Non-integral torsion and 1-dimensional singular sheaves in the Simpson moduli space

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I am a PhD student from Luxembourg in third year, specialized in Algebraic Geometry, Sheaf Theory and Commutative Algebra. For my research I often join work with Dr. Oleksandr Iena.

Objects of study

In my thesis I am interested in the Simpson moduli spaces M_{am+b} of semi-stable sheaves on $\mathbb{P}_2$ with linear Hilbert polynomial $am+b$ where $a,b \in \mathbb{N}$. More precisely I want to know which ones and "how many" of them are locally free on their support. Those which are not are called singular and the subvariety they define is denoted by M'. By determining the codimension of M' in M_{am+b}, we get information about the efficiency of the Simpson compactification.
For small values of a,b as e.g. $3m+1$ and $4m+1$, some work has already been done and O. Iena showed that the codimension is equal to 2. Now I try to generalize this fact to $am+1$. Writing $a=n+1$, it has been shown by M. Maican that M_{am+1} contains a dense open subset U such that all $\mathcal{E} \in U$ is given as a cokernel

$$0 \to n\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} (n-1)\mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{E} \to 0$$

where A has non-zero determinant and the remaining $n \times (n-1)$–matrix A' of linear forms satisfies a certain stability condition. Taking the coefficients of the polynomials, one can introduce coordinates and thus parametrize the sheaves by an open subset $X \subset \mathbb{A}^{3n(n+1)}$. Dividing out the group of isomorphisms of exact sequences, one obtains $X/H \cong U$, so that we can study the properties of the subset $X' \subset X$ which gives singular sheaves. Under the additional hypothesis that all minors of A' are coprime (which is a generic condition), the Hilbert-Burch Theorem allows to represent $\mathcal{E}$ as an extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_Z \to 0$$

where C is the Fitting support of $\mathcal{E}$, i.e. a curve of degree $n+1$ and Z consists of finitely many points (with multiplicities). With this we have shown:

Theorem 1. *If Z only consists of simple points, then $\mathcal{E}$ is singular if and only if Z contains a singular point of C.*

The proof uses the fact that localizations of coordinate rings at smooth points are PIDs and hence freeness and torsion-freeness of a module are equivalent.
Wondering whether the sheaves appearing in the moduli spaces are indeed torsion-free on their support, I also started a study apart to analyze how torsion of a module behaves in the non-integral case. Apparently this has not been done in detail yet. Using the primary decomposition of ideals in Noetherian rings, one can show that

Theorem 2. *A module M on a Noetherian ring R is a torsion module if and only if the codimension of $\operatorname{supp} M$ is positive in each irreducible component of $\operatorname{Spec} R$.*

On the other hand, this does not imply that the torsion subsheaf of a coherent sheaf has support in smaller dimension, the problem being that it may not always be coherent. Indeed we have the criterion

Theorem 3. *The torsion subsheaf of $\widetilde{M}$ is coherent if and only if*

$$\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P) \tag{1}$$

for all prime ideals $P \trianglelefteq R$.

It turned out that this is not always satisfied. I found an example of a torsion-free module over a non-reduced ring which locally has non-trivial torsion and whose torsion subsheaf has dense support. Fortunately

Theorem 4. *If R is a ring with no embedded primes, then* (1) *is satisfied for all M.*

This condition is satisfied for the moduli spaces as the supports are just given by spectra of the coordinate rings.

Open questions

For the rest of my thesis I want to
− generalize theorem 1 to the case where Z may contain points with multiplicities.
− find out if pure-dimensional sheaves are torsion-free in general (even if there are embedded primes).
− find conditions on M and R that are equivalent to (1).
If time permits I also plan to analyze how the open subset U in M_{am+1} is connected to the remaining closed strata and how to pass from one of them to the other one.

References

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