On the High Energy Behavior of Nonlinear Functionals of Random Eigenfunctions on $\mathbb{S}^d$

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Abstract: In this short survey we recollect some of the recent results on the high energy behavior (i.e., for diverging sequences of eigenvalues) of nonlinear functionals of Gaussian eigenfunctions on the d-dimensional sphere $\mathbb{S}^d$, $d \geq 2$. We present a quantitative Central Limit Theorem for a class of functionals whose Hermite rank is two, which includes in particular the empirical measure of excursion sets in the non-nodal case. Concerning the nodal case, we recall a CLT result for the defect on $\mathbb{S}^2$. The key tools are both, the asymptotic analysis of moments of all order for Gegenbauer polynomials, and so-called Fourth-Moment theorems.

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1 Introduction

Let us consider a compact Riemannian manifold $(\mathcal{M}, g)$ and denote by $\Delta_{\mathcal{M}}$ its Laplace-Beltrami operator. There exists a sequence of eigenfunctions $\{f_j\}_{j\in\mathbb{N}}$ and a corresponding non-decreasing sequence of eigenvalues $\{E_j\}_{j\in\mathbb{N}}$

$$\Delta_{\mathcal{M}} f_j + E_j f_j = 0 \ ,$$

such that $\{f_j\}_{j\in\mathbb{N}}$ is a complete orthonormal basis of $L^2(\mathcal{M})$, the space of square integrable measurable functions on $\mathcal{M}$. One is interested in the high energy behavior i.e., as $j \to +\infty$, of eigenfunctions f_j, related to the geometry of both *level sets* $f_j^{-1}(z)$ for $z \in \mathbb{R}$, and connected components of their complement $\mathcal{M} \setminus f_j^{-1}(z)$. One can investigate e.g. the Riemannian volume of these domains: a quantity that can be formally written as a *nonlinear functional* of f_j.

The nodal case corresponding to $z = 0$ has received great attention (for motivating details see [11]).

At least for "generic" chaotic *surfaces* $\mathcal{M}$, Berry's Random Wave Model allows to compare the eigenfunction f_j to a "typical" instance of an isotropic, monochromatic *random* wave with wavenumber $\sqrt{E_j}$ (see [11]). In view of this, much effort has been first devoted to 2-dimensional manifolds such as the torus $\mathbb{T}^2$ (see e.g. [4]) and the sphere $\mathbb{S}^2$ (see e.g. [3], [2], [8], [12]). Spherical random fields have attracted a growing interest, as they model several data sets in Astrophysics and Cosmology, e.g. on Cosmic Microwave Background ([5]).

More recently random eigenfunctions on higher dimensional manifolds have been investigated: e.g. on the hyperspheres ([6]).

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1.1 Random eigenfunctions on $\mathbb{S}^d$

Let us fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denote by $\mathbb{E}$ the corresponding expectation and by $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ the unit d-dimensional sphere ($d \ge 2$); μ_d stands for the Lebeasgue measure of the hyperspherical surface. By real random field on $\mathbb{S}^d$ we mean a real-valued measurable map defined on $(\Omega \times \mathbb{S}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^d))$, where $\mathcal{B}(\mathbb{S}^d)$ denotes the Borel σ-field on $\mathbb{S}^d$. Recall that the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^d}$ on $\mathbb{S}^d$ are integers of the form $-\ell(\ell+d-1) =: -E_\ell$, $\ell \in \mathbb{N}$.

The ℓ-th random eigenfunction T_ℓ on $\mathbb{S}^d$ is the (unique) centered, isotropic real Gaussian field on $\mathbb{S}^d$ with covariance function

$$K_\ell(x,y) := G_{\ell;d}(\cos\tau(x,y)) \quad x,y \in \mathbb{S}^d \ ,$$

where $G_{\ell;d}$ stands for the ℓ-th Gegenbauer polynomial normalized in such a way that $G_{\ell;d}(1) = 1$ and τ is the usual geodesic distance. More precisely, setting $\alpha_{\ell;d} := \binom{\ell+\frac{d}{2}-1}{\ell}$, we have $G_{\ell;d} = \alpha_{\ell;d}^{-1} P_\ell^{(\frac{d}{2}-1,\frac{d}{2}-1)}$, where $P_\ell^{(\alpha,\beta)}$ denote standard Jacobi polynomials. By isotropy (see e.g. [5]) we mean that for every $g \in SO(d+1)$, the random fields $T_\ell = (T_\ell(x))_{x\in\mathbb{S}^d}$ and $T_\ell^g := (T_\ell(gx))_{x\in\mathbb{S}^d}$ have the same law in the sense of finite-dimensional distributions. Here $SO(d+1)$ denotes the group of real $(d+1)\times(d+1)$-matrices A such that $AA' = I$ the identity matrix and $\det A = 1$.

Random eigenfunctions naturally arise as they are the Fourier components of those isotropic random fields on $\mathbb{S}^d$ whose sample paths belong to $L^2(\mathbb{S}^d)$.

Let us consider now functionals of T_ℓ of the form

$$S_\ell(M) := \int_{\mathbb{S}^d} M(T_\ell(x))\, dx \ , \tag{1}$$

where $M : \mathbb{R} \to \mathbb{R}$ is some measurable function such that $\mathbb{E}[M(Z)^2] < +\infty$, $Z \sim \mathcal{N}(0,1)$ a standard Gaussian r.v. In particular, if $M(\cdot) = 1(\cdot > z)$ is the indicator function of the interval $(z, +\infty)$ for $z \in \mathbb{R}$, then (1) coincides with the empirical measure $S_\ell(z)$ of the z-*excursion set* $A_\ell(z) := \{x \in \mathbb{S}^d : T_\ell(x) > z\}$.

1.2 Aim of the survey

We first present a quantitative CLT as $\ell \to +\infty$ for nonlinear functionals $S_\ell(M)$ in (1) on $\mathbb{S}^d$, $d \ge 2$, under the assumption that $\mathbb{E}[M(Z)H_2(Z)] \ne 0$, where $H_2(t) := t^2 - 1$ is the second Hermite polynomial.

For instance the above condition is fullfilled by the empirical measure $S_\ell(z)$ of z-excursion sets for $z \ne 0$. For the nodal case which corresponds to the defect

$$D_\ell := \int_{\mathbb{S}^d} 1(T_\ell(x) > 0)\, dx - \int_{\mathbb{S}^d} 1(T_\ell(x) < 0)\, dx \ , \tag{2}$$

we present a CLT for $d = 2$. Quantitative CLTs for D_ℓ on $\mathbb{S}^d$, $d \ge 2$, will be treated in a forthcoming paper.

We refer to [7], [8] and [6] for the spherical case $d = 2$ and to [6] for all higher dimensions. The mentioned results rely on both, the asymptotic analysis of moments of all order for Gegenbauer polynomials, and Fourth-Moment theorems (see [9], [1]).

2 High energy behavior via chaos expansions

For a function $M : \mathbb{R} \to \mathbb{R}$ as in (1), the r.v. $S_\ell(M)$ admits the chaotic expansion

$$S_\ell(M) = \sum_{q=0}^{+\infty} \frac{J_q(M)}{q!} \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx \qquad (3)$$

(see [9]) in $L^2(\mathbb{P})$ (the space of finite-variance r.v.'s), where H_q is the q-th Hermite polynomial (see e.g. [10]) and $J_q(M) := \mathbb{E}[M(Z)H_q(Z)]$, $Z \sim \mathcal{N}(0,1)$. We have $\mathbb{E}[S_\ell(M)] = J_0(M)\mu_d$; w.l.o.g. $J_0(M) = 0$.

The main idea is first to investigate the asymptotic behavior of each chaotic projection, i.e. of each (centered) r.v. of the form

$$h_{\ell;q,d} := \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx \qquad (4)$$

and then deduce the asymptotic behavior of the whole series (3). Note that $h_{\ell;1,d} = 0$, as T_ℓ has zero mean on $\mathbb{S}^d$. By the symmetry property of Gegenbauer polynomials ([10]), from now on we can restrict ourselves to even multiples ℓ, for which some straightforward computations yield

$$\mathrm{Var}[h_{\ell;q,d}] = 2q!\mu_d\mu_{d-1} \int_0^{\pi/2} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta\ . \qquad (5)$$

2.1 Asymptotics for moments of Gegenbauer polynomials

The proof of the following is in [7], [8] for $d = 2$ and in [6] for $d \geq 3$.

Proposition 1. *As $\ell \to \infty$, for $d = 2$ and $q = 3$ or $q \geq 5$ and for $d, q \geq 3$,*

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta = \frac{c_{q;d}}{\ell^d}(1 + o(1))\ . \qquad (6)$$

The constants $c_{q;d}$ are given by the formula

$$c_{q;d} := \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^{+\infty} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q\left(\frac{d}{2}-1\right)+d-1}\, d\psi\ , \qquad (7)$$

where $J_{\frac{d}{2}-1}$ is the Bessel function ([10]) of order $\frac{d}{2}-1$. The r.h.s. integral in (7) is absolutely convergent for any pair $(d,q) \neq (2,3), (3,3)$ and conditionally convergent for $d = 2, q = 3$ and $d = q = 3$. Moreover for $c_{4;2} := \frac{3}{2\pi^2}$

$$\int_0^{\frac{\pi}{2}} G_{\ell;2}(\cos\vartheta)^4 \sin\vartheta\, d\vartheta = c_{4;2}\frac{\log\ell}{\ell^2}(1 + o(1))\ . \qquad (8)$$

From [10], as $\ell \to +\infty$,

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^2 (\sin\vartheta)^{d-1}\, d\vartheta = 4\mu_d\mu_{d-1}\frac{c_{2;d}}{\ell^{d-1}}(1 + o(1))\ ,\ c_{2;d} := \frac{(d-1)!\mu_d}{4\mu_{d-1}}\ . \qquad (9)$$

Clearly for any $d, q \geq 2$, $c_{q;d} \geq 0$ and $c_{q;d} > 0$ for all even q. Moreover we can give explicit expressions for $c_{3;2}, c_{4;2}$ and $c_{2;d}$ for any $d \geq 2$. We conjecture that the above strict inequality holds for every pair (d,q), and leave this issue as an open question for future research.

2.2 Fourth-Moment Theorems for chaotic projections

Let us recall the usual Kolmogorov d_K, total variation d_{TV} and Wasserstein d_W distances between r.v.'s X, Y: for $\mathcal{D} \in \{K, TV, W\}$

$$d_{\mathcal{D}}(X,Y) := \sup_{h\in H_{\mathcal{D}}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]| \ ,$$

where $H_K = \{1(\cdot \le z), z \in \mathbb{R}\}$, $H_{TV} = \{1_A(\cdot), A \in \mathcal{B}(\mathbb{R})\}$ and H_W is the set of Lipschitz functions with Lipschitz constant one.

The r.v. $h_{\ell;q,d}$ in (4) belongs to the so-called qth Wiener chaos. The Fourth-Moment Theorem ([9]) states that if $Z \sim \mathcal{N}(0,1)$, for $\mathcal{D} \in \{K, TV, W\}$ we have

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}}, Z\right) \le C_{\mathcal{D}}(q)\sqrt{\frac{\mathrm{cum}_4(h_{\ell;q,d})}{\mathrm{Var}[h_{\ell;q,d}]^2}} \ , \tag{10}$$

where $C_{\mathcal{D}}(q) > 0$ is some explicit constant and $\mathrm{cum}_4(h_{\ell;q,d})$ is the fourth cumulant of the r.v. $h_{\ell;q,d}$. An application of (10) together with upper bounds for cumulants leads to the following result (see [6]).

Theorem 1. *For all $d, q \ge 2$ and $\mathcal{D} \in \{K, TV, W\}$ we have, as $\ell \to +\infty$,*

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}}, Z\right) = O\left(\ell^{-\delta(q;d)}(\log \ell)^{-\eta(q;d)}\right) \ , \tag{11}$$

where $\delta(q;d) \in \mathbb{Q}$, $\eta(q;d) \in \{-1, 0, 1\}$ and $\eta(q;d) = 0$ but for $d = 2$ and $q = 4, 5, 6$.

The exponents $\delta(q;d)$ and $\eta(q;d)$ can be given explicitly (see [6]), turning out in particular that if $(d, q) \ne (3,3), (3,4), (4,3), (5,3)$ and $c_{q;d} > 0$,

$$\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}} \stackrel{\mathcal{L}}{\to} Z \ , \qquad \text{as } \ell \to +\infty \ , \tag{12}$$

where from now on, $\to^{\mathcal{L}}$ denotes convergence in distribution and $Z \sim \mathcal{N}(0,1)$.

Remark 1. For $d = 2$, the CLT (12) was already proved in [8]; nevertheless Theorem 1 improves the existing bounds on the rate of convergence to the asymptotic Gaussian distribution.

2.3 Quantitative CLTs for Hermite rank 2 functionals

Proposition 1 states that whenever M is such that $J_2(M) \ne 0$ in (3), i.e. the functional $S_\ell(M)$ in (1) has Hermite rank two, then

$$\lim_{\ell\to+\infty} \frac{\mathrm{Var}[S_\ell(M)]}{\mathrm{Var}\left[\frac{J_2(M)}{2} h_{\ell;2,d}\right]} = 1 \ . \tag{13}$$

Hence, loosely speaking, $S_\ell(M)$ and its 2nd chaotic projection $\frac{J_2(M)}{2} h_{\ell;2,d}$ have the same high energy behaviour. The main result presented in this survey is the following, whose proof is given in [6].

Theorem 2. *Let $M : \mathbb{R} \to \mathbb{R}$ in (1) be s.t. $\mathbb{E}[M(Z)H_2(Z)] =: J_2(M) \neq 0$, then*

$$d_W\left(\frac{S_\ell(M)}{\sqrt{\operatorname{Var}[S_\ell(M)]}}, Z\right) = O\left(\ell^{-\frac{1}{2}}\right) , \qquad \textit{as } \ell \to \infty , \tag{14}$$

where $Z \sim \mathcal{N}(0,1)$. In particular, as $\ell \to +\infty$,

$$\frac{S_\ell(M)}{\sqrt{\operatorname{Var}[S_\ell(M)]}} \xrightarrow{\mathcal{L}} Z . \tag{15}$$

3 Geometry of high energy excursion sets

Consider the empirical measure $S_\ell(z)$ of the z-excursion set $A_\ell(z)$ for $z \in \mathbb{R}$, as in §1.1. It is easy to check that in (3) $\mathbb{E}[S_\ell(z)] = \mu_d(1 - \Phi(z))$ and for $q \geq 1$, $J_q(1(\cdot > z)) = H_{q-1}(z)\phi(z)$, where Φ and ϕ denote respectively the cdf and the pdf of the standard Gaussian law. Since $J_2(1(\cdot > z) = z\phi(z)$, Theorem 2 immediately entails that, as $\ell \to \infty$, if $z \neq 0$

$$d_W\left(\frac{S_\ell(z) - \mu_d(1 - \Phi(z))}{\sqrt{\operatorname{Var}[S_\ell(z)]}}, Z\right) = O\left(\ell^{-\frac{1}{2}}\right) .$$

The nodal case $z = 0$ requires different arguments: in the chaos expansion for the defect (2) D_ℓ only odd chaoses occur but each of them "contributes" by Proposition 1. Asymptotics for the defect variance on $\mathbb{S}^2$ have been given in [7]:

$$\operatorname{Var}[D_\ell] = \frac{C}{\ell^2}(1 + o(1)) , \quad \text{as } \ell \to +\infty ,$$

for $C > \frac{32}{\sqrt{27}}$. Moreover in [8] a CLT has been proved: as $\ell \to +\infty$,

$$\frac{D_\ell}{\sqrt{\operatorname{Var}[D_\ell]}} \xrightarrow{\mathcal{L}} Z ,$$

where $Z \sim \mathcal{N}(0,1)$. In a forthcoming paper, we will provide quantitative CLTs for the defect on $\mathbb{S}^d$, $d \geq 2$.

Remark 2. The volume of excursion sets is just one instance of Lipschtz-Killing curvatures. In the 2-dimensional case, these are completed by the Euler-Poincaré characteristic ([3]) and the length of level curves ([4],[12] for the nodal variances). In forthcoming papers jointly with D. Marinucci, G. Peccati and I. Wigman, we will investigate the asympotic distribution of the latter on both the sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2$. For future research, we would like to characterize the high energy behavior of all Lipschitz-Killing curvatures on every "nice" compact manifold.

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