. However, given a family over B', it is possible to construct a family over B by using f. In other words, the functor $\mathcal{F}$ is actually contravariant. We denote this map by

$$f^* := \mathcal{F}(f) \;:\; \mathcal{F}(B') \to \mathcal{F}(B)$$

f^* is called the *pullback of f* and it is given as follows. Let $\pi' : X \to B'$ be a family over B'. We define an object f^*X and a family $p : f^*X \to B$ by the following universal property :

1) f^*X is an object of the same type as X, i.e. both belong to the same category
2) there exists a morphism $h : f^*X \to X$ in this category such that the following diagram commutes

$$\begin{tikzcd} f^*X \arrow[r, "h"] \arrow[d, "p"'] & X \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

3) for any other family $q : D \to B$ with a morphism $k : D \to X$ such that $\pi' \circ k = f \circ q$ (i.e. for any other family satisfying the same conditions), there exists a unique morphism $D \to f^*X$ such that

$$\begin{tikzcd} & & X \arrow[dr, "\pi'"] & \\ D \arrow[urr, "k"] \arrow[r, dashed, "\exists!"] \arrow[drr, "q"'] & f^*X \arrow[ur, "h"] \arrow[dr, "p"] & & B' \\ & & B \arrow[ur, "f"'] & \end{tikzcd}$$

f^*X is called the *pullback of X along f*. One can show that it exists in most of the interesting cases. As a universal object, it is also uniquely given up to isomorphism (if it exists). Here below we give some examples of how f^*X may e.g. look like.

Fiber bundles :
Let $\pi' : X \to B'$ be a fiber bundle over a smooth base manifold B' and $f : B \to B'$ a smooth map between smooth manifolds. Then one can define

$$f^*X = X \times_{B'} B := \left\{ (x,b) \in X \times B \;\middle|\; \pi'(x) = f(b) \right\}$$

together with the projection onto the second factor $p_2 : f^*X \to B$. If $\{U_i\}_i$ is a trivializing open covering of B', then $\{f^{-1}(U_i)\}_i$ is a trivializing cover of B, so that f^*X is indeed a fiber bundle over B. By definition,

$$\begin{tikzcd} f^*X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

And for $b \in B$, the fiber of f^*X over b is

$$(f^*X)_b = p_2^{-1}(b) = \left\{ (x,b) \;\middle|\; x \in X,\ \pi'(x) = f(b) \right\} \cong \left\{ x \in X \;\middle|\; \pi'(x) = f(b) \right\} = \pi'^{-1}\big(f(b)\big) = X_{f(b)}$$

Affine schemes :
Let $S = \operatorname{Spec} R$ be an affine scheme. We define a family $X \to S$ over S by saying that X is an affine scheme over S. Let Y be another affine scheme together with a morphism of schemes $f : Y \to S$. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for some commutative rings A, B, the pullback of X along f is given by $f^*X = \operatorname{Spec}(A \otimes_R B)$:

$$\begin{tikzcd} \operatorname{Spec}(A \otimes_R B) \arrow[r, "h"] \arrow[d, "p"'] & X = \operatorname{Spec} A \arrow[d] \\ Y = \operatorname{Spec} B \arrow[r, "f"] & S = \operatorname{Spec} R \end{tikzcd}$$

where the morphisms h and p are induced by the natural homomorphisms $A \to A \otimes_R B$ and $B \to A \otimes_R B$. One can show that a similar construction is also possible if S, X, Y are general schemes (for this one has to work locally and then glue the corresponding affine schemes). This is denoted by $f^*X = X \times_S Y$ and called the *fibred product* of X and Y over S. We do not go into further details of this construction. More information can e.g. be found in [Ha] and [U2].

Finally we constructed the following contravariant functor $\mathcal{F}$, called the *moduli functor* of our family :

$$\mathcal{F} \;:\; \mathtt{LRS}^{\mathrm{op}} \longrightarrow \mathtt{Set} \;:\; B \longmapsto \mathcal{F}(B)$$
$$(\, f : B \to B' \,) \longmapsto \Big(f^* : \mathcal{F}(B') \to \mathcal{F}(B) : \; (X \to B') \mapsto (f^*X \to B) \Big)$$

Since this functor describes all the families, one sometimes also says that $\mathcal{F}$ is the moduli problem.

4.2.2 Definition

The first fundamental question to answer in studying a moduli problem is to determine whether the moduli functor is representable. Recall that $\mathcal{F} : \mathtt{LRS}^{\mathrm{op}} \to \mathtt{Set}$ is representable if there exists a locally ringed space M such that we have an isomorphism of functors $\mathcal{F} \cong \operatorname{Hom}_{\mathtt{LRS}}(\,\cdot\,, M)$. If this is the case, we say that M is a *fine moduli space* for the moduli problem described by $\mathcal{F}$.

We start by giving some recalls about representability. Denote

$$h_M := \operatorname{Hom}_{\mathtt{LRS}}(\,\cdot\,, M) \;:\; \mathtt{LRS}^{\mathrm{op}} \longrightarrow \mathtt{Set} \;:\; N \longmapsto \operatorname{Hom}_{\mathtt{LRS}}(N, M)$$

h_M is called the *functor of points* of M. If $f : N \to N'$ is a morphism of locally ringed spaces, then

$$h_M(f) = \circ f \;:\; \mathrm{Hom}_{\mathtt{LRS}}(N', M) \to \mathrm{Hom}_{\mathtt{LRS}}(N, M) \;:\; g \mapsto g \circ f$$

By the Yoneda Lemma, we have the following bijection, functorially in X :

$$\mathrm{Hom}_{\mathtt{Fct}(\mathtt{LRS}^{\mathrm{op}},\mathtt{Set})}\big(h_X, \mathcal{F}\big) \xrightarrow{\sim} \mathcal{F}(X)$$

If the moduli functor $\mathcal{F}$ is representable by a locally ringed space M, there is an isomorphism of functors $h_M \cong \mathcal{F}$, which therefore corresponds to a unique element $\mathcal{U} \in \mathcal{F}(M)$. By definition, this $\mathcal{U}$ is a family over M. It is called the *universal family* over M. Now we have the following important result :

4.2.3 Theorem

Let M be a fine moduli space for the moduli problem described by $\mathcal{F}$. If $\pi : X \to B$ is any family over a locally ringed space B, then there exists a unique morphism of locally ringed spaces $f : B \to M$ such that $X = f^*\mathcal{U}$, i.e. every family over B is uniquely given by a pullback of the universal family.

$$\begin{tikzcd} X \;=\; f^*\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ B \arrow[r, "f"] & M \end{tikzcd}$$

Moreover M is unique up to canonical isomorphism and "parameterizes" all equivalence classes of objects of the given type we want to study.

Proof. taken from [HM]. Let $f : B \to B'$ be a morphism of locally ringed spaces. $\mathcal{F}$ being representable by M, we have the commutative diagram

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathtt{LRS}}(B, M) \\ \mathcal{F}(B') \arrow[u, "f^*"] \arrow[r, "\cong"] & \mathrm{Hom}_{\mathtt{LRS}}(B', M) \arrow[u, "\circ f"'] \end{tikzcd}$$

In particular, $\mathcal{F}(M)$ is the set of all families (up to equivalence) over M. Again by representability, we have $\mathcal{F}(M) \cong \mathrm{Hom}_{\mathtt{LRS}}(M, M)$ and id_M corresponds to the universal family $\mathcal{U} \in \mathcal{F}(M)$.
Now let $X \in \mathcal{F}(B)$ be an arbitrary family over the base B. Since $\mathcal{F}(B) \cong \mathrm{Hom}_{\mathtt{LRS}}(B, M)$, we know that X corresponds to a unique morphism of locally ringed spaces $f : B \to M$. Thus we have

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathtt{LRS}}(B, M) \\ \mathcal{F}(M) \arrow[u, "f^*"] \arrow[r, "\cong"] & \mathrm{Hom}_{\mathtt{LRS}}(M, M) \arrow[u, "\circ f"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, mapsto, "\simeq"] & f \\ \mathcal{U} \arrow[u, mapsto, "f^*"] \arrow[r, mapsto, "\simeq"] & \mathrm{id}_M \arrow[u, mapsto, "\circ f"'] \end{tikzcd}$$

and by commutativity of the diagram, we obtain that $f^*\mathcal{U} = X$ since both spaces correspond to f.
It follows that M is unique up to a canonical isomorphism. Indeed, if M' is another fine moduli space with another universal family $\mathcal{U}' \in \mathcal{F}(M')$, there are unique morphisms $f : M' \to M$ and $g : M \to M'$ such that

$$\mathcal{U}' = f^*\mathcal{U} \quad , \quad \mathcal{U} = g^*\mathcal{U}' \quad \Rightarrow \mathcal{U}' = f^*g^*\mathcal{U}' = (g \circ f)^*\mathcal{U}' \quad \text{and} \quad \mathcal{U} = g^*f^*\mathcal{U} = (f \circ g)^*\mathcal{U}$$

since $f^* = \mathcal{F}(f)$ and $\mathcal{F}$ is contravariant. By uniqueness, we get $g \circ f = \mathrm{id}_{M'}$ and $f \circ g = \mathrm{id}_M$, i.e. $M' \cong M$.

Finally to see that M describes all isomorphism classes of our objects, note that

$$\mathcal{F}(\mathrm{pt}) \cong \mathrm{Hom}_{\mathtt{LRS}}\big(\{\mathrm{pt}\}, M\big) \cong M$$

But if $\pi : X \to \{\mathrm{pt}\}$ is a family over a point, then $X = X_{\mathrm{pt}}$ is itself a fiber and must therefore be an object of the type that we fixed for our moduli problem. Since we only consider equivalence classes, $\mathcal{F}(\mathrm{pt}) \cong M$ is the set of all equivalence classes of objects that we want to study. $\square$

Conclusion :
If there exists a fine moduli space M for a moduli problem $\mathcal{F}$, this space M classifies all (equivalence classes of) objects we are interested in. Moreover we have a 1-to-1 correspondence between (equivalence classes of) families over a locally ringed space B and morphisms of locally ringed spaces $B \to M$. Hence we are able to translate between information about the geometry of families of the moduli problem and the geometry of the moduli space M itself.

4.2.4 Problem

Unfortunately, very few moduli functors are representable by simple locally ringed spaces as e.g. schemes or manifolds. The reason for this failure is that fine moduli spaces do not exist if the objects we want to classify have non-trivial automorphisms. The following example is taken from [Ho].

Suppose that we want to study isomorphism classes of vector bundles of a fixed rank r over smooth manifolds (these are examples of families of vector spaces). Hence the moduli functor is

$$\mathcal{F} \;:\; \texttt{Diff}^{\mathrm{op}} \to \texttt{Set} \;:\; B \mapsto \big\{ \text{isomorphism classes of vector bundles of rank } r \text{ over } B \big\}$$

This functor is not representable by a manifold. Suppose e.g. that $\mathcal{F}$ is represented by some smooth manifold M. So we know that every vector bundle $X \in \mathcal{F}(B)$ over a manifold B uniquely writes as $X = f^*\mathcal{U}$ for some smooth map $f : B \to M$.

$$\begin{tikzcd} X \;=\; f^*\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ B \arrow[r, "f"] & M \end{tikzcd}$$

We claim that a bundle $X \to B$ is trivial if and only if the corresponding map $f : B \to M$ is constant. Indeed, if $\{U_i\}_i$ is a trivializing open covering for the universal family, then $\{f^{-1}(U_i)\}_i$ is a trivializing cover for its pullback. If f is constant, then either $f^{-1}(U_i) = \emptyset$ or $f^{-1}(U_i) = B$, which means that $\{B\}$ is itself a trivializing cover, i.e. $X \to B$ is trivial. Conversely, if either $f^{-1}(U_i) = \emptyset$ or $f^{-1}(U_i) = B$ for each i, then f must be constant, otherwise $\exists\, a, b \in f(B)$ such that $a \neq b$. Since manifolds are Kolmogorov spaces, there exists an open neighborhood U of one of them which does not contain the other point, say $a \in U$ and $b \notin U$. But then $f^{-1}(U)$ is neither empty, nor equal to the whole space, which is a contradiction.

All trivial vector bundles of rank r over the same base B are isomorphic to the trivial bundle $p_1 : B \times \mathbb{K}^r \to B$, to which corresponds a unique smooth map $B \to M$ which must in addition be constant (as shown above), i.e. we obtain a unique point $x \in M$. Thus any trivial bundle of rank r over B is uniquely given as a pullback along the constant map $B \to \{x\}$.
Now let $\pi : E \to B$ be a non-trivial vector bundle of rank r over B (such bundles exist) with trivializing open covering $\{U_i\}_i$. It corresponds to a unique smooth map $f : B \to M$. Since each $E_{|U_i} \to U_i$ is trivial, the restrictions of this map must be the constant maps $f_{|U_i} : U_i \to \{x\}$. Hence f itself must be constant with image $\{x\}$, which is impossible since E has been chosen to be non-trivial. This contradiction shows that $\mathcal{F}$ cannot be representable.

The problem is that, even if the vector bundle E is trivial on each U_i, we can glue these trivial bundles in a non-trivial way because each vector space has non-trivial automorphisms. Similar constructions can be done each time we are dealing with objects and families of objects that admit non-trivial automorphisms. Hence fine moduli spaces do in general not exist and we have to find alternative methods for studying moduli problems and classifying our objects of interest.

4.3 Coarse moduli spaces

4.3.1 Motivation

As observed in the previous section, fine moduli spaces rarely exist, and if they exist, they are usually difficult to construct. Hence the necessity to develop alternative tools for studying our moduli problems. We now explain 3 possible solutions to this problem, which can be found in [Ho] as well.

Solution 1 : Rigidifying the problem
The first solution is to change the given moduli problem by requiring objects and morphisms to satisfy some additional conditions, namely in such a way that the objects with additional structure do no longer have non-trivial automorphisms. The idea is to study only a certain subcategory of the objects we started with. This is of course a possible solution, but not a very satisfying one since we do not classify our objects of interest. Nevertheless it gives a first success and one may get some ideas of how to study the problem in general.

Solution 2 : Algebraic stacks
The second solution, and probably the most satisfying one, is to look for more general classifying spaces by extending the category of locally ringed spaces. The moduli problem may not be representable by a scheme or a manifold, but maybe by a more general space, called an *algebraic stack*. We do not give a precise definition of this notion.
Algebraic stacks are generalizations of algebraic varieties, schemes and, more generally, algebraic spaces. They have been introduced by the mathematicians **Pierre Deligne** and **David Mumford** in 1969 in order to define the (fine) moduli space of compact Riemann surfaces of genus g (see also section 4.4). Similarly, vector bundles can also be represented by stacks rather than by schemes. Such a moduli space is usually obtained by constructing a larger space which parameterizes our objects and then to quotient by a group action which takes care of the automorphisms.

Solution 3 : Coarse moduli spaces
The solution we are going to analyze are the so-called coarse moduli spaces. Here we follow the ideas of [HM]. The idea is to ask for a weaker condition than representability by leaving the moduli problem unchanged. Instead of asking for an isomorphism $\mathcal{F} \cong h_M$, we only require that there should exist a natural transformation $\psi : \mathcal{F} \to h_M$ for a suitable locally ringed space M. In this case, we can still associate to every family $X \in \mathcal{F}(B)$ a morphism $f : B \to M$ by setting $f = \psi_B(X) \in \mathrm{Hom}_{\mathtt{LRS}}(B, M)$.
This requirement is however not sufficient. We need ψ to satisfy some additional conditions, otherwise the coarse moduli space may not be unique (up to isomorphism), e.g. if (M, ψ) is a solution and if we have another locally ringed space $N \ncong M$ with a morphism $\varphi : M \to N$, then $(N, \varphi \circ \psi)$ is a solution as well.

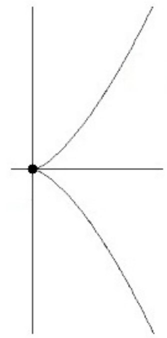
$$\mathcal{F} \xrightarrow{\psi} h_M = \mathrm{Hom}_{\mathtt{LRS}}(\,\cdot\,, M) \xrightarrow{\varphi\circ} \mathrm{Hom}_{\mathtt{LRS}}(\,\cdot\,, N) = h_N$$

In particular, we always obtain a "pathological solution" by composing with the constant map $M \to \{\mathrm{pt}\}$, which would imply that every point $\{\mathrm{pt}\}$ was a moduli space for our moduli problem.
This situation can be avoided by requiring in addition that we want the points of our moduli space to be in 1-to-1 correspondence with the equivalence classes of objects that we study. However, this does still not guarantee uniqueness up to isomorphism. Assume e.g. that we want to study the moduli problem with families $X \to B$ (where X, B are complex manifolds) being defined by the condition that the fiber X_b ($b \in B$) is a line in $\mathbb{C}^2$ passing through the origin $(0, 0)$. If $\sim$ is the trivial relation, then $\mathcal{F}(\mathrm{pt})$ is the set of all lines in $\mathbb{C}^2$ through the origin, so that the projective line $\mathbb{P}^1$ is indeed a possible choice for a moduli space. But $\mathbb{P}^1$ is also in 1-to-1 correspondence with the cuspidal curve $C \subset \mathbb{P}^2$ defined by the equation $y^2 z = x^3$ via the bijection

$$\mathbb{P}^1 \xrightarrow{\sim} C = \left\{ (x : y : z) \in \mathbb{P}^2 \;\middle|\; x^3 - y^2 z = 0 \right\} \;:\; (a : b) \longmapsto (a^2 b : a^3 : b^3)$$

Hence C would also be a possible choice for a moduli space. But C is not isomorphic to $\mathbb{P}^1$, i.e. there is no biholomorphic map between them, because $(0 : 0 : 1)$ is a singular point of C (all partial derivatives of the homogeneous polynomial $X^3 - Y^2 Z$ vanish at this point), hence C is not a submanifold of $\mathbb{P}^2$. Figure 4.3 shows how the cuspidal curve looks in $\mathbb{R}^2$ if we put $z = 1$: the origin is an angular point of the graph.

Figure 4.3: the cuspidal curve $y^2 = x^3$ in $\mathbb{R}^2$



4.3.2 Definition

The way out of non-uniqueness is to require that the natural transformation ψ should be universal among all natural transformations which satisfy the same conditions. More precisely :
Let $\mathcal{F} : \texttt{LRS}^{\mathrm{op}} \to \texttt{Set}$ be the moduli functor of a given moduli problem. We say that a locally ringed space M is a *coarse moduli space* for the functor $\mathcal{F}$ if there exists a natural transformation $\psi : \mathcal{F} \to h_M$ such that

1) the map $\psi_{\{\mathrm{pt}\}} : \mathcal{F}(\mathrm{pt}) \to \mathrm{Hom}_{\texttt{LRS}}\big(\{\mathrm{pt}\}, M\big) \cong M$ is a set bijection.
2) for any other locally ringed space N with a natural transformation $\phi : \mathcal{F} \to h_N$, there exists a unique morphism of functors $H : h_M \to h_N$ such that $\phi = H \circ \psi$.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & h_M \arrow[dl, "\exists!\, H"] \\ & h_N & \end{tikzcd}$$

Usually one says that the pair (M, ψ) is the coarse moduli space for the moduli functor. Condition 1) says that two objects are equivalent with respect to the relation $\sim$ if and only if they correspond to the same point in M. And condition 2) assures uniqueness up to canonical isomorphism of such a coarse moduli space.
Note that the Yoneda Lemma and equation (1.4) imply that the morphism of functors $H : h_M \to h_N$ is induced by a unique morphism of locally ringed spaces $h : M \to N$ with $H = h\,\circ$.

Examples :
1) A fine moduli space is also a coarse moduli space. The first condition is clear because $\psi : \mathcal{F} \xrightarrow{\sim} h_M$ is an isomorphism and for the second one, it suffices to set $H := \phi \circ \psi^{-1}$.
2) One can show (see e.g. [HM]) that $\mathbb{P}^1$ is a coarse moduli space, and even a fine moduli space, for the above moduli problem of lines in $\mathbb{C}^2$ passing through the origin. Since the cuspidal curve $C \subset \mathbb{P}^2$ is not isomorphic to the projective line, it follows that C cannot define a coarse moduli space for this problem.

4.3.3 Consequences

Unfortunately, the requirements for a coarse moduli space do not imply the existence of a universal family as in the case of fine moduli spaces and we do not have such a nice result as in theorem 4.2.3. But we can still deduce some weaker consequences from the definition.

a) Let $\pi' : X \to B'$ be a family over B' and $f : B \to B'$ a morphism of locally ringed spaces.

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, "\psi_B"] & \mathrm{Hom}_{\texttt{LRS}}(B, M) \\ \mathcal{F}(B') \arrow[u, "f^*"] \arrow[r, "\psi_{B'}"] & \mathrm{Hom}_{\texttt{LRS}}(B', M) \arrow[u, "\circ f"'] \end{tikzcd}$$

As usual, we can define the pullback family $p: f^*X \to B$. If M is a coarse moduli space with a morphism $\psi: \mathcal{F} \to h_M$, then $\psi_B(f^*X) = \psi_{B'}(X) \circ f$ because $X \in \mathcal{F}(B')$, $f^*X \in \mathcal{F}(B)$ and ψ is a natural transformation.

b) Let M be a coarse moduli space for a given moduli problem $\mathcal{F}$ with morphism $\psi: \mathcal{F} \to h_M$ and N be any locally ringed space with a natural transformation $\phi: \mathcal{F} \to h_N$. Hence for any family $X \in \mathcal{F}(B)$, we can construct the morphisms $\psi_B(X): B \to M$ and $\phi_B(X): B \to N$. By the defining property of a coarse moduli space, we know there exists a unique morphism $H: h_M \to h_N$, induced by a morphism of locally ringed spaces $h: M \to N$, such that $\phi = H \circ \psi$. Thus we have the commutative diagrams

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & h_M \arrow[dl, "H"] \\ & h_N & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(B) \arrow[rr, "\psi_B"] \arrow[dr, "\phi_B"'] & & \mathrm{Hom}_{\mathtt{LRS}}(B,M) \arrow[dl, "h\circ"] \\ & \mathrm{Hom}_{\mathtt{LRS}}(B,N) & \end{tikzcd}$$

and it follows that $\phi_B(X) = h \circ \psi_B(X)$, $\forall\, X \in \mathcal{F}(B)$.

c) Now consider the particular case where we have a locally ringed space B and a point $p \in B$ with inclusion map $f_p: \{p\} \hookrightarrow B$. If $X \in \mathcal{F}(B)$, the pullback $f_p^*X \in \mathcal{F}(\{p\})$ is a family over the point p and corresponds to a unique point $x \in M$ in our coarse moduli space because of bijectivity of $\psi_{\{p\}}$. Moreover

$$\begin{tikzcd} \mathcal{F}\big(\{p\}\big) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathtt{LRS}}\big(\{p\}, M\big) \cong M \\ \mathcal{F}(B) \arrow[u, "f_p^*"] \arrow[r, "\psi_B"] & \mathrm{Hom}_{\mathtt{LRS}}(B,M) \arrow[u, "\circ f_p"'] \end{tikzcd}$$

where the map $\mathrm{Hom}(B,M) \to \mathrm{Hom}(\{p\},M): g \mapsto g \circ f_p$ actually corresponds to the evaluation map

$$\mathrm{Hom}(B,M) \to M \;:\; g \mapsto g(p)$$

So the diagram tells us that $\psi_{\{p\}}(f_p^*X) = \psi_B(X)(p)$, $\forall\, p \in B$.

d) Combining the 2 previous results, we obtain the following formula :

$$\phi_{\{p\}}(f_p^*X) = \phi_B(X)(p) = \big(h \circ \psi_B(X)\big)(p) = h\big(\psi_B(X)(p)\big) = (h \circ \psi_{\{p\}})(f_p^*X)$$

for any family $X \to B$ and every point $p \in B$.

4.4 Moduli spaces of curves

4.4.1 Definition

We finish this last chapter by describing the very important moduli space $\mathcal{M}_g$, which as a set is defined as

$$\mathcal{M}_g := \big\{ \text{isomorphism classes of compact Riemann surfaces of genus } g \big\}$$

Here we follow the text of [Sch]. In particular, we also adopt the convention of switching between the algebraic and the analytic language. For example a compact Riemann surface (a 1-dimensional complex manifold), is also called an *algebraic curve*. We know that isomorphism classes of such algebraic curves can be classified by their algebraic *genus*, which is defined as the dimension of the vector space of global holomorphic differential forms. If M is a compact Riemann surface, then $g(M) := \dim_{\mathbb{C}} \Omega(M)$.

Now let the genus g of our curves be fixed. We want to endow the set $\mathcal{M}_g$ with a geometric structure that reflects the possibilities in which Riemann surfaces of genus g can show up, e.g. a path in $\mathcal{M}_g$ should correspond to a change of the complex structure on the Riemann surface. In particular, we are interested in the dimension of this "space", which shall indicate how many parameters we need to describe the isomorphism classes of our curves.

4.4.2 $\mathcal{M}_g$ as a moduli space

Since we consider isomorphism classes of objects of a certain type (curves of genus g), the above developed theory suggests to reformulate the problem as a moduli problem. Indeed :
A family of Riemann surfaces (curves) of genus g consists of a surjective map $\pi : X \to B$ where X, B are complex manifolds (we also call them algebraic varieties), π is a holomorphic (or, algebraic) map and the fibers $X_b = \pi^{-1}(b)$ are curves of genus g for all $b \in B$.

Let $\pi : X \to B$ be such a family of curves (of fixed genus g). Then we can define the map

$$\psi_{B,X} \ : \ B \longrightarrow \mathcal{M}_g \ : \ b \longmapsto [X_b]$$

which assigns to $b \in B$ the isomorphism class of its fiber X_b (which is a compact Riemann surface of genus g). The structure of $\mathcal{M}_g$ should be such that $\psi_{B,X} : B \to \mathcal{M}_g$ is a holomorphic map between complex manifolds. Moreover we want $\mathcal{M}_g$ to be a universal such space. The explanation of this requirement recalls some techniques we already discovered in section 4.3.3. First we fix the base manifold B and consider all possible families of curves $\pi : X \to B$ over B. This allows to define an assignment

$$\begin{aligned} \psi_B \ : \ \mathcal{F}(B) = \big\{ \text{isomorphism classes of families over } B \big\} &\longrightarrow \big\{ \text{algebraic maps } B \to \mathcal{M}_g \big\} \\ [\, \pi : X \to B \,] &\longmapsto \psi_{B,X} \end{aligned}$$

This is well-defined since the definition of an isomorphism of families (section 4.1.6) implies that $X_b \cong X'_b$, i.e. $[X_b] = [X'_b]$, $\forall\, b \in B$ whenever two families $X \to B$ and $X' \to B$ are isomorphic. Hence $\psi_{B,X} = \psi_{B,X'}$.

If $\pi' : X \to B'$ is a family of curves over a base B' and $f : B \to B'$ is a holomorphic map of complex manifolds, we can define the pullback family $p_2 : f^*X \to B$ as in 4.2.1 and have the commutative diagram

$$\begin{tikzcd} B \arrow[rr, "\psi_{B,f^*X}"] \arrow[dr, "f"'] & & \mathcal{M}_g \\ & B' \arrow[ur, "\psi_{B',X}"'] & \end{tikzcd}$$

because $\psi_{B,f^*X}(b) = \big[(f^*X)_b\big] = \big[X_{f(b)}\big] = \psi_{B',X}\big(f(b)\big)$ because $(f^*X)_b \cong X_{f(b)}$, $\forall\, b \in B$. Finally we let the base B vary and this induces a map $\psi : B \mapsto \psi_B$.
Also note that $\mathcal{M}_g = \mathcal{F}(\mathrm{pt})$ as sets because $\mathcal{F}(\mathrm{pt})$ is the set of isomorphism classes of families of curves over 1 point, i.e. the set of all isomorphism classes of compact Riemann surfaces of the given genus g. Hence

$$\psi_{\{\mathrm{pt}\}} \ : \ \mathcal{F}(\mathrm{pt}) \longrightarrow \big\{ \text{algebraic maps } \{\mathrm{pt}\} \to \mathcal{M}_g \big\} \cong \mathcal{M}_g$$

is a set bijection. Hence the first criterion (see section 4.3.2) for $\mathcal{M}_g$ to be a coarse moduli space for this moduli problem of curves is already satisfied.

Now let N be any complex manifold and consider a map ϕ with similar properties than ψ, i.e. ϕ defines a map ϕ_B for every complex manifold B that assigns an algebraic map $\phi_{B,X} : B \to N$ to every isomorphism class of families over this base B. In addition, this assignment should again satisfy $\phi_{B,f^*X} = \phi_{B',X} \circ f$ for any holomorphic map $f : B \to B'$. In particular if $B = \{\mathrm{pt}\}$ is again a single point, we obtain a map

$$\phi_{\{\mathrm{pt}\}} \ : \ \big\{ \text{isomorphism classes of families over } \{\mathrm{pt}\} \big\} \longrightarrow \big\{ \text{algebraic maps } \{\mathrm{pt}\} \to N \big\}$$

i.e. a map $\phi_{\{\mathrm{pt}\}} : \mathcal{M}_g \to N$. For $\mathcal{M}_g$ to be a universal space, we require the complex structure of $\mathcal{M}_g$ to be such that these are holomorphic maps for all possible such pairs (N, ϕ). If such a structure exists, then $\mathcal{M}_g$ will indeed be a coarse moduli space for the moduli problem.

Of course, it would be even better if $\mathcal{M}_g$ was also a fine moduli space together and there was a universal family $\pi : \mathcal{U} \to \mathcal{M}_g$ such that every family of curves $X \to B$ is uniquely given by a pullback along the holomorphic map $\psi_{B,X} : B \to \mathcal{M}_g$.

$$\begin{tikzcd} X \ = \ \psi^*_{B,X}(\mathcal{U}) \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ B \arrow[r, "\psi_{B,X}"] & M \end{tikzcd}$$

Unfortunately this is not the case (again by existence of non-trivial automorphisms) : $\mathcal{M}_g$ is never a fine moduli space and there does not exists a universal family of curves over $\mathcal{M}_g$. However one can show that

4.4.3 Results

$\mathcal{M}_g$ can be endowed with a complex structure as required above. In particular :
1) $\mathcal{M}_g$ exists as a coarse moduli space and is an irreducible quasi-projective variety with

$$\dim \mathcal{M}_g = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g-3 & \text{if } g \geq 2 \end{cases}$$

2) For every genus $g > 2$, one can find an open dense subset of $\mathcal{M}_g$ where there exists a universal family.

These results are taken from [Sch] ; the first two cases are clear since we know that the Riemann sphere $\hat{\mathbb{C}} \cong S^2$ is the only compact Riemann surface of genus 0 and that genus 1 curves are given by complex tori and classified by the j-invariant of their defining lattice, so that $\mathcal{M}_1$ is parameterized by $\mathbb{C}$.

4.4.4 Example : family of complex tori

To close the thesis, we give an example of such a family of curves of genus 1. It can also be found [Sch].

Let $U \subseteq \mathbb{C}$ be open and consider the upper half-plane $\mathcal{H} := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$. Let $\tau : U \to \mathcal{H}$ be a given holomorphic map. We want to construct a family over U such that the fiber over the point $z \in U$ is given by the torus T_z which is defined by the lattice $\mathbb{Z} \oplus \tau(z)\,\mathbb{Z}$, i.e.

$$T_z = \mathbb{C} \big/ \langle 1 \, , \, \tau(z) \rangle$$

For this, we define $X := (U \times \mathbb{C}) \,/ \sim$ where the equivalence relation $\sim$ is defined as

$$(z, w) \sim (z', w') \ \Leftrightarrow \ z' = z \ \text{ and } \ \exists\, n, m \in \mathbb{Z} \text{ such that } w' = w + n + \tau(z)\, m \tag{4.3}$$

X is a complex manifold endowed with the quotient topology with respect to the projection $U \times \mathbb{C} \to X$. Next we define the holomorphic projection map $\pi : X \to U : [z, w] \mapsto z$, whose fibers are

$$\pi^{-1}(z) = \big\{ [z, w] \bigm| w \in \mathbb{C} \big\} \cong \big\{ w \in \mathbb{C} \text{ up to a shift as in (4.3)} \big\} = \mathbb{C} \big/ \langle 1 \, , \, \tau(z) \rangle = T_z$$

Hence the fibers are exactly given by the tori we were looking for and $\pi : X \to U$ indeed defines a family of curves of genus 1 over the open subset $U \subseteq \mathbb{C}$.

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