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THE SUBGROUP MEASURING THE DEFECT OF THE ABELIANIZATION OF $\mathrm{SL}_2(\mathbb{Z}[i])$

ALEXANDER D. RAHM

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Abstract

There is a natural inclusion of $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\mathbb{Z}[i])$, but it does not induce an injection of commutator factor groups (Abelianizations).

In order to see where and how the 3-torsion of the Abelianization of $\mathrm{SL}_2(\mathbb{Z})$ disappears, we study a double cover of the amalgamated product decomposition $\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) *_{(\mathbb{Z}/2\mathbb{Z})} (\mathbb{Z}/6\mathbb{Z})$ inside $\mathrm{SL}_2(\mathbb{Z}[i])$; and then compute the homology of the covering amalgam.

1. Introduction

Recall the classical decomposition of $\mathrm{SL}_2(\mathbb{Z})$ as an amalgamated product $C_4 *_{C_2} C_6$, where C_n denotes the cyclic group with n elements, along the modular tree [7]. When we take $\mathrm{SL}_2(\mathbb{Z})$ modulo its center $\{\pm 1\}$, which is at the same time the kernel of its action on the modular tree, then we obtain the decomposition $\mathrm{PSL}_2(\mathbb{Z}) \cong C_2 *_{\{1\}} C_3$. Here, we easily see how the subgroup $\{\pm 1\}$ is divided out in each factor of the amalgamated product. In this article, we want to go in the other direction : Instead of collapsing the center of the action, we want to extend it. Then the extended group Γ is going to admit an amalgamated decomposition, the quotient of which is the decomposition of $\mathrm{SL}_2(\mathbb{Z})$.

To see this, we are going to reflect the modular tree on the real axis, and then identify the reflected copy with the original by passing to the quotient modulo complex conjugation. Then our extended group Γ acts also by Möbius transformations, but, as we allow the matrices in Γ to have complex entries, they can send the upper half-plane to the lower half-plane. We therefore need to choose Γ small enough and suitable such that in the quotient modulo complex conjugation, it preserves the image of the modular tree.

$$\text{Let } \Gamma := \mathrm{SL}_2(\mathbb{Z}) \cup \{iB \mid B \in \mathrm{GL}_2(\mathbb{Z}) \text{ and } \det B = -1\},$$

where $i := \sqrt{-1}$. Note that Γ is a group and contains $\mathrm{SL}_2(\mathbb{Z})$ as a subgroup of index 2. Our main result is the following amalgamated decomposition.

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Theorem 1. *There is an isomorphism $\Gamma \cong Q_8 *_{C_4} (C_3 \rtimes C_4)$, where Q_8 is the quaternion group of order 8 and C_m is the cyclic group of order m.*

Conclusion. Computing the homology of Γ, which we do in a corollary to this Theorem, we can see precisely where the element of order 3 in the Abelianization of $\mathrm{SL}_2(\mathbb{Z})$, given by the conjugacy class of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, vanishes when mapping to the Abelianization of of the Picard modular group $\mathrm{SL}_2(\mathbb{Z}[i])$; the latter Abelianization containing no 3–torsion, whilst it contains the 2-torsion of $H_1(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z})$ [6], [5]. Namely, this vanishing occurs when extending the Abelian subgroup C_6 in $\mathrm{SL}_2(\mathbb{Z})$ to the non-Abelian subgroup $(C_3 \rtimes C_4)$ in $\mathrm{SL}_2(\mathbb{Z}[i])$.

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2. Decomposing the extended group

For the proof of Theorem 1, we define an action of Γ on the quotient of the union of upper half-plane and lower half-plane modulo complex conjugation, by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \{z, \bar{z}\} := \left\{ \frac{az+b}{cz+d}, \overline{\left(\frac{az+b}{cz+d}\right)} \right\}.$$

The orbit of the points $P := \{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\}$ and $Q := \{\pm i\}$ under the action of the subgroup $\mathrm{SL}_2(\mathbb{Z})$ can be made in an obvious fashion the vertex set of a tree X which is canonically isomorphic to the well-known tree that can be found in [7]. We will now proceed in six steps and show that

1. We have $\Gamma \cdot X = X$, so Γ acts on the tree X.
2. Let T be the oriented segment from P to Q. Then T is a fundamental domain for this action.
3. The isotropy group Γ_P is generated by
$$\rho := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } \tau := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$
4. The isotropy group Γ_Q is generated by $\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and τ.
5. The isotropy group Γ_T is the cyclic group generated by τ, so we have canonical inclusions into Γ_P and Γ_Q.
6. The matrix τ generates C_4; the matrices τ and σ generate Q_8;
τ and ρ generate $(C_3 \rtimes C_4)$.

Having arrived at this stage, applying the following classical theorem of Bass–Serre theory implies Theorem 1.

Theorem 2 ([7],Theorem 4.6)**.** *Let G be a group acting on a graph X, and let T be an oriented segment of X, with vertices P and Q. Suppose that T is a fundamental* *domain of X mod G. Let G_P, G_Q and G_T be the isotropy groups of their indexing objects. Then the following properties are equivalent:*

- *X is a tree.*
- *The homomorphism $G_P *_{G_T} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is an isomorphism.*

Proof of Theorem 1.
Step 1. We need to consider the imaginary elements $\gamma \in \Gamma$. They are of the form $\gamma = \begin{pmatrix} -ia & -ib \\ ic & id \end{pmatrix}$ with a, b, c, d in $\mathbb{Z}$ and $ad - bc = -1$. The image under γ of a point $\{z, \bar{z}\}$ is

$$\begin{pmatrix} -ia & -ib \\ ic & id \end{pmatrix} \cdot \{z, \bar{z}\} = \left\{ -\left(\frac{az+b}{cz+d}\right), -\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right) \right\}.$$

So, the action of Γ is the action of $\mathrm{SL}_2(\mathbb{Z})$ twisted by the reflecting on the imaginary axis (which is not carried out by $\mathrm{SL}_2(\mathbb{Z})$, as the latter preserves orientation); and the modular tree is preserved under this action because it admits this reflection symmetry axis.

Step 2. Follows from the analogous property of the original modular tree.

Step 3. Consider an imaginary element $\gamma = \left(\begin{smallmatrix} -ia & -ib \\ ic & id \end{smallmatrix}\right)$ with a, b, c, d in $\mathbb{Z}$, $ad - bc = -1$ and $\gamma \cdot P$ containing $-\frac{1}{2} + \frac{\sqrt{3}}{2} i \in P$. Then we obtain the fixed point equation

$$-a\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right) - b = \left(c\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right) + d\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right),$$

which we decompose into real and imaginary part. Then we can solve to $b = (\frac{1}{2} \mp \frac{1}{2})a \pm c$ and $d = \mp a + (\frac{1}{2} \pm \frac{1}{2})c$. Using $ad - bc = -1$, we deduce the equation $\pm 1 = a^2 + c^2 - ac$, which admits six integer solutions. We obtain exactly one matrix of determinant 1 for each of these solutions. Namely,

$$\pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix} \quad \pm \begin{pmatrix} -i & 0 \\ i & i \end{pmatrix}.$$

Together with the isotropy matrices from $\mathrm{SL}_2(\mathbb{Z})$,

$$\pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain the stabilizer of P in Γ as a set. Now step 3 is completed by checking that the group generated by τ and ρ under matrix multiplication consists exactly of these twelve matrices.

Step 4. We proceed analogously to step 3 and obtain the stabilizing matrices

$$\pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Checking that the group generated by matrix multiplication of τ and σ consists exactly of these eight matrices completes this step.

Step 5. As we have defined T to be the oriented segment with vertices P and Q, its isotropy group consists of the elements of Γ fixing both P and Q (which gives us the canonical inclusions into Γ_P and Γ_Q):

$$\Gamma_T = \Gamma_P \cap \Gamma_Q = \left\{ \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

It is easy to check that this group is generated by the matrix τ.

Step 6. This is straightforward by establishing the respective multiplication tables.

Applying Theorem 2 now completes the proof of Theorem 1. □

Corollary to Theorem 1. *The homology of* Γ *with trivial* $\mathbb{Z}$*-coefficients is*

$$\mathrm{H}_q(\Gamma) \cong \mathrm{H}_q(Q_8 *_{C_4} (C_3 \rtimes C_4)) \cong \begin{cases} \mathbb{Z}/8 \oplus \mathbb{Z}/3, & q \equiv 3 \mod 4, \\ 0, & q \equiv 0 \mod 2, \\ (\mathbb{Z}/2)^2, & q \equiv 1 \mod 4, \\ \mathbb{Z}, & q = 0. \end{cases}$$

Proof. From Theorem 1, we know that we can use the equivariant spectral sequence associated to the action of the amalgamated product $Q_8 *_{C_4} (C_3 \rtimes C_4)$ on its tree. This spectral sequence is concentrated in the first two columns ($p = 0$ and $p = 1$); and its even rows (q non-zero and even) vanish. To compute the differentials of degree 1,

$$\mathbb{Z}/8 \oplus (\mathbb{Z}/4 \oplus \mathbb{Z}/3) \leftarrow \mathbb{Z}/4$$

in the rows $q \equiv 3 \mod 4$ and

$$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4 \leftarrow \mathbb{Z}/4$$

in the rows $q \equiv 1 \mod 4$, we use the following resolutions.

- A resolution for the eight-elements quaternion group Q_8 that we construct from the action of Q_8 on the unit quaternions $\mathbb{H}^*$ by left multiplication in the quaternion field $\mathbb{H}$. The unit quaternions $\mathbb{H}^*$ are homeomorphic to the 3-sphere, and we put a Q_8-equivariant cell structure on them.
  Alternatively, we can use the resolutions for Q_8 provided in the literature by Cartan and Eilenberg [3], Brown [2], Adem and Milgram [1].
- The resolution for $C_3 \rtimes C_4$ given by C.T.C. Wall [8] (we are in the case $r = 3$, $s = 4$, $t = 2$. Beware of the typographical misprint on page 254 of [8], which has turned the symbol $\sum\limits_{j=0}^{s-1}$ into $\sum\limits_{j=1}^{s-1}$).

Wall also examines the Lyndon–Hochschild–Serre spectral sequence converging to

$$\mathrm{H}_n(C_3 \rtimes C_4) \cong \bigoplus_{p+q=n} \mathrm{H}_p(C_4; \mathrm{H}_q(C_3)),$$

and concentrated in the edges where either p or q is zero. This gives a natural inclusion

$$\mathrm{H}_q(C_3 \rtimes C_4) \hookleftarrow \mathrm{H}_q(C_4)$$

in all degrees q, so we obtain the claimed result. $\square$

One can check this corollary on the computer with Homological Algebra Programming (HAP) [4].

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Alexander D. Rahm `Alexander.Rahm@nuigalway.ie`

National University of Ireland at Galway, Department of Mathematics
`http://www.maths.nuigalway.ie/~rahm/`