Spécialisation du groupe de monodromie et ℓ-indépendance

SPECIALIZATION OF MONODROMY GROUP AND ℓ-INDEPENDENCE

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Abstract. Soit E un schéma abélien sur une variété lisse et géométriquement connexe X, définie sur un corps k de type fini sur $\mathbb{Q}$. Soit η le point générique de X et soit $x \in X$ un point fermé. Si $\mathfrak{g}_\ell$ et $(\mathfrak{g}_\ell)_x$ sont les algèbres de Lie des représentations ℓ-adiques de Galois des variétés abéliennes E_η et E_x, alors $(\mathfrak{g}_\ell)_x$ est plongée dans $\mathfrak{g}_\ell$ par spécialisation. Nous démontrons que l'ensemble $\{x \in X \text{ point fermé} \mid (\mathfrak{g}_\ell)_x \subsetneq \mathfrak{g}_\ell\}$ est indépendant de ℓ, ce qui confirme la Conjecture 5.5 de [3].

Let E be an abelian scheme over a geometrically connected, smooth variety X defined over k, a finitely generated field over $\mathbb{Q}$. Let η be the generic point of X and $x \in X$ a closed point. If $\mathfrak{g}_\ell$ and $(\mathfrak{g}_\ell)_x$ are the Lie algebras of the ℓ-adic Galois representations for abelian varieties E_η and E_x, then $(\mathfrak{g}_\ell)_x$ is embedded in $\mathfrak{g}_\ell$ by specialization. We prove that the set $\{x \in X \text{ closed point} \mid (\mathfrak{g}_\ell)_x \subsetneq \mathfrak{g}_\ell\}$ is independent of ℓ and confirm Conjecture 5.5 in [3].

§0. Introduction

Let E be an abelian scheme of relative dimension n over a geometrically connected, smooth variety X defined over k, a finitely generated field over $\mathbb{Q}$. If K is the function field of X and η is the generic point of X, then $A := E_\eta$ is an abelian variety of dimension n defined over K. The structure morphism $X \to \mathrm{Spec}(k)$ induces at the level of *étale* fundamental groups a short exact sequence of profinite groups:

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k := \mathrm{Gal}(\overline{k}/k) \to 1. \tag{0.1}$$

Any closed point $x : \mathrm{Spec}(\mathbf{k}(x)) \to X$ induces a splitting $x : \Gamma_{\mathbf{k}(x)} \to \pi_1(X_{\mathbf{k}(x)})$ of exact sequence (0.1) for $\pi_1(X_{\mathbf{k}(x)})$.

Let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. For each prime number ℓ, we have the Galois representation $\rho_\ell : \Gamma_K \to \mathrm{GL}(T_\ell(A))$ where $T_\ell(A)$ is the ℓ-adic Tate module of A. This representation is unramified over X and factors through $\rho_\ell : \pi_1(X) \to \mathrm{GL}(T_\ell(A))$ (still denote the map by ρ_ℓ for simplicity). The image of ρ_ℓ is a compact ℓ-adic Lie subgroup of $\mathrm{GL}(T_\ell(A)) \cong \mathrm{GL}_{2n}(\mathbb{Z}_\ell)$. Any closed point $x : \mathrm{Spec}(\mathbf{k}(x)) \to X$ induces an ℓ-adic Galois representation of $\Gamma_{\mathbf{k}(x)}$ by restricting ρ_ℓ to $x(\Gamma_{\mathbf{k}(x)})$. This representation is isomorphic to the Galois representation of $\Gamma_{\mathbf{k}(x)}$ on the ℓ-adic Tate module of E_x, the abelian variety over $\mathbf{k}(x)$ that is the specialization of E at x.

For the sake of simplicity, we write $G_\ell := \rho_\ell(\pi_1(X))$, $\mathfrak{g}_\ell := \mathrm{Lie}(G_\ell)$, $(G_\ell)_x := \rho_\ell(x(\Gamma_{\mathbf{k}(x)}))$ and $(\mathfrak{g}_\ell)_x := \mathrm{Lie}((G_\ell)_x)$. We have $(\mathfrak{g}_\ell)_x \subset \mathfrak{g}_\ell$. We set X^{cl} the set of closed points of X and define the exceptional set

$$X_{\rho_{E,\ell}} := \{x \in X^{cl} | (\mathfrak{g}_\ell)_x \subsetneq \mathfrak{g}_\ell\}.$$

The main result (Theorem 1.5) of this note is that the exceptional set $X_{\rho_{E,\ell}}$ is independent of ℓ. Conjecture 5.5 in [Cadoret & Tamagawa 3] is then a direct application of our theorem.

§1. ℓ-independence of $X_{\rho_{E,\ell}}$

Theorem 1.1. (Serre [6 §1]) Let A be an abelian variety defined over a field K finitely generated over $\mathbb{Q}$ and let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. If $\rho_\ell : \Gamma_K \to \mathrm{GL}(T_\ell(A))$ is the ℓ-adic representation of Γ_K, then the Lie algebra $\mathfrak{g}_\ell$ of $\rho_\ell(\Gamma_K)$ is algebraic and the rank of $\mathfrak{g}_\ell$ is independent of the prime ℓ.

Remark 1.2. When K is a number field, the algebraicity of the ℓ-adic Lie algebra $\mathfrak{g}_\ell$ was proven by Bogomolov [1]. When K is a global field of finite characteristic > 2, the rank independence on ℓ was proven by Zarhin [7].

Since $V_\ell := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a semisimple Γ_K-module (Faltings [4]), the action on V_ℓ of the Zariski closure $\mathfrak{G}_\ell$ of $\rho_\ell(\Gamma_K)$ in GL_{V_ℓ} is also semisimple. Therefore $\mathfrak{G}_\ell$ is a reductive algebraic group (Borel [2]). By Theorem 1.1, $\mathfrak{g}_\ell$ is algebraic. So the rank of $\mathfrak{g}_\ell$ is just the dimension of maximal tori in $\mathfrak{G}_\ell$. We need two more theorems, the first one is the Tate conjecture for abelian varieties proved by G. Faltings and the second one is a result of Yu. G. Zarhin on algebraic reductive Lie algebras.

Theorem 1.3. (Faltings [4]) Let A be an abelian variety defined over a field k that is finitely generated over $\mathbb{Q}$ and let $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Then the map $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{End}_{\Gamma_k}(V_\ell(A))$ is an isomorphism.

Theorem 1.4. (Zarhin [8 §5]) Let V be a finite dimensional vector space over a field of characteristic 0. Let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathrm{End}(V)$ be Lie algebras of reductive subgroups of GL_V. Let us assume that the centralizers of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in $\mathrm{End}(V)$ are equal and that the ranks of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are equal. Then $\mathfrak{g}_1 = \mathfrak{g}_2$.

We are now able to prove our main theorem.

Theorem 1.5. The set $X_{\rho_{E,\ell}}$ is independent of ℓ.

Proof. Suppose $x \in X^{cl} \backslash X_{\rho_\ell}$, then $(\mathfrak{g}_\ell)_x = \mathfrak{g}_\ell$. It suffices to show $\mathfrak{g}_{\ell'} = (\mathfrak{g}_{\ell'})_x := \mathrm{Lie}(\rho_{\ell'}(x(\Gamma_{\mathbf{k}(x)})))$ for any prime number ℓ'. Since base change with finite field extension of $\mathbf{k}(x)$ does not change the Lie algebras, $\mathrm{End}_{\overline{k}}(E_x)$ is finitely generated, and we have the exponential map from Lie algebras to Lie groups, we may assume that $\mathrm{End}_{\overline{k}}(E_x) = \mathrm{End}_k(E_x)$ and $\mathrm{End}_{\Gamma_k}(V_\ell(E_x)) = \mathrm{End}_{(\mathfrak{g}_\ell)_x}(V_\ell(E_x))$. We do the same for the abelian variety E_η/K. We therefore have

$$\dim_{\mathbb{Q}_{\ell'}}(\mathrm{End}_{\mathfrak{g}_{\ell'}}(V_{\ell'}(E_\eta))) \stackrel{1}{=} \dim_{\mathbb{Q}_{\ell'}}(\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell'})$$

$$\stackrel{2}{=} \dim_{\mathbb{Q}_\ell}(\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \stackrel{3}{=} \dim_{\mathbb{Q}_\ell}(\mathrm{End}_{\mathfrak{g}_\ell}(V_\ell(E_\eta)))$$

$$\stackrel{4}{=} \dim_{\mathbb{Q}_\ell}(\mathrm{End}_{(\mathfrak{g}_\ell)_x}(V_\ell(E_x))) \stackrel{5}{=} \dim_{\mathbb{Q}_\ell}(\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$$

$$\stackrel{6}{=} \dim_{\mathbb{Q}_{\ell'}}(\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell'}) \stackrel{7}{=} \dim_{\mathbb{Q}_{\ell'}}(\mathrm{End}_{(\mathfrak{g}_{\ell'})_x}(V_{\ell'}(E_x))).$$

Theorem 1.3 implies the first, third, fifth and seventh equality. The dimensions of $\mathbb{Q}_\ell$-vector spaces $\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ and $\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ do not depend on ℓ; this implies the second and the sixth equality. Equality $\mathfrak{g}_\ell = (\mathfrak{g}_\ell)_x$ implies the fourth equality.

We have $\mathrm{End}_{\mathfrak{g}_{\ell'}}(V_{\ell'}(E_\eta)) = \mathrm{End}_{(\mathfrak{g}_{\ell'})_x}(V_{\ell'}(E_x))$ because the left one is contained in the right one. In other words, the centralizer of $(\mathfrak{g}_{\ell'})_x$ is equal to the centralizer of $\mathfrak{g}_{\ell'}$. We know that $(\mathfrak{g}_{\ell'})_x \subset \mathfrak{g}_{\ell'}$ are both reductive, thanks to the semisimplicity of the corresponding Galois representaions (Faltings [4]). By Theorem 1.1 on ℓ-independence of reductive ranks and equality $\mathfrak{g}_\ell = (\mathfrak{g}_\ell)_x$,

$$\mathrm{rank}(\mathfrak{g}_{\ell'}) = \mathrm{rank}(\mathfrak{g}_\ell) = \mathrm{rank}(\mathfrak{g}_\ell)_x = \mathrm{rank}(\mathfrak{g}_{\ell'})_x.$$

Therefore, by Theorem 1.4 we conclude that $(\mathfrak{g}_{\ell'})_x = \mathfrak{g}_{\ell'}$ and thus prove the theorem. □

Corollary 1.6 (Conjecture 5.5 of [3]). Let k be a field that is finitely generated over $\mathbb{Q}$, X a smooth, separated, geometrically connected curve over k with the field of rational function K. Let η be the generic point of X and E an abelian scheme over X. Let $\rho_\ell : \pi_1(X) \to \mathrm{GL}(T_\ell(E_\eta))$ be the corresponding ℓ-adic representation. Then there exists a finite subset $X_E \subset X(k)$ such that for any prime ℓ, $X_{\rho_{E,\ell}} = X_E$, where $X_{\rho_{E,\ell}}$ is the set of all $x \in X(k)$ such that $(G_\ell)_x$ is not open in $G_\ell := \rho_\ell(\pi_1(X))$.

Proof. The uniform open image theorem for GLP (geometrically Lie perfect) representations [3 Thm. 1.1] implies the finiteness of $X_{\rho_{E,\ell}}$. Theorem 1.5 implies ℓ-independence. □

Corollary 1.7. Let A be an abelian variety of dimension $n \geq 1$ defined over a field K that is finitely generated over $\mathbb{Q}$. Let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K. For each prime number ℓ, we have the Galois representation $\rho_\ell : \Gamma_K \to \mathrm{GL}(T_\ell(A))$ where $T_\ell(A)$ is the ℓ-adic Tate module of A. If the Mumford-Tate conjecture for abelian varieties over number fields is true, then there is an algebraic subgroup H of GL_{2n} defined over $\mathbb{Q}$ such that the identity component of the Zariski closure of $\rho_\ell(\Gamma_K)$ in $\mathrm{GL}_{2n}(V_\ell(A))$ is equal to $H \times_{\mathbb{Q}} \mathbb{Q}_\ell$ for all ℓ.

Proof. There exists an abelian scheme E over a variety X defned over a number field k such that the function field of X is K and $E_\eta = A$ where η is the generic point of X (see, e.g., Milne [5 §20]). By [6 §1], there exists a closed point $x \in X$ such that $(\mathfrak{g}_\ell)_x = \mathfrak{g}_\ell$. Therefore, we have $(\mathfrak{g}_\ell)_x = \mathfrak{g}_\ell$ for any prime ℓ by Theorem 1.5. Since all Lie algebras are algebraic (Theorem 1.1), if we take H as the Mumford-Tate group of E_x, then the identity component of the Zariski closure of $\rho_\ell(\Gamma_K)$ in $\mathrm{GL}_{2n}(V_\ell(A))$ is equal to $H \times_{\mathbb{Q}} \mathbb{Q}_\ell$ for all ℓ. $\square$

Question. Is the algebraic group H in Corollary 1.7 isomorphic to the Mumford-Tate group of the abelian variety A?

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References

1. F. A. Bogomolov, *Sur l'algébricité des représentations ℓ-adiques*, C.R.A.S. **290** (1980), 701-703.
2. A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, 126 (2nd ed.), Springer-Verlag 1991.
3. A. Cadoret and A. Tamagawa, *A uniform open image theorem for ℓ-adic representations I*, preprint.
4. G. Faltings, Complements to Mordell. Chapter 6 in: G. Faltings and G. Wüstholz, *Rational Points*, Seminar Bonn/Wuppertal 1983-1984, Vieweg 1984.
5. J. S. Milne, *"Jacobian Varieties". Arithmetic Geometry.* New York: Springer-Verlag 1986.
6. J.-P. Serre, *Letter to K. A. Ribet*, Jan. 1, 1981, reproduced in Coll. Papers, vol. IV, no 133.
7. Yu. G. Zarhin, *Abelian varieties, ℓ-adic representations and Lie algebras. Rank independence on ℓ*, Invent. Math. **55** (1979), 165-176.
8. Yu. G. Zarhin, *Abelian varieties having a reduction of $K3$ type*, Duke Math. J. **65** (1992), 511-527.

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