An extension result of CR functions by a general Schwarz Reflection Principle

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We study a general Schwarz Reflection Principle in one complex variable and obtain a holomorphic extension result of continuous CR functions defined on a real analytic, generic, CR submanifold in $\mathbb{C}^N$.

1 A general Schwarz Reflection Principle

The classical Schwarz Reflection Principle can be stated as follows:

Theorem 1.1. *Suppose that Ω is a connected domain, symmetric with respect to the real axis, and that $L = \Omega \cap \mathbb{R}$ is an interval. Let $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$. Suppose that $f \in A(\Omega^+)$, a function holomorphic on Ω^+ and that $Im(f)$ has a continuous extension to $\Omega^+ \cup L$ that vanishes on L. Then there is a $F \in A(\Omega)$ such that $F = f$ in Ω^+ and $F(z) = \overline{f(\overline{z})}$ in $\Omega - \Omega^+$.*

Our generalized version replaces the vanishing of $Im(f)$ on L by the real analyticity of $Im(f)$ on L which is a necessary condition for the holomorphic extension. We simplify our version as follows:

Theorem 1.2. *Let $D = D(0,1)$ be the open disc centered at the origin with radius 1, $D^+ = \{z \in D\colon Im(z) > 0\}$, $L = D \cap \mathbb{R}$. Suppose that $f \in A(D^+)$ and $Im(f)$ has a continuous extension to $D^+ \cup L$ such that $v(x,0) = Im(f)|_L$ is real analytic at 0 with radius of convergence r. Denote $D_r = D(0,r)$, then there exists $F \in A(D^+ \cup D_r)$ such that $F = f$ in D^+ and $F(z) = \overline{f(\overline{z})} + 2iv(z,0)$ $\forall z \in D_r - D_r^+$ where $v(z,0)$ denotes a holomorphic function on D_r.*

Proof: Since $v(x,0) = Im(f)|_L$ is real analytic at 0 with radius of convergence r, we can write $v(x,0) = \sum a_n x^n$ $\forall |x| < r$. Thus we get a holomorphic function $v(z,0) = \sum a_n z^n$ on D_r by the complexification of the variable x. Now $Im(f(z) - iv(z,0))|_{(-r,r)} \equiv 0$, by the Schwarz Reflection Principle, the function

$$g(z) := \begin{cases} f(z) - iv(z,0) & \text{for} \quad z \in D_r^+ \\ \overline{f(\overline{z}) - iv(\overline{z},0)} & \text{for} \quad z \in D_r - D_r^+ \end{cases}$$

is holomorphic on D_r. Thus the function

$$F(z) := g(z) + iv(z,0) = \begin{cases} f(z) & \text{for} \quad z \in D^+ \\ \overline{f(\overline{z}) - iv(\overline{z},0)} + iv(z,0) & \text{for} \quad z \in D_r - D_r^+ \end{cases}$$

is holomorphic on $D^+ \cup D_r$ and is the desire extension. Since a_n are real, $F(z) = \overline{f(\overline{z})} + 2iv(z, 0)$ $\forall z \in D_r - D_r^+$. ■

Theorem 1.2 leads to a reflection principle of harmonic functions if we impose some real analyticity condition on the boundary.

Corollary 1.1. *Let D, D^+, L as above. Suppose that $v(x, y)$ is harmonic in D^+ and has a continuous extension to $D^+ \cup L$ such that $v(x, 0)$ is real analytic at 0 with radius of convergence r. Denote $D_r = D(0, r)$, then there exists V harmonic in $D^+ \cup D_r$ such that $V = v$ in D^+ and $V(x, y) = 2Re(v(z, 0)) - v(x, -y)$ $\forall (x, y) \in D_r - D_r^+$ where $v(z, 0)$ denotes a holomorphic function on D_r.*

Proof: Since we can find $u(x, y)$ in D^+ such that $u + iv \in A(D^+)$, we then apply Theorem 1.2 to get the extension $V(x, y)$ in D_r, $V(x, y) = Im(\overline{f(\overline{z})} + 2iv(z, 0)) = 2Re(v(z, 0)) - v(x, -y)$. ■

Proposition 1.3. *Every regular real analytic curve S in $\mathbb{C}$ is locally biholomorphic equivalent to the real line, i.e. $\forall p_0 \in S$, $\exists$ a local biholomorphic map ϕ from a neighbohoood U of 0 in $\mathbb{C}$ that takes 0 to p_0 and $U \cap \mathbb{R}$ to $\phi(U) \cap S$.*

Proof: WLOG, assume $p_0 = 0$, since S is regular real analytic, then S can be locally parametrized by $(\sum_{n=1}^{\infty} a_n t^n, \sum_{n=1}^{\infty} b_n t^n)$, $t \in (-r, r)$ with $(a_1, b_1) \neq (0, 0)$. Then the local biholomorphic mapping $\phi(z) = \sum_{n=1}^{\infty}(a_n + ib_n)z^n$ defined on $U = D(0, \frac{r}{\sqrt{2}})$ maps $U \cap \mathbb{R}$ to $\phi(U) \cap S$ and takes 0 to 0. ■

By Proposition 1.3, Theorem 1.2 can be generalized as follows: if a function f holomorphic on one side of S, $Im(f)$ has a contiuous extension to a neighborhood U of p_0 in S such that the restriction to U is real analytic, then f can be holomorphically extended to a neighborhood of p_0 in $\mathbb{C}$. A generalized Corollary 1.1 will follow easily.

2 Holomorphic extension of CR functions on real analytic, generic CR submanifold in $\mathbb{C}^N$

We first introduce the necessary notations and definitions needed in the sequel. We mainly follow [BER]. A smooth (real analytic) real submanifold of $\mathbb{C}^N$ of codimension d is a subset M of $\mathbb{C}^N$ such that $\forall p_0 \in M$, there is a neighborhood U of p_0 and a smooth (real analytic) real vector-valued function $\rho = (\rho_1, ..., \rho_d)$ defined in U such that

$$M \cap U = \{Z \in U : \rho(Z, \overline{Z}) = 0\},$$

with differentials $d\rho_1, ..., d\rho_d$ linearly independent in U.

Definition 2.1. A real submanifold $M \subset \mathbb{C}^N$ is CR if the complex rank of the complex differentials $\partial\rho_1, ..., \partial\rho_d$ is constant for $p \in M$. It is generic if $\partial\rho_1, ..., \partial\rho_d$ are $\mathbb{C}$ linearly independent for $p \in M$.

It should be noted that the above definitions are independent of choice of local defining functions ρ. Let $M \subset \mathbb{C}^N$ be a germ of real analytic, generic, CR submanifold of codimension d at p_0, write $N = n + d$, $n \geq 1$. After a local holomorphic change of coordinates, we can assume that there exists Ω, a sufficiently small open neighborhood of 0 in $\mathbb{C}^{n+d}$, such that M is given in Ω by

$$Im(w) = \phi(z, \overline{z}, Re(w)),$$

with $z \in \mathbb{C}^n$, $w \in \mathbb{C}^d$, ϕ a real-valued analytic function (power series) of $z, \overline{z}, Re(w)$ such that $\phi(0,0,0) = 0$ and $d\phi(0,0,0) = 0$. Such a choice of coordinates is called regular coordinates.

Now suppose M is a real analytic, generic submanifold, locally parametrized by the regular coordinates near $0 \in M$. Since the function $\phi(z, \overline{z}, s)$ is real analytic near $0 \in \mathbb{R}^{2n+d}$, the variable s can be complexified. Thus, we extend the parametrization $\Psi(z, \overline{z}, s) = (z, s + i\phi(z, \overline{z}, s) \in \mathbb{C}^{n+d}$ to a local real analytic diffeomorphism $\widetilde{\Psi}(z, \overline{z}, s + it) = (z, s + it + i\phi(z, \overline{z}, s + it) \in \mathbb{C}^{n+d}$, defined in a neighborhood of $0 \in \mathbb{R}^{2n+2d}$.

By the above notation, we have the following holomorphic extension result, see [BER].

Proposition 2.1. *Let M be a real analytic generic submanifold of $\mathbb{C}^{n+d}$ of codimension d in regular coordinates near $0 \in M$. If h be a CR function on M, then h extends holomorphically to a full neighborhood of 0 if and only if there exists $\epsilon > 0$ such that for every $z \in \mathbb{C}^n$, $|z| < \epsilon$, the function $s \longmapsto h \circ \Psi(z, \overline{z}, s)$ extends holomorphically to the open set $\{s + it \in \mathbb{C}^d,\ |s| < \epsilon,\ |t| < \epsilon\}$ in such a way that the extension $H := h \circ \Psi(z, \overline{z}, s + it)$ is a bounded, measurable function of all its variables.*

The following important corollary is an easy consequence of the above proposition.

Corollary 2.1. *Let $M \subset \mathbb{C}^N$ be a real analytic generic submanifold and f a CR function in a neighborhood of $p \in M$. Then f extends as a holomorphic function in a neighborhood of p in $\mathbb{C}^N$ if and only if f is real analytic in a neighborhood of p in M.*

We shall describe an open wedge with a generic edge in $\mathbb{C}^{n+d}$. Let M be a generic submanifold of $\mathbb{C}^{n+d}$ of codimension d and $p_0 \in M$. Let $\rho = (\rho_1, ..., \rho_d)$ be a defining functions of M near p_0 and Ω a small neighborhood of p_0 in $\mathbb{C}^{n+d}$ in which ρ is defined. If Γ is an open convex cone with vertex at the origin in $\mathbb{R}^d$, we set

$$W(\Omega, \rho, \Gamma) := \{Z \in \Omega : \rho(Z, \overline{Z}) \in \Gamma\}.$$

The above set is an open subset of $\mathbb{C}^{n+d}$ whose boundary contains $M \cap \Omega$. Such a set is called a wedge of edge M in the direction Γ centered at p_0.

The following shows that $W(\Omega, \rho, \Gamma)$ is in a sense independent of the choice of ρ, which will allow us to change defining functions freely. [BER]

Proposition 2.2. *Let ρ and ρ' be two defining functions for M near p_0, where M and p_0 are as above. Then there is a $d \times d$ real invertible matrix B such that for every Ω and Γ as above the following holds. For any open convex cone $\Gamma_1 \subset \mathbb{R}^d$ with $B\Gamma_1 \cap S^{d-1}$ relatively compact in $\Gamma \cap S^{d-1}$ (where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$), there exists Ω_1, an open neighborhood of p_0 in $\mathbb{C}^{n+d}$, such that*

$$W(\Omega_1, \rho', \Gamma_1) \subset W(\Omega, \rho, \Gamma).$$

Below is an Edge-of-the-Wedge Theorem, see [Kr], [Ru]. This theorem together with Proposition 2.1 are essential to the proof of the main theorem.

Theorem 2.3. *If $\Gamma \subset \mathbb{R}^d$ is an open convex cone, $R > 0$, $d \geq 2$. Let $V = \Gamma \cap B(0, R)$. Let $E \subset \mathbb{R}^d$ be a nonempty neighborhood of 0. Define $W^+ \subset \mathbb{C}^d$, $W^- \subset \mathbb{C}^d$ by*

$$W^+ = E + iV, \quad W^- = E - iV$$

Then there exists a fixed neighborhood U of $0 \in \mathbb{C}^d$ such that the following property holds: For any continuous function $g : W^+ \cup W^- \cup E \longrightarrow \mathbb{C}$ that is holomorphic on $W^+ \cup W^-$, there is a holomorphic G on U such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$.

Proof: Let $s \in E$. We may assume that $s = 0$. After composition with a linear isomorphism A in $\mathbb{C}^d$ with real coefficients, we may assume that $i\{(t_1, ..., t_d) \in \mathbb{R}^d : t_j > 0,\ j = 1, ..., d\} \subset A^{-1}(i\Gamma)$, there exists $B(0, R') \subset \mathbb{R}^d$ with $R' > 6\sqrt{d}$ and $E' = \{s \in \mathbb{R}^d : |s_j| < 6,\ j = 1, ..., d\}$ such that $E' + iB(0, R') \subset A^{-1}(E + iB(0, R))$. So we reduce the case to the following.

CLAIM: Let
$E = \{s \in \mathbb{R}^d : |s_j| < 6,\ j = 1, ..., d\}$,
$V = \{t \in \mathbb{R}^d : 0 < t_j < 6,\ j = 1, ..., d\}$,
$W^+ = E + iV,\ \ W^- = E - iV$.
Let $U = D^d(0, 1)$. If $g : W^+ \cup W^- \cup E \longrightarrow \mathbb{C}$ is continuous and g is holomorphic on $W^+ \cup W^-$, then there is a holomorphic G on U such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$.

Let $c = \sqrt{2} - 1$ and define

$$\begin{aligned} \varphi : \overline{D}^2(0,1) &\longrightarrow \mathbb{C} \\ (w, \lambda) &\longmapsto \frac{w + \lambda/c}{1 + c\lambda w} \end{aligned}$$

Then

$$\varphi(w, \lambda) = \frac{w + \lambda/c + |\lambda|^2 \overline{w} + c\overline{\lambda}|w|^2}{|1 + c\lambda w|^2};$$

hence

$$Im\varphi(w,\lambda) = \frac{(1-|\lambda|^2)Im(cw)+(1-|cw|^2)Im(\lambda)}{c|1+c\lambda w|^2}$$

Notice that

1. $sgn(Im\varphi) = sgn(Im\lambda)$ if $|\lambda| = 1$;
2. $sgn(Im\varphi) = sgn(Im\lambda)$ if $w \in \mathbb{R}$;
3. $\varphi(w, 0) = w$;
4. $|\varphi(w,\lambda)| \leq (1 + 1/c)(1 - c) < 6$. It follows that the function

$$\Phi : D^d(0,1) \times \overline{D} \longrightarrow \mathbb{C}^d$$
$$(w,\lambda) \longmapsto (\varphi(w_1,\lambda), ..., \varphi(w_d,\lambda))$$

satisfies

5. $\Phi(w, e^{i\theta}) \in W^+$ if $0 < \theta < \pi$;
6. $\Phi(w, e^{i\theta}) \in W^-$ if $\pi < \theta < 2\pi$;
7. $\Phi(w, e^{i\theta}) \in E$ if $\theta = 0$ or $\theta = \pi$;

In short, $\Phi(z, e^{i\theta}) \in \mathrm{Dom}(g)$ for all $0 \leq \theta < 2\pi$, all $w \in D^d(0,1)$. So we may define

$$G(w) = \frac{1}{2\pi}\int_0^{2\pi} g(\Phi(w, e^{i\theta}))d\theta,\ w \in D^d(0,1).$$

We claim that this is what we seek. First, G is holomorphic by an application of Morera's theorem. Next, we note that for fixed $s \in E \cap D^d(0,1)$, the function $g(\Phi(s,\cdot))$ is continuous on $\overline{D}$ and holomorphic on $\{\lambda \in D : Im\lambda > 0 \text{ or } < 0\}$. Again, by Morera, $g(\Phi(s,\cdot))$ is holomorphic on all of D. It follows by Mean Value Property that

$$G(s) = \frac{1}{2\pi}\int_0^{2\pi} g(\Phi(s, e^{i\theta}))d\theta = g(\Phi(s, 0)) = g(s);$$

hence $G = g$ on $E \cap D^d(0,1)$. If $s + it \in \mathbb{C}^d$ is fixed, $|s + it| < 1/2$, $t > 0$, then the function

$$\xi \longmapsto G(s + \xi t) - g(s + \xi t)$$

is holomorphic for $\xi \in \mathbb{C}$ small and $\equiv 0$ when ξ is real. If follows that G and g are identical on $W^+ \cup W^-$. If follows that $G \equiv g$ on $W^+ \cup W^- \cup E$.

If we return to our original case, the desire extension of g is

$$G(w) = \frac{1}{2\pi}\int_0^{2\pi} g \circ A(\Phi(A^{-1}(w), e^{i\theta}))d\theta,\ w \in A^{-1}(D^d(0,1)).$$

since it agrees with our g in some open set. Our desire U is $A^{-1}(D^d(0,1))$ ∎

According to Corollary 2.1, a CR fucntion $f = u + iv$ on a real analytic, generic submanifold can be holomorphically extended to a full neighborhood if and only if u, v are real analytic on M. Our main theorem, to some extent, reduces the case to u continuous and v real analytic on M.

Theorem 2.4. *Let $M \subset \mathbb{C}^{n+d}$ be a real analytic, generic submanifold. $0 \in M$, in a neighborhood Ω of 0 in $\mathbb{C}^{n+d}$, M is given by regular coordinates, $\rho := Im(w) - \phi(z, \overline{z}, Re(w)) = 0$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$. Let $f = u + iv$ be a holomorphic function defined in the wedge $W = W(\Omega, \rho, \Gamma)$, where $W(\Omega, \rho, \Gamma)$ is as above defined with Γ an open convex cone in $\mathbb{R}^d$. If f extends continuously up to the edge $M \cap \Omega$ with $v = Im(f)$ is real analytic on the edge $M \cap \Omega$. Then there exists a holomorphic function F in a neighborhood $\widetilde{\Omega}$ of 0 in $\mathbb{C}^{n+d}$ such that $F|_{W\cap\widetilde{\Omega}} = f$.*

Before proving the theorem, we need a lemma which ensures a wedge with edge as an open subet of $\mathbb{R}^{2n+d}$ contained in $\widetilde{\Psi}^{-1}(\Omega \cap W)$.

Lemma 2.5. *There exists an neighborhood U_1 of 0 in $\mathbb{R}^{2n+2d}$, an neighborhood Q_1 of 0 in $\mathbb{R}^{2n+d}$, an open convex cone Γ' with vertex at 0 in $\mathbb{R}^d$ such that $\widetilde{\Psi}$ is a real analytic diffeomorphism from U_1 to $\widetilde{\Psi}(U_1) \subset \Omega$ and $\widetilde{\Psi}((Q_1 \times \Gamma') \cap U_1) \subset \Omega \cap W$.*

Proof of the lemma: Take an neighborhood U_1 of 0 in $\mathbb{R}^{2n+2d}$ such that $\widetilde{\Psi}$ is a real analytic diffeomorphism from U_1 to $\widetilde{\Psi}(U_1) \subset \Omega$. $\widetilde{\Psi}(z, s + it) = (z, s + it + i\phi(z, \overline{z}, s + it)) = (\widetilde{z}, \widetilde{w})$, we can write $t = \Theta(\widetilde{z}, \overline{\widetilde{z}}, \widetilde{w}, \overline{\widetilde{w}})$ where Θ is $\mathbb{R}^d$ valued, real analytic in all its variables. Note that $t := \Theta(\widetilde{z}, \overline{\widetilde{z}}, \widetilde{w}, \overline{\widetilde{w}})$ can be taken as the defining functions for M in $\widetilde{\Psi}(U)$. By Proposition 2.2, there exists a neighborhood of 0 in $\mathbb{C}^{n+d}$(we may take it as a subset of U_1 and still denote it as U_1), a convex open cone Γ' such that

$$W(U_1, t, \Gamma') \subset W(\Omega, \rho, \Gamma),$$

Take $Q_1 = U_1 \cap \mathbb{R}^{2n+d}$, we then have $\widetilde{\Psi}((Q_1 \times \Gamma') \cap U_1) \subset \Omega \cap W$. ∎

Proof of the theorem: $f|_{M\cap\Omega}$ is a continuous CR function on $M \cap \Omega$. By the lemma, the function $f \circ \widetilde{\Psi}(z, \overline{z}, s + it) = f(z, \overline{z}, s + it + i\phi(z, \overline{z}, s + it))$ is defined on $\{(z, s + it) \in (Q_1 \times \Gamma') \cap U_1\}$, note that it is holomorphic in the variables $w = s + it$. To use Proposition 2.1, it suffices to show that $f \circ \widetilde{\Psi}$ can be continuously extended to a full neighborhood of 0 in $\mathbb{R}^{2n+2d}$ such that the extension is holomorphic in variables $w = s + it$.

Now, denote the restriction of the function $v \circ \widetilde{\Psi}(z, s + it)$ on Q_1 to be $v(z, \overline{z}, s)$. By the real analyticity of v and Ψ, we can treat $v(z, \overline{z}, s)$ as a power series expansion at $0 \in \mathbb{R}^{2n+d}$ and complexify the variable s to w. Denote the new function to be $v(z, \overline{z}, w)$ defined in a product neighborhood $U_2 = U_3 \times U_4 \subset \mathbb{C}^n \times \mathbb{C}^d$ of 0. Note that $v(z, \overline{z}, w)$ is continuous in U_2 and holomorphic in variables w.

Following the idea of Theorem 1.2, the function

$$g(z,\overline{z},s+it) :=$$
$$\begin{cases} \underline{f\circ\widetilde{\Psi}(z,\overline{z},s+it)-iv(z,\overline{z},s+it)} & \text{for} \quad (z,s+it)\in(Q_1\times(\Gamma'\cup\{0\}))\cap U_2 \\ f\circ\widetilde{\Psi}(z,\overline{z},s-it)-iv(z,\overline{z},s-it) & \text{for} \quad (z,s+it)\in(Q_1\times-\Gamma')\cap U_2 \end{cases}$$

is holomorphic in variables $w=s+it$ for fixed z and continuously up to $Q_1\cap U_2$.

We shall apply the construction of holomorphic extension in the proof of Theorem 2.3 to $g(z,\overline{z},s+it)$ for every fixed z. According to the proof, we have a linear isomorphism A in $\mathbb{C}^d$ with real coefficients and $\Phi : D^d(0,1)\times\overline{D}\longrightarrow\mathbb{C}^d$. Now we extend A to a linear isomorphism in $\mathbb{C}^{n+d}$ which maps $(z,w)\in\mathbb{C}^{n+d}$ to $(z,A(w))$, still denote it as A. We also extend Φ to a mapping which takes $(z,w,\lambda)\in U_3\times D^d(0,1)\times\overline{D}$ to $(z,\Phi(w,\lambda))\in\mathbb{C}^n\times\mathbb{C}^d$, still denote it as Φ. Since the choice of A depends only on U_4 and the cone Γ', so we have the extension

$$G(z,\overline{z},w)=\frac{1}{2\pi}\int_0^{2\pi} g\circ A(\Phi(A^{-1}(z,w),e^{i\theta}))d\theta,\ \ \forall (z,w)\in A^{-1}(U_3\times D^d(0,1))$$

by Theorem 2.3 and Lemma 2.5.

From the construction above, the extension $G(z,\overline{z},s+it)$ is continuous on $U_3\times U_5=A^{-1}(U_3\times D^d(0,1))$ and holomorphic in variables w. Thus, the function $F(z,\overline{z},s+it):=G+iv(z,\overline{z},s+it)$ is a continuous extension of $f\circ\widetilde{\Psi}(z,\overline{z},s+it)$ in $U_3\times U_5$, holomorphic in $w=s+it$. Thus by Proposition 2.1, f can be holomorphically extended to a full neighborhood $\widetilde{\Omega}$ of 0 in $\mathbb{C}^{n+d}$. ■

The following theorem states that minimality is a sufficient condition for holomorphic extension of all CR functions from a generic submanifold M in $\mathbb{C}^N$ into an open wedge. See [BER], [T].

Theorem 2.6. *Let M be a generic submanifold of $\mathbb{C}^{n+d}$ of codimension d and $p_0\in M$. If M is minimal at p_0, then for every open neighborhood U of p_0 in M there exists a wedge W with edge M centered at p_0 such that every continuous CR function in U extends holomorphically to the wedge W.*

Since a real analytic CR submanifold M in $\mathbb{C}^N$ is finite type at $p_0\in M$ if and only if it is minimal at p_0. Thus, together Theorem 2.6, we have the following corollary.

Corollary 2.2. *Let $M\subset\mathbb{C}^N$ be a real analytic, generic CR submanifold, finite type at $p_0\in M$. If $f=u+iv$ is a continuous CR function defined in a neighborhood of p_0 in M with v real analytic. Then f can be holomorphically extended to a full neighborhood of p_0 in $\mathbb{C}^N$. (Or u is also real analytic in a neighborhood of p_0 in M by Corollary 2.1)*

Definition 2.2. A CR submanifold of the form

$$M = \{(z, s+it) \in \mathbb{C}^n \times \mathbb{C}^d;\ t = \phi(z, \overline{z})\}$$

where $\phi : \mathbb{C}^n \longmapsto \mathbb{R}^d$ is smooth with $\phi(0) = 0$ and $d\phi(0) = 0$ is called rigid.

Corollary 2.3. *Let M be a real analytic, generic submanifold, $0 \in M$, given by $\{(z, s+it) \in U \subset \mathbb{C}^n \times \mathbb{C}^d : t = \phi(z, \overline{z}, s)\}$. Let M' be a real analytic, generic, rigid submanifold, $0 \in M'$, given by $\{(z', s'+it') \in U' \subset \mathbb{C}^{n'} \times \mathbb{C}^{d'} : t' = \phi'(z', \overline{z'})\}$. If $H = (f_1, ..., f_{n'}, g_1, ..., g_{d'})$ is a holomorphic mapping defined in the wedge $W = W(U, \rho, \Gamma)$ with Γ an open convex cone. If H extends continuously to the edge $M \cap U$ with $H(0) = 0$, $H(M) \subset M'$. Then $(f_1, ..., f_{n'})$ can be holomorphically extended near 0 implies H can be holomorphically extended near 0.*

Proof: Denote $Im(g_i)(z, \overline{z}, s) := Im(g_i) \circ \Psi(z, \overline{z}, s)$, $1 \leq i \leq d'$, similarily for $f_j(z, \overline{z}, s)$, $1 \leq j \leq n'$. Now $Im(g_i)(z, \overline{z}, s) = \phi'(f(z, \overline{z}, s), \overline{f(z, \overline{z}, s)})$ and f_j real analytic. Write $f_j(z, \overline{z}, s) = \sum a^j_{\alpha\beta\gamma} z^\alpha \overline{z}^\beta s^\gamma$ for all j in a neighborhood of 0 in $\mathbb{R}^{2n+d}$, similarily $\overline{f_j}(z, \overline{z}, s) = \sum \overline{a^j_{\alpha\beta\gamma}} \overline{z}^\alpha z^\beta s^\gamma$ for all j in the same neighborhood. We extend $f_j(z, \overline{z}, s)$ to $f_j(z, \chi, w) = \sum a^j_{\alpha\beta\gamma} z^\alpha \chi^\beta w^\gamma$, $\overline{f_j}(z, \overline{z}, s)$ to $\overline{f_j}(z, \chi, w) = \sum \overline{a^j_{\alpha\beta\gamma}} \chi^\alpha z^\beta w^\gamma$ in a neighborhood of 0 of $\mathbb{C}^{2n+d}$.

Note that the function $\phi'_i(f_1(z, \chi, w), ..., f_{n'}(z, \chi, w), \overline{f_1}(z, \chi, w), ..., \overline{f_{n'}}(z, \chi, w))$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n+d}$ by the real analyticity of ϕ', it is therefore real analytic in that neighborhood. The restriction of such function to the plane $\{(z, \chi, w) \in \mathbb{C}^{2n+d};\ \chi = \overline{z}, w \in \mathbb{R}^d\}$ is also real analytic and equals $Im(g_i)(z, \overline{z}, s)$, thus $Im(g_i)$ are real analytic for all i, by Theorem 2.4, g_i can be holomorphically extended to a neighborhood, we get the desire result. ∎