MONODROMY OF GALOIS REPRESENTATIONS AND EQUAL-RANK SUBALGEBRA EQUIVALENCE

CHUN YIN HUI

ABSTRACT. Let K be a number field, $\mathscr{P}$ the set of prime numbers, and $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ a compatible system (in the sense of Serre [19]) of semisimple, n-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$. Denote the Zariski closure of $\rho_\ell(\mathrm{Gal}(\overline{K}/K))$ in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ by G_ℓ and its Lie algebra by $\mathfrak{g}_\ell$. It is known that the identity component G_ℓ° is reductive and the formal character of the tautological representation $G_\ell^\circ \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is independent of ℓ (Serre). We use the theory of abelian ℓ-adic representations to prove that the formal character of the tautological representation of the derived group $(G_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is likewise independent of ℓ. By investigating the geometry of weights of this faithful representation, we prove that the semisimple parts of $\mathfrak{g}_\ell \otimes \mathbb{C}$ satisfy an equal-rank subalgebra equivalence for all ℓ which is equivalent to the number of $A_n := \mathfrak{sl}_{n+1,\mathbb{C}}$ factors for $n \in \{6, 9, 10, 11, ...\}$ and the parity of the number of A_4 factors in $\mathfrak{g}_\ell \otimes \mathbb{C}$ are independent of ℓ.

§1. Introduction

Let K be a number field, $\overline{K}$ its algebraic closure and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. Let $\mathscr{P}$ be the set of prime numbers. A compatible system (Definition 3.4) of ℓ-adic representations $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ of G_K is a collection of continuous representations

$$\rho_\ell : G_K \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell),$$

indexed by $\mathscr{P}$, such that for any $\ell \in \mathscr{P}$, ρ_ℓ is unramified at all but finitely many non-Archimedean places of K, and for any $\ell, \ell' \in \mathscr{P}$, the characteristic polynomials of $\rho_\ell(\mathrm{Frob}_w)$ and $\rho_{\ell'}(\mathrm{Frob}_w)$ (well-defined if ρ_ℓ and $\rho_{\ell'}$ are unramified at v, and w is a valuation of $\overline{K}$ extending v) are equal with rational coefficients for all but finitely many non-Archimedean places v of K. Such a compatible system arises, for example, from the Galois action of G_K on the ℓ-adic Tate modules $T_\ell(A) \otimes \mathbb{Q}_\ell$ of an abelian variety A defined over K (see [19, Chap. 1]) or more generally, on the ℓ-adic étale cohomology groups $H_{et}^k(X_{\overline{K}}, \mathbb{Q}_\ell)$ of a complete nonsingular variety X defined over K (Deligne [3]). We may assume our representations ρ_ℓ are semisimple for all ℓ by semi-simplification since the characteristic polynomials of matrices only carry information from the semisimple parts.

For a compatible system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ of semisimple ℓ-adic representations of G_K, the Zariski closure of $\rho_\ell(G_K)$ in the algebraic group $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ is a reductive algebraic group and is called the *algebraic monodromy group* at ℓ. Denote it by G_ℓ, its connected component by G_ℓ°, and its Lie algebra by $\mathfrak{g}_\ell$. Let $\Phi_\ell : G_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ be the tautological representation. Consider the following conjectures:

Conjecture 1.1. There exists a faithful representation $\Phi : G \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$ of a reductive group G over $\mathbb{Q}$, such that for all $\ell \in \mathscr{P}$, $(G_\ell, \Phi_\ell) \cong (G, \Phi) \times_{\mathbb{Q}} \mathbb{Q}_\ell$.

Conjecture 1.2. There exists a faithful representation $\Phi : G \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$ of a connected reductive group G over $\mathbb{Q}$, such that for all $\ell \in \mathscr{P}$, $(G_\ell^\circ, \Phi_\ell) \cong (G, \Phi) \times_{\mathbb{Q}} \mathbb{Q}_\ell$.

Conjecture 1.1 is true if $\rho_\ell(G_K)$ is abelian for one ℓ, see chapter 3 of [19]. Weaker variants of this conjecture have appeared many times in the literature. For example, if our compatible system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ comes from the Galois action on the Tate modules of an abelian variety, then Conjecture 1.2 follows immediately from the semisimplicity of ρ_ℓ (Faltings [4]) and the Mumford-Tate conjecture (see Mumford [15], Serre [18, §9]). If $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ comes from the Galois action on the étale cohomology groups of a complete non-singular variety, then Conjecture 1.2 is implied (see [12, §5]) by the well-known semisimplicity conjecture [18, §9] and the general Tate conjecture [22]. By the method of Frobenius tori, Serre [16, p. 6,17],[17, §2.2.3] has proved the following ℓ-independence results.

Theorem 1.3. The open subgroup of finite index $\rho_\ell^{-1}(G_\ell^\circ(\mathbb{Q}_\ell)) \subset G_K$ is independent of ℓ.

Theorem 1.4. The pair (T_ℓ, Φ_ℓ) consisting of a maximal torus T_ℓ of G_ℓ° and the tautological representation $\Phi_\ell : T_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is independent of ℓ. Therefore, the formal character of $G_\ell^\circ \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ and hence the rank of G_ℓ° are independent of ℓ.

Remark 1.5. Serre originally stated these results for representations associated to abelian varieties, but his proofs work for arbitrary compatible systems.

Remark 1.6. Theorem 1.3 implies there is a smallest finite extension K^{conn} of K such that the Zariski closure of $\rho_\ell(G_{K^{\mathrm{conn}}})$ in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ is equal to G_ℓ° for all ℓ. For more results about K^{conn}, see Silverberg and Zarhin [20],[21] and Larsen and Pink [13].

   We study ℓ-independence of compatible systems of ℓ-adic representations of G_K, assuming semisimplicity. We are not able to prove Conjecture 1.1 and Conjecture 1.2, but using the theory of abelian ℓ-adic representations, we extend Theorem 1.4 by proving that the formal character of the representation of the derived group $(G_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is independent of ℓ. Using these data, we prove that the semisimple parts of the complexified Lie algebras $\mathfrak{g}_\ell \otimes \mathbb{C}$ satisfy an equal-rank subalgebra equivalence (Definition 2.18). The results of this paper are summarized as follows.

(1) (Proposition 3.18) The dimension of the center of G_ℓ° is bounded by $d_{K^{\mathrm{conn}}}$, the common dimension of the Serre groups $S_{\mathfrak{m}}$ associated to the number field K^{conn}.

(2) (Theorem 3.19) The triple $(((G_\ell^\circ)^{\mathrm{der}} \cap T_\ell)^\circ, T_\ell, \Phi_\ell)$ is independent of ℓ, where T_ℓ is a maximal torus of G_ℓ° and Φ_ℓ is the embedding of T_ℓ into $\mathrm{GL}_{n,\mathbb{Q}_\ell}$. Therefore, the formal character of the tautological representation $(G_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ and hence the semisimple rank of G_ℓ° are independent of ℓ.

(3) (Theorem 3.21) Consider the free abelian group of virtual complex simple Lie algebras which contains semisimple Lie algebras naturally. We divide by the subgroup generated by all expressions $\mathfrak{g} - \mathfrak{h}$ where $\mathfrak{h} \subset \mathfrak{g}$ are equal rank semisimple Lie algebras. The semisimple parts of the complexified Lie algebras $\mathfrak{g}_\ell \otimes \mathbb{C}$ have the same image in this quotient group (satisfy equal-rank subalgebra equivalence in Definition 2.18) for all ℓ. This is equivalent to the number of $A_n := \mathfrak{sl}_{n+1,\mathbb{C}}$ factors for $n \in \{6, 9, 10, 11, ...\}$ and the parity of the number of A_4 factors in $\mathfrak{g}_\ell \otimes \mathbb{C}$ are independent of ℓ.

(4) (Theorem 4.1) Let K be a field, finitely generated over $\mathbb{Q}$ and G_K its absolute Galois group. If the system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ arises from the Galois action of G_K on the Tate modules of an abelian variety X defined over a field K, then (3) holds for $\mathfrak{g}_\ell \otimes \mathbb{C}$.

Remark 1.7. Larsen and Pink [11] studied compatible systems of representations of profinite groups that are endowed with a dense subset of "Frobenius" elements. They gave an example in [11, §10] that the semisimple rank of G_ℓ (the algebraic monodromy group at ℓ) depends on ℓ, contrary to (2).

Remark 1.8. (1), (2), (3) above also hold for any semisimple compatible system of λ-adic [19, Chap. 1 §2.3] representations. This will be explained at the end of section 3.

Remark 1.9. We study ℓ-independence of mod ℓ Galois representations that arise from étale cohomology in a subsequent paper [9] and obtain "mod ℓ" versions of (2) and (3) when $\ell \gg 0$.

Let us sketch the proofs of our results. We may assume $K = K^{\text{conn}}$ so that $G_\ell = G_\ell^\circ$. Since the Lie algebra of $[G_\ell, G_\ell]$ is the semisimple part of $\mathfrak{g}_\ell$. Hence, we could study the dimension of the center of $\mathfrak{g}_\ell$ by considering the dimension of the image of following semisimple ℓ-adic representation

$$G_K \xrightarrow{\rho_\ell} G_\ell(\mathbb{Q}_\ell) \to G_\ell/[G_\ell, G_\ell](\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_m(\mathbb{Q}_\ell).$$

This map factors through the quotient $G_K \to G_K^{\text{ab}}$, so we get $\Psi_\ell : G_K^{\text{ab}} \to \mathrm{GL}_m(\mathbb{Q}_\ell)$. Let $\sum_K$ be the set of finite places of K. Since Ψ_ℓ is unramified at all except finitely many places S of K, if F_w is a Frobenius element of a valuation w of $\overline{K}$ extending a place $v \in \sum_K \backslash S$, then the eigenvalues of $\Psi_\ell(F_w)$ are algebraic numbers. This is the key observation which leads to result (1). By imitating the proof of Theorem 1.10, we can find an integer N such that $\Psi_\ell^N : G_K^{\text{ab}} \to \mathrm{GL}_m(\mathbb{Q}_\ell)$ is locally algebraic.

Theorem 1.10. (See Serre [19, Chap. 3 §3], Waldschmidt [23], Henniart [7, §5])
If $\rho : G_K^{\text{ab}} \to \mathrm{GL}_m(\mathbb{Q}_\ell)$ is a rational, semisimple, ℓ-adic abelian representation of K, then ρ is locally algebraic.

Therefore, by the theory of abelian ℓ-adic representation, Ψ_ℓ^N arises from some abelian ℓ-adic representation attached to some Serre group $S_\mathfrak{m}$. In other words, $S_\mathfrak{m}(\mathbb{Q}_\ell)$ surjects onto $\Psi_\ell^N(G_K^{\text{ab}})$. Since the dimension of $\Psi_\ell^N(G_K^{\text{ab}})$, $\Psi_\ell(G_K^{\text{ab}})$ and the center of $\mathfrak{g}_\ell$ are the same, we get (1). By the above techniques, the upper bound of the dimension of center in (1) and Theorem 1.4, we construct an auxiliary compatible system of representations of G_K to prove (2). The restriction of the formal character to the derived group of G_ℓ° are independent of

ℓ by (2). Therefore, we only need to study the semisimple part of $\mathfrak{g}_\ell$. We have for each ℓ a faithful representation of $(\mathfrak{g}_\ell)_{ss} \otimes \mathbb{C}$ which gives the same formal character. So we need to answer the question below.

Q: To what extent is a complex semisimple Lie algebra $\mathfrak{g}$ determined if the formal character of a faithful representation of $\mathfrak{g}$ is given?

Larsen and Pink have answered this question in the case that the representation is irreducible [10, §4]. The difficulty of the question can be illustrated by the following example. We know that $E_7 \times A_1$ and $A_4 \times A_4$ are subalgebras of maximal rank in E_8, so if we restrict a representation of E_8 to $E_7 \times A_1$ and $A_4 \times A_4$, the formal characters of these two representations are the same. We address this question by investigating the geometry of the roots and weights in the formal characters. Actually, we will first prove that the number of A_n factors for $n \in \{6, 9, 10, 11, ...\}$ and the parity of the number of A_4 factors in $\mathfrak{g}$ are invariants (Theorem 2.14, 2.17). Then, it follows easily that the image of $\mathfrak{g}$ in the quotient group in (3) is invariant (Theorem 2.19). This together with Proposition 2.20 imply (3). Finally, (4) is a direct consequence of (3) and a result on ℓ-independence of specialization of monodromy groups of abelian varieties (Hui [8, Thm. 2.5]).

The structure of this paper is as follows. Section 2 is devoted to answering question **Q**, which is purely representation theoretic; we only use the representation theory of complex semisimple Lie algebras. We will prove $(1), (2)$, and (3) in section 3. This section relies heavily on the theory of abelian ℓ-adic representation (Serre [19]). In section 4, we consider systems $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ coming from abelian varieties and prove (4).

§2 **Geometry of weights in formal characters**

This section is purely representation theoretic and self-contained. It is devoted to answering the question below.

Q: To what extent is a complex semisimple Lie algebra $\mathfrak{g}$ determined if the formal character of a faithful representation of $\mathfrak{g}$ is given?

(**2.1**) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $t_\mathfrak{g}$ some Cartan subalgebra of $\mathfrak{g}$. Denote the roots, the weight lattice and the Weyl group by $\Phi_\mathfrak{g}$, $\Lambda_\mathfrak{g}$ and $W_\mathfrak{g}$ respectively. If $\Theta : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on some n-dimensional complex vector space V, then the action of $t_\mathfrak{g}$ on V can be diagonalized and then we have n weight vectors $\alpha_1, ..., \alpha_n \in t_\mathfrak{g}^*$. Let $\mathbb{Z}[\Lambda_\mathfrak{g}]$ be the group ring over $\mathbb{Z}$ generated by the free abelian group $\Lambda_\mathfrak{g}$. We define the *formal character* of Θ to be $\mathrm{Char}_\Theta(V) := \alpha_1 + \cdots + \alpha_n \in \mathbb{Z}[\Lambda_\mathfrak{g}]$. We know that $\mathrm{Char}_\Theta(V)$ is invariant under the Weyl group $W_\mathfrak{g}$. Let $\Psi : \mathfrak{h} \to \mathfrak{gl}(V')$ be a representation of another complex semisimple $\mathfrak{h}$ on an n-dimensional complex vector space V'. We say $\mathfrak{h}$ and $\mathfrak{g}$ *have the same formal character* if there is an isomorphism F between $t_\mathfrak{h}^*$ and $t_\mathfrak{g}^*$ such that $F(\mathrm{Char}_\Psi(V))$(defined in an obvious way) is equal to $\mathrm{Char}_\Theta(V)$. We can now state our main theorems of this section.

Theorem 2.14 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then the number of A_n factors of $\mathfrak{g}$ and $\mathfrak{h}$ are the same when $n \in \{6, 9, 10, ...\}$.

Theorem 2.17 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then the parities of the numbers of A_4 factors of $\mathfrak{g}$ and $\mathfrak{h}$ are the same.

Theorem 2.19 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then $\mathfrak{g}$ and $\mathfrak{h}$ satisfy equal-rank subalgebra equivalence (Definition 2.18).

(**2.2**) If $\Theta : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation, then $\mathfrak{g}$ is embedded as a subspace in $\mathrm{End}(V) = V \otimes V^*$ through Θ. It is easy to check the representation of $\mathfrak{g}$ on this subspace $\mathfrak{g} \subset \mathrm{End}(V) = V \otimes V^*$ is the adjoint representation [5, Chap. 13.1]. Therefore, the representation of $\mathfrak{g}$ on $V \otimes V^*$ contains the adjoint representation of $\mathfrak{g}$ as a subrepresentation and is also faithful. The formal character of $V \otimes V^*$ depends only on the formal character of V and is just the sum of all the differences of weights in $\mathrm{Char}_\Theta(V)$. All the roots of $\mathfrak{g}$ appear in the formal character $\mathrm{Char}(V \otimes V^*)$ because the adjoint representation is a subrepresentation. So from now on, we may further assume that $\mathrm{Char}_\Theta(V)$ contains all the roots of $\mathfrak{g}$. The advantage of this assumption is that there are strong geometric connections among roots and weights once we introduce a suitable Euclidean metric on $\Lambda_\mathfrak{g} \otimes \mathbb{R}$.

(**2.3**) Suppose $\mathrm{Char}_\Theta(V) = \alpha_1 + \cdots + \alpha_n$, we define an inner product on $(\Lambda_\mathfrak{g} \otimes \mathbb{R})^*$ in terms of the formal character by setting

$$(x_1, x_2) = \sum_{i=1}^{n} \alpha_i(x_1)\alpha_i(x_2).$$

We denote the dual inner product on $\Lambda_\mathfrak{g} \otimes \mathbb{R}$ by $\langle \ , \ \rangle$. Since Θ is faithful, $\{\alpha_i\}$ spans $\Lambda_\mathfrak{g} \otimes \mathbb{R}$, so $(\ , \ )$ and $\langle \ , \ \rangle$ are positive definite. Since $\mathrm{Char}_\Theta(V)$ is $W_\mathfrak{g}$ invariant, so is $\langle \ , \ \rangle$. This determines $\langle \ , \ \rangle$ up to a positive scalar factor on each simple root system of $\mathfrak{g}$. $W_\mathfrak{g}$ is then a subgroup of the orthogonal group $\mathrm{O}(\Lambda_\mathfrak{g} \otimes \mathbb{R})$ under this Euclidean inner product. Note that $\langle \ , \ \rangle$ is defined solely by the formal character $\mathrm{Char}_\Theta(V)$.

(**2.4**) Now if V and V' are faithful representations of $\mathfrak{g}$ and $\mathfrak{h}$ with the same formal character, then we could assume that the roots of $\mathfrak{g}$ and $\mathfrak{h}$ appear in the formal character by (**2.2**). We can define an Euclidean inner product (which depends only on the formal character) on $\Lambda_\mathfrak{g} \otimes \mathbb{R}$ and on $\Lambda_\mathfrak{h} \otimes \mathbb{R}$ respectively by (**2.3**). Since the formal characters are the same, $\Lambda_\mathfrak{g} \otimes \mathbb{R}$ and $\Lambda_\mathfrak{h} \otimes \mathbb{R}$ are isometric.

Let $v \in \Phi_\mathfrak{g}$ and $u \in \Phi_\mathfrak{h}$. We claim that the angle θ between them with respect to the Euclidean metric above belongs to the set $\{0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°\}$. Indeed, since u (a root of $\mathfrak{h}$) is also a weight of $\mathfrak{g}$ while v (a root of $\mathfrak{g}$) is also a weight of $\mathfrak{h}$, we still have the

following relations [5, §14.1]

$$2\frac{||u||\cos\theta}{||v||} = 2\frac{\langle u,v\rangle}{\langle v,v\rangle} \in \mathbb{Z},$$

$$2\frac{||v||\cos\theta}{||u||} = 2\frac{\langle u,v\rangle}{\langle u,u\rangle} \in \mathbb{Z}.$$

The product of the left hand side is $4\cos^2\theta \in \mathbb{Z}$ and we obtain our claim.

Now, we can determine the ratio $\frac{||u||}{||v||}$.

If $\theta = 0°$ or $180°$, then $\frac{||u||}{||v||}, \frac{||v||}{||u||} \in \frac{\mathbb{Z}}{2}$. We conclude that $\frac{||v||}{||u||} \in \{\frac{1}{2}, 1, 2\}$;

If $\theta = 30°$ or $150°$, then $\frac{||u||}{||v||}, \frac{||v||}{||u||} \in \frac{\mathbb{Z}}{\sqrt{3}}$. We conclude that $\frac{||v||}{||u||} \in \{\frac{1}{\sqrt{3}}, \sqrt{3}\}$;

If $\theta = 45°$ or $135°$, then $\frac{||u||}{||v||}, \frac{||v||}{||u||} \in \frac{\mathbb{Z}}{\sqrt{2}}$. We conclude that $\frac{||v||}{||u||} \in \{\frac{1}{\sqrt{2}}, \sqrt{2}\}$;

If $\theta = 60°$ or $120°$, then $\frac{||u||}{||v||}, \frac{||v||}{||u||} \in \mathbb{Z}$. We conclude that $\frac{||v||}{||u||} = 1$.

Lemma 2.5. Every semisimple Lie algebra $\mathfrak{g}$ contains a semisimple subalgebra $\mathfrak{g}'$ of maximal rank such that every simple factor of $\mathfrak{g}'$ is of type A_n and the number of A_n factors of $\mathfrak{g}$ and $\mathfrak{g}'$ are the same for $n \in \{4, 6, 9, 10, 11, 12, ...\}$.

Proof. By **Table** 1 [6, Table 5] at the end of this section and the following facts [5, Chap. 18, 19]:

$$\mathfrak{so}(3) = A_1,$$
$$\mathfrak{so}(4) = A_1 \times A_1,$$
$$\mathfrak{so}(6) = A_3,$$
$$\mathfrak{sp}(2) = A_1,$$
$$\mathfrak{so}(4) \subset \mathfrak{so}(5) = \mathfrak{sp}(4),$$

one can choose a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ of maximal rank such that every simple factor of $\mathfrak{g}'$ is of type A_n and that the number of A_n factors of $\mathfrak{g}$ and $\mathfrak{g}'$ are the same for $n \in \{4, 6, 9, 10, 11, 12, ...\}$. $\square$

Since $\mathfrak{g}' \subset \mathfrak{g}$ are of same rank, one may assume they have the same Cartan subalgebra. If $\Theta : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ and we take the restriction to $\mathfrak{g}'$, then $\mathrm{Char}_\Theta(V) = \mathrm{Char}_{\Theta|_{\mathfrak{g}'}}(V)$. To prove Theorem 2.14 and Theorem 2.17, we can reduce to the case that every simple factor of our Lie algebras $\mathfrak{g}, \mathfrak{h}$ is of type A_n.

Lemma 2.6. Suppose $\mathfrak{q}$ is a simple factor of $\mathfrak{g}$ of type A_n, $n \geq 2$. Let u be a root of $\mathfrak{h}$ such that $u \notin (\Lambda_\mathfrak{q} \otimes \mathbb{R}) \cup (\Lambda_\mathfrak{q} \otimes \mathbb{R})^\perp \subset \Lambda_\mathfrak{g} \otimes \mathbb{R}$. Then the angle θ between u and any root of $\mathfrak{g}$ can only be $45°$, $60°$, $90°$, $120°$, $135°$.

Proof. We decompose the semisimple Lie algebra $\mathfrak{g}$ into simple factors with $\mathfrak{q}_1 = \mathfrak{q}$.

$$\Lambda_\mathfrak{g} \otimes \mathbb{R} = (\Lambda_{\mathfrak{q}_1} \otimes \mathbb{R}) \perp (\Lambda_{\mathfrak{q}_2} \otimes \mathbb{R}) \perp \cdots \perp (\Lambda_{\mathfrak{q}_m} \otimes \mathbb{R})$$

Since $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp}$, $u \notin \Lambda_{\mathfrak{q}_i} \otimes \mathbb{R}$ for all i. Thus the angle cannot be $0°$ or $180°$. Assume v is a root of $\mathfrak{g}$ such that the angle θ between u and v is $30°$ (if $\theta = 150°$, then choose root $-v$). Since $\mathfrak{q} = \mathfrak{q}_1$ is of type A_n $(n \geq 2)$, we can always choose some root $w \in \Phi_{\mathfrak{q}_1}$ such that $\{u, v, w\}$ are linearly independent and the angle between u and w is not $90°$. Now, consider the group G generated by R_u, R_v, and R_w, the three reflections of hyperplanes corresponding to roots u, v, w. Since each reflection is either in the Weyl group of $\mathfrak{g}$ or in the Weyl group of $\mathfrak{h}$, the formal character $\mathrm{Char}_{\Theta}(V)$ is invariant under G. It follows that G is finite because G permutes weights in $\mathrm{Char}_{\Theta}(V)$ and those weights span $\Lambda_{\mathfrak{g}} \otimes \mathbb{R}$. So we conclude that G is a finite subgroup of $\mathrm{O}(\Lambda_{\mathfrak{g}} \otimes \mathbb{R})$, the orthogonal group of $\Lambda_{\mathfrak{g}} \otimes \mathbb{R}$. Also note that the three-dimensional space spanned by $\{u, v, w\}$ is actually invariant under G. Denote the determinant 1 subgroup of G by G^+, it is a finite subgroup of $\mathrm{SO}(3)$. By the classification of finite subgroups of $\mathrm{SO}(3)$ [1, Chap. 5 Thm. 9.1], G^+ is one of the following:

C_k : The cyclic group of rotations by multiples of $360°/k$ about a line;

D_k : The dihedral group of symmetries of regular k-gon;

T : The tetrahedral group of 12 rotations carrying a regular tetrahedron to itself;

O : The octahedral group of 24 rotations carrying a cube to itself;

I : The icosahedral group of 60 rotations carrying a regular icosahedron to itself.

Note that both the angle between u, v and the angle between u, w are not $90°$, and this means G^+ can only be T, O or I. But the angle between u, v is $30°$. This implies G^+ contains $R_u \circ R_v$, a rotation of order 6 which is impossible. This finishes the proof. $\qquad\square$

(**2.7**) Let $\mathfrak{q}$ be a complex semisimple Lie algebra of type A_n, $n \geq 1$, then $\Lambda_{\mathfrak{q}} \otimes_{\mathbb{Z}} \mathbb{R}$ has dimension n. There exist weights $e_1, ..., e_n, e_{n+1} \in \Lambda_{\mathfrak{q}}$ such that (See [6, Table 1])

$$\Lambda_{\mathfrak{q}} \otimes \mathbb{R} = \mathrm{span}_{\mathbb{R}}\{e_1, ..., e_n, e_{n+1}\},$$

$$e_1 + e_2 + ... + e_n + e_{n+1} = 0,$$

$$\Lambda_{\mathfrak{q}} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n.$$

If we normalize so that the length of roots is $\sqrt{2}$, then

$$\langle e_i, e_j \rangle = \begin{cases} \frac{n}{n+1} & \text{if } 1 \leq i = j \leq n \\ \frac{-1}{n+1} & \text{if } 1 \leq i \neq j \leq n \end{cases}$$

The roots $\Phi_{\mathfrak{q}}$ comprise the set $\{e_i - e_j : 1 \leq i \neq j \leq n+1\}$. We can take $\{e_1, ..., e_n\}$ as a basis of $\Lambda_{\mathfrak{q}} \otimes \mathbb{R}$. The picture for A_2 looks like

The square of the length of a weight $w = a_1 e_1 + \cdots + a_n e_n$ is

$$\langle a_1 e_1 + \cdots + a_n e_n, a_1 e_1 + \cdots + a_n e_n \rangle = \frac{\sum_{i=1}^{n} n a_i^2 - 2 \sum_{i<j} a_i a_j}{n+1}$$

$$(\Omega) \qquad = \frac{\sum_{i=1}^{n} a_i^2 + \sum_{i<j}(a_i - a_j)^2}{n+1} \geq \frac{(n-k) + k(n-k)}{n+1},$$

if k is the number of zero a_i. Thus if $w \neq 0$ satisfies $||w|| < 1$, then

$$1 > \frac{(n-k) + k(n-k)}{n+1},$$

so

$$1 > k(n-k-1),$$

which implies $k = 0$ or $n-1$. Thus, one sees easily that the shortest weight is of length $\sqrt{\frac{n}{n+1}}$. Since n (the "numerator") is the biggest integer smaller than $n+1$ (the "denominator"), it is easy to check that $\sqrt{\frac{n}{n+1}}$ is the only length of weight that is less than 1.

If $||w|| = 1$, then it is not hard to see by investigating inequality (Ω) that

$$||w|| = \frac{(n-k) + k(n-k)}{n+1}.$$

So

$$1 = k(n-k-1),$$

which implies $k = 1$ and $n = 3$.

Proposition 2.8. Suppose $\mathfrak{q}$ is a simple factor of $\mathfrak{g}$ of type A_n, normalized as in (**2.7**). Let u be a root of $\mathfrak{h}$. The orthogonal projection of u to $\Lambda_{\mathfrak{q}} \otimes \mathbb{R}$, denoted by u', belongs to $\Lambda_{\mathfrak{q}}$. We write $u' = a_1 e_1 + \cdots + a_n e_n$. If $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp} \subset \Lambda_{\mathfrak{g}} \otimes \mathbb{R}$, then the following are true.

(1) If the angle θ between u and some root of $\mathfrak{q}$ is $60°$, then either all $a_i \in \{0, 1\}$ or all $a_i \in \{0, -1\}$.
(2) If the angle θ between u and some root of $\mathfrak{q}$ is $45°$, then all $a_i \in \{0, 1\}$, or all $a_i \in \{0, -1\}$, or all $a_i \in \{0, 2\}$, or all $a_i \in \{0, -2\}$.

Proof. Suppose $\mathfrak{g} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_m$ is the decomposition of $\mathfrak{g}$ into a direct sum of simple factors with $\mathfrak{q}_1 = \mathfrak{q}$. We know that the representation ring of $\mathfrak{g}$ is the tensor product over $\mathbb{Z}$ of the representation rings of $\mathfrak{q}_i$ for all i [5, Exercise 23.42]. In other words, any representation of $\mathfrak{g}$ is given by the direct sum of some tensor products of irreducible representation of $\mathfrak{q}_i$. Therefore, if u is a weight of $\mathfrak{g}$ appearing in $\mathrm{Char}_{\Theta}(V)$, then u' is some weight appearing in some irreducible representation of $\mathfrak{q}$, so u' belongs to $\Lambda_{\mathfrak{q}}$.

Since $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp}$ and the roots $\Phi_{\mathfrak{q}}$ span $\Lambda_{\mathfrak{q}} \otimes \mathbb{R}$, there exist some root of $\mathfrak{q}$ such that $\theta \in \{30°, 45°, 60°, 120°, 135°, 150°\}$ by (**2.4**). I claim that if the angle θ between u and some root of $\mathfrak{q}$ is $60°$, then the angle between u and any root of $\mathfrak{q}$ belongs to $\{60°, 90°, 120°\}$, and if $\theta = 45°$, then the angle between u and any root of $\mathfrak{q}$ belongs to $\{45°, 90°, 135°\}$. Suppose not, let $v_1, v_2 \in \Phi_{\mathfrak{q}}$ such that the angle between u and v_1 is $45°$ while the angle between u and v_2 is $60°$ (WLOG). Since $\mathfrak{q}$ is of type A_n, we have

$$\frac{||u||}{||v_1||} = \frac{||u||}{||v_2||}.$$

By (**2.4**), the right hand number and the left hand number belong to $\mathbb{Z} - \{0\}$ and $\frac{\mathbb{Z}}{\sqrt{2}} - \{0\}$ respectively but the two sets are disjoint. So this is impossible.

Let v be a root of $\mathfrak{q}$ and θ the angle between u and v, and write $u' = a_1 e_1 + \cdots + a_n e_n$. First consider $v = e_i - e_j$ where $i, j \leq n$. By (**2.7**), we have

$$\langle u', v \rangle = \langle a_1 e_1 + \cdots + a_n e_n, e_i - e_j \rangle = \frac{a_i(n+1) - \sum_{k=1}^{n} a_k}{n+1} - \frac{a_j(n+1) - \sum_{k=1}^{n} a_k}{n+1} = a_i - a_j.$$

Then consider $v = e_i - e_{n+1}$ where $i \leq n$. By (**2.7**), we have $v = e_1 + e_2 + \cdots + e_{i-1} + 2e_i + e_{i+1} + \cdots + e_n$, so

$$\langle u', v \rangle = \langle a_1 e_1 + \cdots + a_n e_n, e_1 + e_2 + \cdots + e_{i-1} + 2e_i + e_{i+1} + \cdots + e_n \rangle$$

$$= \frac{a_i(n+1) - \sum_{k=1}^{n} a_k}{n+1} + \sum_{j=1}^{n} \frac{a_j(n+1) - \sum_{k=1}^{n} a_k}{n+1} = a_i.$$

If we are in the $60°$ case, then $||u|| = ||v|| = \sqrt{2}$. Therefore, $|a_i|$ and $|a_i - a_j|$ are both of the form

$$|\langle u', v \rangle| = |2\frac{\langle u', v \rangle}{\langle v, v \rangle}| = |2\frac{\langle u, v \rangle}{\langle v, v \rangle}| = |2\cos\theta| = \begin{cases} 1 & \text{if } \theta = 60°, 120° \\ 0 & \text{if } \theta = 90° \end{cases}$$

From this, it is easy to see that either all $a_i \in \{0, 1\}$ or all $a_i \in \{0, -1\}$.

If we are in the $45°$ case, then $||u||$ is either 2 or 1. If $||u|| = 2$, then

$$|a_i|, |a_i - a_j| = |\langle u', v \rangle| = |2\frac{\langle u', v \rangle}{\langle v, v \rangle}| = |2\frac{\langle u, v \rangle}{\langle v, v \rangle}| = |2\sqrt{2}\cos\theta| = \begin{cases} 2 & \text{if } \theta = 45°, 135° \\ 0 & \text{if } \theta = 90°. \end{cases}$$

When $||u|| = 1$, then

$$|a_i|, |a_i - a_j| = |\langle u', v \rangle| = |2\frac{\langle u', v \rangle}{\langle v, v \rangle}| = |2\frac{\langle u, v \rangle}{\langle v, v \rangle}| = |\sqrt{2}\cos\theta| = \begin{cases} 1 & \text{if } \theta = 45°, 135° \\ 0 & \text{if } \theta = 90°. \end{cases}$$

From this, it is easy to see either all $a_i \in \{0, 1\}$ or all $a_i \in \{0, -1\}$ or all $a_i \in \{0, 2\}$ or all $a_i \in \{0, -2\}$. $\qquad\square$

(**2.9**) Again, suppose $\mathfrak{q}$ is a simple factor of $\mathfrak{g}$ of type A_n. Let u be a root of $\mathfrak{h}$ such that $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp} \subset \Lambda_{\mathfrak{g}} \otimes \mathbb{R}$. Let $\mathfrak{q}'$ be a simple factor of $\mathfrak{g}$ of rank n' such that the orthogonal projection u' of u to $\Lambda_{\mathfrak{q}'} \otimes \mathbb{R}$ is non-zero. Write $u' = a_1 e_1 + \cdots + a_{n'} e_{n'}$ as in (**2.7**). Let k' be the number of zero coefficients. Normalize so that $||u||^2 = 2$. We can use Proposition 2.8 (coordinates computation) to compute $||u'||^2$. Consider the following 3 cases:

(1) If the angle between u and some root of $\mathfrak{q}'$ is $60°$, then by (**2.4**) the length of roots of $\mathfrak{q}'$ is $\sqrt{2}$. By using (**2.7**) and Proposition 2.8, we have

$$||u'||^2 = ||a_1 e_1 + \cdots + a_{n'} e_{n'}||^2 = \frac{\sum_{i=1}^{n'} a_i^2 + \sum_{1 \leq i < j \leq n'} (a_i - a_j)^2}{n' + 1} = \frac{(n' - k')(k' + 1)}{n' + 1}.$$

(2) If the angle between u and some root of $\mathfrak{q}'$ is $45°$ and assuming the length of roots of $\mathfrak{q}'$ is 2 (see $(\mathbf{2.4})$), then we multiply by 2 and use $(\mathbf{2.7})$ and Proposition 2.8 to get

$$||u'||^2 = 2||a_1e_1 + \cdots + a_{n'}e_{n'}||^2 = \frac{2(\sum_{i=1}^{n'} a_i^2 + \sum_{1\leq i<j\leq n'}(a_i - a_j)^2)}{n' + 1} = \frac{2(n' - k')(k' + 1)}{n' + 1}.$$

(3) If the angle between u and some root of $\mathfrak{q}'$ is $45°$ and assuming the length of roots of $\mathfrak{q}'$ is 1, then we multiply $1/2$ and use $(\mathbf{2.7})$ and Proposition 2.8 to get

$$||u'||^2 = \frac{||a_1e_1 + \cdots + a_{n'}e_{n'}||^2}{2} = \frac{\sum_{i=1}^{n'} a_i^2 + \sum_{1\leq i<j\leq n'}(a_i - a_j)^2}{2(n' + 1)}$$
$$= \frac{4(n' - k')(k' + 1)}{2(n' + 1)} = \frac{2(n' - k')(k' + 1)}{n' + 1}.$$

Note that $(n' - k') + (k' + 1) = n' + 1$, so it is easy to see $||u'||^2 \geq \frac{1}{2}$ for the $60°$ case and $||u'||^2 \geq 1$ for the $45°$ case.

$(\mathbf{2.10})$ The computations in $(\mathbf{2.9})$ imply strong restrictions on the lengths of various projections of the root u. Suppose we are in the situation of $(\mathbf{2.9})$ and $\mathfrak{q} = \mathfrak{q}_1$ is of type A_n, $n \geq 4$. Consider the situation that there exists another simple factor $\mathfrak{q}_2$ of rank n_2 of $\mathfrak{g}$ such that the projection u_2 of u to $\Lambda_{\mathfrak{q}_2} \otimes \mathbb{R}$ is non-zero.

Let's first show that the angle between u and any root of $\mathfrak{q}_2$ cannot be $30°$ or $150°$. Suppose not, choose root v_2 of $\mathfrak{q}_2$ that makes angle $30°$ with u. Choose a root v_1 of $\mathfrak{q}_1$ so that the angle θ between v_1 and u is less than $90°$ by $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp}$. $(\mathbf{2.4})$ implies θ is $30°$ or $45°$ or $60°$. By considering the Euclidean space spanned by u, v_1, v_2, one sees easily that the first two cases are impossible because v_1 and v_2 are perpendicular. For the case $\theta = 60°$, one observes that u lies on the plane spanned by roots v_1, v_2. Since $n \geq 2$ and $u \notin (\Lambda_{\mathfrak{q}} \otimes \mathbb{R}) \cup (\Lambda_{\mathfrak{q}} \otimes \mathbb{R})^{\perp}$, the angle between any root of $\mathfrak{g}_1$ and u belongs to $\{60°, 90°, 120°\}$ by $(\mathbf{2.4})$ and one can choose a root v_1' of $\mathfrak{q}_1$ making an angle of $60°$ with u by the second paragraph of the proof of Proposition 2.8. This is also absurd if one consider the Euclidean 3-space spanned by v_1, v_1', v_2. Therefore, we can apply the results in $(\mathbf{2.9})$.

Let u_1 be the projection of u to $\Lambda_{\mathfrak{q}_1} \otimes \mathbb{R}$. Then we have

$$2 = ||u||^2 \geq ||u_1||^2 + ||u_2||^2.$$

If the angle between u and some root of $\mathfrak{q}_1$ is $45°$, then by $(\mathbf{2.9})$ and above

$$2 \geq ||u_1||^2 + ||u_2||^2 \geq \frac{2n}{n + 1} + \frac{n_2}{n_2 + 1} \geq \frac{2n}{n + 1} + \frac{1}{2},$$

which is impossible when $n \geq 4$.

If the angle between u and some root of $\mathfrak{q}_1$ is $60°$ and the angle between u and some root of $\mathfrak{q}_2$ is $45°$, then by the calculations above,

$$2 \geq ||u_1||^2 + ||u_2||^2 \geq ||u_1||^2 + \frac{2n_2}{n_2 + 1} \geq ||u_1||^2 + 1.$$

By (**2.7**) and $n \geq 4$, the only length of weight that is less than or equal to 1 is $\sqrt{\frac{n}{n+1}}$. Therefore, $||u_1||^2 = \frac{n}{n+1}$. But then

$$2 \geq ||u_1||^2 + ||u_2||^2 \geq \frac{n}{n+1} + \frac{2n_2}{n_2 + 1} \geq \frac{n}{n+1} + 1.$$

This implies

$$||u||^2 - (||u_1||^2 + ||u_2||^2) \leq 2 - (\frac{n}{n+1} + 1) = \frac{1}{n+1}.$$

Note that $\frac{1}{n+1} < \frac{1}{2}$ when $n \geq 2$. Therefore, by the last line of (**2.9**), we conclude that

$$||u||^2 = ||u_1||^2 + ||u_2||^2,$$

and hence,

$$2 = \frac{n}{n+1} + \frac{2(n_2 - k_2)(k_2 + 1)}{n_2 + 1}.$$

It is easy to check that this is not solvable for $n \geq 1$. Therefore, there does not exist a root $v \in \Phi_\mathfrak{g}$ such that the angle between v and u is $45°$. Using similar arguments, we see if n is big (at least ≥ 4), then u can only have two components. i.e.

$$2 = ||u||^2 = ||u_1||^2 + ||u_2||^2.$$

Since no weight of A_n has length 1 for $n \geq 4$, one of the $||u_1||^2, ||u_2||^2$ is smaller than 1 and the other is bigger than 1. Since $\sqrt{\frac{m}{m+1}}$ is the only length of weight smaller than 1 in A_m, together with (**2.9**), this gives the following Diophantine equation for positive integers m, l, k:

(D) $$2 = \frac{m}{m+1} + \frac{(l-k)(k+1)}{l+1}.$$

Proposition 2.11. The solutions of Diophantine equation (D) are $(m, l, k) = (1, 5, 2)$, $(1, 7, 1)$, $(1, 7, 5)$, $(2, 5, 1)$, $(2, 5, 3)$, $(4, 4, 1)$ and $(4, 4, 2)$.

Proof. Write $a = l - k$ and $b = k + 1$. Since m is positive, thus we have

$$\frac{ab}{a + b} < 2$$

which is equivalent to

$$(a - 2)(b - 2) \leq 3.$$

If $a - 2$ and $b - 2$ are non-zero, then there are only finitely many possible pairs (a, b). In this case, there exists only one solution $(m, l, k) = (1, 5, 2)$ which corresponds to $(a, b) = (3, 3)$.

If $a - 2 = 0$, then the equation becomes

$$2 = \frac{m}{m+1} + \frac{2b}{b+2}$$

which is equivalent to

$$m(b - 2) = 4.$$

Therefore, we see $(m, b) = (1, 6), (2, 4), (4, 3)$ and we deduce easily that $(m, l, k) = (1, 7, 1)$, $(1, 7, 5)$, $(2, 5, 1)$, $(2, 5, 3)$, $(4, 4, 1)$, $(4, 4, 2)$. $\qquad\square$

($\mathbf{2.12}$) From Proposition 2.11, we see that the m, l coordinates of solutions of (D) do not belong to $\{6, 8, 9, 10, 11, ...\}$. Therefore, there does not exist a root $u \in \Phi_\mathfrak{h}$ such that $u \notin (\Lambda_\mathfrak{q} \otimes \mathbb{R}) \cup (\Lambda_\mathfrak{q} \otimes \mathbb{R})^\perp$ for any A_n factor $\mathfrak{q}$ of $\mathfrak{g}$ where $n \in \{6, 8, 9, 10, 11, ...\}$. In other words, any root $u \in \Phi_\mathfrak{h}$ must either be contained in or perpendicular to $\Lambda_\mathfrak{q} \otimes \mathbb{R} \subset \Lambda_\mathfrak{g} \otimes \mathbb{R}$. On the other hand, since any simple root system is irreducible, there exist simple factors $\mathfrak{p}_1, ..., \mathfrak{p}_s$ of $\mathfrak{h}$ such that $\Lambda_\mathfrak{q} \otimes \mathbb{R} = (\Lambda_{\mathfrak{p}_1} \otimes \mathbb{R}) \oplus \cdots \oplus (\Lambda_{\mathfrak{p}_s} \otimes \mathbb{R})$. If we consider the sum of all the weights in the formal character sitting in this subspace of $t_\mathfrak{g}^*$, then it is actually a formal character coming from some faithful representations of $\mathfrak{q}$ and $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$ [5, Exercise 23.42]. Thus, we reduce our problem to the case that $\mathfrak{g}$ is a simple complex Lie algebra of type A_n, $n \in \{6, 8, 9, 10, 11, ...\}$. If we can prove that $s = 1$, then $\mathfrak{p}_1 = \mathfrak{q}$ and the number of A_n factors of $\mathfrak{g}$ and $\mathfrak{h}$ for $n \in \{6, 8, 9, 10, 11, ...\}$ are the same.

($\mathbf{2.13}$) Suppose $\mathfrak{g}$ is simple of type A_n, $n \in \{4, 5, 6, 9, 10, 11, ...\}$ and $\mathfrak{h} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$ is a direct sum of simple factors of type A_n. Assume the length of roots of $\mathfrak{g}$ is $\sqrt{2}$. Let u be a root of $\mathfrak{h}$ and θ the angle between u and some root $v \in \Phi_\mathfrak{g}$. Consider the following 3 cases:

$\theta = 30°$ This is impossible because we can always choose some root v' (since $n \geq 3$) of $\mathfrak{g}$ that makes an angle of $30°$ with u such that $\{u, v, v'\}$ spans a 3-dimensional space and then use the argument in Lemma 2.6.

$\theta = 45°$ Then u is either of length 1 or 2 by ($\mathbf{2.4}$). It is impossible for u and some root of $\mathfrak{g}$ to be parallel because $\theta = 45°$. Hence, the angle between u and any root of $\mathfrak{g}$ belongs to $\{45°, 90°, 135°\}$. By Proposition 2.8 and computations in ($\mathbf{2.9}$), we deduce that

$$4 = ||u||^2 = \frac{4(n-k)(k+1)}{n+1} \qquad \text{or} \qquad 1 = ||u||^2 = \frac{(n-k)(k+1)}{n+1}.$$

Hence, we for both cases obtain the equation

$$k(n - k - 1) = 1$$

which implies $(n, k) = (3, 1)$. This is impossible since $3 \notin \{4, 5, 6, 9, 10, 11, ...\}$.

$\theta = 60°$ Length of u is $\sqrt{2}$ by ($\mathbf{2.4}$). If u is not equal to any root of $\mathfrak{g}$, then the angle between u and any root of $\mathfrak{g}$ belongs to $\{60°, 90°, 120°\}$. By Proposition 2.8 and computations in ($\mathbf{2.9}$),

$$2 = ||u||^2 = \frac{(n-k)(k+1)}{n+1}.$$

$$2(n+1) = (n-k)(k+1).$$

Let $N = n+1$ and $K = k+1$, we have

$$K^2 - NK + 2N = 0.$$

Since $N^2 - 8N$, the discriminant of the left hand side is a perfect square Δ^2, thus we have

$$(N - \Delta)(N + \Delta) = 8N.$$

Assume Δ is non-negative, we have $1 \le (N - \Delta) \le 8$. So we only have 8 cases to consider: $(N - \Delta) \in \{1, 2, ..., 8\}$. We find that N can only be $8, 9$, thus n can only be $7, 8$ which do not belong to $\{4, 5, 6, 9, 10, ...\}$.

So $\theta = 0°$ or $180°$ and we conclude that any root u of $\mathfrak{h}$ lies on the line spanned by a root $\mathfrak{g}$. If we can prove that $\mathfrak{h}$ is simple, then $\mathfrak{g} = \mathfrak{h}$ and we are done. Write $\mathfrak{h} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_s$ and let n_i be the rank of $\mathfrak{p}_i$. If we take a base S_i of $\mathfrak{p}_i$ for all $1 \le i \le s$, then S_i is a subset of $\Phi_{\mathfrak{g}}$ for all i and S_1 is orthogonal to S_i for $i \ge 2$. Note that the union of all S_i forms a basis of $\Lambda_{\mathfrak{g}} \otimes \mathbb{R}$, therefore $S_2 \cup \cdots \cup S_s$ spans a space of dimension $n - n_1$. By $(\mathbf{2.7})$, the set of roots of $\mathfrak{g}$ is given by $\{e_i - e_j : 1 \le i \ne j \le n+1\}$. Without loss of generality, we may assume

$$S_1 = \{c(e_1 - e_2), c(e_2 - e_3), ..., c(e_{n_1} - e_{n_1+1})\}$$

for some constant $c \in \{1/2, 1, 2\}$ by $(\mathbf{2.4})$. If $a_1 e_1 + \cdots + a_n e_n$ is orthogonal to S_1, then $a_1 = a_2 = \cdots = a_{n_1} = a_{n_1+1}$ by $(\mathbf{2.7})$. Therefore, the set of roots of $\mathfrak{g}$ that are orthogonal to S_1 is

$$\{e_i - e_j : n_1 + 1 \le i \ne j \le n+1\}$$

which spans a space of dimension $n - n_1 - 1$. It contradicts that $S_2 \cup \cdots \cup S_s$ spans a space of dimension $n - n_1$. So we conclude that $\mathfrak{h}$ is simple and is equal to $\mathfrak{g} = A_n$ if $n \in \{4, 5, 6, 9, 10, 11, ...\}$. If we consider the intersection of the two sets $\{6, 8, 9, 10, 11, ...\}$ and $\{4, 5, 6, 9, 10, 11, ...\}$ in $(\mathbf{2.12})$ and $(\mathbf{2.13})$, then we can get the following theorem.

Theorem 2.14 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then the number of A_n factors of $\mathfrak{g}$ and $\mathfrak{h}$ are the same when $n \in \{6, 9, 10, ...\}$.

Proof. First, we may assume each faithful representation contains the adjoint representation as a subrepresentation by $(\mathbf{2.2})$. Hence the roots $\Phi_{\mathfrak{g}}$, $\Phi_{\mathfrak{h}}$ really appear in the formal character. Since we only care about the number of A_n factors for $n \in \{6, 9, 10, ...\}$, we assume $\mathfrak{g}$ and $\mathfrak{h}$ only have simple factors of type A_n by Lemma 2.5. Now if $\mathfrak{q}$ is a simple factor of $\mathfrak{g}$ with rank $n \in \{6, 9, 10, ...\}$ and $u \in \Phi_{\mathfrak{h}}$, $(\mathbf{2.12})$ says that u is either contained in or perpendicular to $\Lambda_{\mathfrak{q}} \otimes \mathbb{R}$. So we may reduce to the case that $\mathfrak{g}$ is $\mathfrak{q}$ and if we can prove that $\mathfrak{h}$ is also simple, then we are done. Finally $(\mathbf{2.13})$ finishes that part. $\square$

$(\mathbf{2.15})$ We are going to prove that the parities of the number of A_4 factors of $\mathfrak{g}$ and $\mathfrak{h}$ are equal. Again we may assume our Lie algebras only have simple factors of type A_n by Lemma 2.5 and the roots of $\mathfrak{h}$ appear in the formal character by $(\mathbf{2.2})$. Therefore, the roots of $\mathfrak{h}$

belong to $\Lambda_{\mathfrak{g}}$. Suppose

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$
$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

are decompositions into a direct sum of proper ideals such that $\Lambda_{\mathfrak{g}_1} \otimes \mathbb{R} = \Lambda_{\mathfrak{h}_1} \otimes \mathbb{R}$. Then we also have $\Lambda_{\mathfrak{g}_2} \otimes \mathbb{R} = \Lambda_{\mathfrak{h}_2} \otimes \mathbb{R}$ by orthogonality. If this is the case, then we say the pair $(\mathfrak{g} \curvearrowright V, \mathfrak{h} \curvearrowright V')$ of representations is *reducible*. It is because the weights of the original formal character sitting inside $\Lambda_{\mathfrak{g}_i} \otimes \mathbb{R} = \Lambda_{\mathfrak{h}_i} \otimes \mathbb{R}$ come from some faithful representations of $\mathfrak{g}_i$ and $\mathfrak{h}_i$, $i = 1, 2$. We can reduce our problem to some pairs $(\mathfrak{g}_i \curvearrowright V_i, \mathfrak{h}_i \curvearrowright V_i')$ for $i = 1, 2$. Now we assume that $(\mathfrak{g} \curvearrowright V, \mathfrak{h} \curvearrowright V')$ is not reducible. If $v \in \Phi_{\mathfrak{g}}$ and $u \in \Phi_{\mathfrak{h}}$, then by (**2.4**) we have

$$\frac{\langle v, v \rangle}{\langle u, u \rangle} \in 2^a 3^b$$

for some integers a, b.

(**2.16**) Assume the pair $(\mathfrak{g} \curvearrowright V, \mathfrak{h} \curvearrowright V')$ is not reducible. Let n be their common rank and write

$$\mathfrak{g} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_m$$
$$\mathfrak{h} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$$

as direct sums of simple factors. Let the rank of $\mathfrak{q}_i$, $\mathfrak{p}_j$ be n_i, r_j respectively. The standard basis for the weight lattice of $\mathfrak{q}_i$, $\Lambda_{\mathfrak{q}_i}$, $1 \le i \le m$ is given by

$$\{e_1^i, e_2^i, ..., e_{n_i}^i\}.$$

So a basis for the weight lattice of $\mathfrak{g}$, $\Lambda_{\mathfrak{g}}$ is given by

$$B = \{e_1^1, e_2^1, ..., e_{n_1}^1, e_1^2, e_2^2, ..., e_{n_2}^2, ..., e_1^m, e_2^m, ..., e_{n_m}^m\}.$$

If we normalize the inner product $\langle \; , \; \rangle$ so that the length of roots of $\mathfrak{q}_1$ is $\sqrt{2}$, then by (**2.15**) the positive definite $n \times n$ matrix Q defining $\langle \; , \; \rangle$ on $\Lambda_{\mathfrak{g}} \otimes \mathbb{R}$ under the basis B is

$$Q = \begin{pmatrix} Q_1 & 0 & ... & 0 \\ 0 & \gamma_2 Q_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \gamma_m Q_m \end{pmatrix},$$

where $\gamma_i \in 2^{\mathbb{Z}} 3^{\mathbb{Z}}$ by (**2.15**) and Q_i is an $n_i \times n_i$ matrix in the following form by (**2.7**) for $1 \le i \le m$:

$$Q_i = \begin{pmatrix} \frac{n_i}{n_i+1} & \frac{-1}{n_i+1} & \cdots & \frac{-1}{n_i+1} \\ \frac{-1}{n_i+1} & \frac{n_i}{n_i+1} & \cdots & \frac{-1}{n_i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n_i+1} & \frac{-1}{n_i+1} & \cdots & \frac{n_i}{n_i+1} \end{pmatrix}.$$

If we take a base $S = \{u_1, ..., u_n\}$ of root system of $\mathfrak{h}$, then it defines an $n \times n$ matrix A whose i-th column is u_i in terms of the basis B. Since all u_i belong to the weight lattice of

$\mathfrak{g}$, A is a matrix of integral entries. If we enumerate S in a proper way, then we have the following equation

$$A^t Q A = \begin{pmatrix} \mu_1 C_1 & 0 & \ldots & 0 \\ 0 & \mu_2 C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_s C_s \end{pmatrix},$$

where $\mu_j \in 2^{\mathbb{Z}} 3^{\mathbb{Z}}$ by (**2.15**) and C_j is the Cartan matrix of $\mathfrak{p}_j$ for $1 \le j \le s$. We know that the determinant of the Cartan matrix C_j is $r_j + 1$ and it is easy to check that the determinant of Q_i is $\frac{1}{n_i+1}$. Therefore, by taking determinant of the matrix equation, we get

$$\frac{\gamma_2 \ldots \gamma_m \det(A)^2}{(n_1 + 1) \ldots (n_m + 1)} = \mu_1 \ldots \mu_s (r_1 + 1) \ldots (r_s + 1).$$

We see $\det(A)$ is an integer and $\gamma_i, \mu_j \in 2^{\mathbb{Z}} 3^{\mathbb{Z}}$. If we reduce this equation modulo $(\mathbb{Q}^*)^2$, then by the unique factorization of primes we see the parities of $\mathrm{ord}_5(n_1 + 1) \ldots (n_m + 1)$ and $\mathrm{ord}_5(r_1 + 1) \ldots (r_s + 1)$ are the same. By Theorem 2.14, the number of A_n factors for $n \ge 9$ is invariant, so this implies the parity of number of A_4 factors is an invariant because $4 + 1 = 5$. Thus, we have just proved the following theorem.

Theorem 2.17 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then the parities of the numbers of A_4 factors of $\mathfrak{g}$ and $\mathfrak{h}$ are the same.

Definition 2.18 Consider the free abelian group $\mathfrak{F}$ of virtual complex simple Lie algebras. Semisimple Lie algebras are subset $\mathfrak{F}$ in a natural way. We divide by the subgroup $\mathfrak{D}$ generated by all expressions $\mathfrak{g} - \mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{g}$ are semisimple of same rank. We say that two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ satisfy *equal-rank subalgebra equivalence*, denoted by $\mathfrak{g} \approx \mathfrak{h}$, if they have same image in $\mathfrak{F}/\mathfrak{D}$.

Theorem 2.19 follows easily from the results above.

Theorem 2.19 If faithful representations of two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same formal character, then $\mathfrak{g}$ and $\mathfrak{h}$ satisfy equal-rank subalgebra equivalence.

Proof. By Lemma 2.5 we may assume our Lie alegbras consist of simple factors of type A_n. By Theorem 2.14, we further assume the simple factors are of types A_n, $n \in \{1, 2, 3, 4, 5, 7, 8\}$. From **Table** 1, the following Lie algebras are equivalent:

  (1) $A_4 \times A_4 \approx A_8 \approx A_2 \times A_2 \times A_2 \times A_2$;
  (2) $A_7 \approx A_2 \times A_5$;
  (3) $A_1 \times A_5 \approx A_2 \times A_2 \times A_2$;
  (4) $A_2 \approx A_1 \times A_1$.

Hence by Theorem 2.17 and the list above, we may assume $n \in \{1, 3\}$ and it suffices to prove that $A_3 \approx A_1 \times A_1 \times A_1$. But we know that

$$A_3 \times A_1 \approx \mathfrak{so}(6) \times \mathfrak{so}(3) \approx \mathfrak{so}(4) \times \mathfrak{so}(5) \approx \mathfrak{so}(4) \times \mathfrak{so}(4) \approx A_1 \times A_1 \times A_1 \times A_1.$$

So we are done. $\square$

Proposition 2.20 Two semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ satisfy equal-rank subalgebra equivalence if and only if the number of A_n factors when $n \in \{6, 9, 10, 11, ...\}$ and the parity of the number of A_4 factors of them are the same.

Proof. We have seen that equal-rank subalgebra equivalence is implied by the conditions on A_n factors in the proof of the above theorem. Suppose two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of the same rank satisfy equal-rank subalgebra equivalence, then in the free group $\mathfrak{F}$ we have

$$\mathfrak{g} - \mathfrak{h} = \mathfrak{h}_1 - \mathfrak{g}_1 + \mathfrak{h}_2 - \mathfrak{g}_2 + \cdots + \mathfrak{h}_k - \mathfrak{g}_k$$

where $\mathfrak{h}_i \subset \mathfrak{g}_i$ or $\mathfrak{g}_i \subset \mathfrak{h}_i$ and are of the same rank for all i. Thus we obtain

$$\mathfrak{g} + \mathfrak{g}_1 + \cdots + \mathfrak{g}_k = \mathfrak{h} + \mathfrak{h}_1 + \cdots + \mathfrak{h}_k$$

By Theorem 2.14 and 2.17, one sees easily by taking a faithful representation of $\mathfrak{g}_i$ or $\mathfrak{h}_i$ that for each i, $\mathfrak{g}_i$ and $\mathfrak{h}_i$ have the the number of A_n factors for $n \in \{6, 9, 10, ...\}$ and the parity of the numbers of A_4 factors. So do $\mathfrak{g}$ and $\mathfrak{h}$ by the equation above. $\square$

Table 1 [6, Table 5]. The table lists some semisimple maximal subalgebras $\mathfrak{f}$ of maximal rank in simple complex Lie algebra $\mathfrak{g}$ (up to conjugacy in $\mathfrak{g}$).

$\mathfrak{g}$	$\mathfrak{f}$
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $l \geq 2$	$\mathfrak{so}_{2k}(\mathbb{C}) \oplus \mathfrak{so}_{2(l-k)+1}(\mathbb{C})$ $2 \leq k \leq l$
$\mathfrak{sp}_{2l}(\mathbb{C})$ $l \geq 3$	$\mathfrak{sp}_{2k}(\mathbb{C}) \oplus \mathfrak{sp}_{2(l-k)}(\mathbb{C})$ $1 \leq k \leq [\frac{l}{2}]$
$\mathfrak{so}_{2l}(\mathbb{C})$ $l \geq 4$	$\mathfrak{so}_{2k}(\mathbb{C}) \oplus \mathfrak{so}_{2(l-k)}(\mathbb{C})$ $2 \leq k \leq \frac{l+1}{2}$
E_6	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_6(\mathbb{C})$ $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$
E_7	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_{12}(\mathbb{C})$ $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_6(\mathbb{C})$ $\mathfrak{sl}_8(\mathbb{C})$
E_8	$\mathfrak{sl}_2(\mathbb{C}) \oplus E_7$ $\mathfrak{sl}_3(\mathbb{C}) \oplus E_6$ $\mathfrak{sl}_5(\mathbb{C}) \oplus \mathfrak{sl}_5(\mathbb{C})$ $\mathfrak{so}_{16}(\mathbb{C})$ $\mathfrak{sl}_9(\mathbb{C})$
F_4	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_6(\mathbb{C})$ $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$ $\mathfrak{so}_9(\mathbb{C})$
G_2	$\mathfrak{sl}_3(\mathbb{C})$ $\mathfrak{so}_4(\mathbb{C})$

§3 Compatible system of ℓ-adic representations and ℓ-independence

(**3.1**) We follow the terminology and notations of Serre [19]. Let K be a number field, G_K its absolute Galois group, $\sum_K$ the set of all finite places of K and ℓ a prime number. An *ℓ-adic representation of K* is a continuous homomorphism $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ for some n. Here G_K is equipped with the profinite topology and $\mathrm{GL}_n(\mathbb{Q}_\ell)$ is an ℓ-adic Lie group. An *abelian ℓ-adic representation of K* is an ℓ-adic representation of G_K that factors through G_K^{ab}.

(**3.2**) Let $v \in \sum_K$ and p_v the characteristic of the residue field k_v of the place v. If w is a valuation of $\overline{K}$ extending v, we denote the decomposition group, inertia group and Frobenius element of w by D_w, I_w and F_w respectively. D_w and I_w are closed subgroups of G_K. We say that ρ is unramified at v if $\rho(I_w)$ is trivial for any valuation w of $\overline{K}$ extending v. If the representation ρ is unramified at v, then the restriction of ρ to D_w factors through D_w/I_w and $\rho(F_w)$ is defined for any $w|v$. We denote it by $F_{w,\rho}$ and the conjugacy class of $F_{w,\rho}$ in $\mathrm{GL}_n(\mathbb{Q}_\ell)$ by $F_{v,\rho}$.

Definition 3.3. An ℓ-adic representation ρ is said to be *rational* if there exists a finite subset S of $\sum_K$ such that:

  (a) Any element of $\sum_K \backslash S$ is unramified with respect to ρ.
  (b) If $v \notin S$, the coefficients of $P_{v,\rho}(T) := \det(1 - F_{v,\rho}T)$ belongs to $\mathbb{Q}$.

Definition 3.4. Let ℓ' be a prime, ρ' an ℓ'-adic representation of K, and assume that ρ, ρ' are rational. Then ρ, ρ' are said to be *compatible* if there exists a finite subset S of $\sum_K$ such that ρ and ρ' are unramified outside of S and $P_{v,\rho}(T) = P_{v,\rho'}(T)$ for $v \in \sum_K \backslash S$.

Definition 3.5. Let $\mathscr{P}$ be the set of prime numbers. For each prime ℓ let ρ_ℓ be a rational ℓ-adic representation of K. The system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ is said to be *compatible* if ρ_ℓ, $\rho_{\ell'}$ are compatible for any two primes ℓ, ℓ'. The system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ is said to be *strictly compatible* if there exists a finite subset S of $\sum_K$ such that:

  (a) Let $S_\ell = \{v :\ p_v = \ell\}$. Then, for every $v \notin S \cup S_\ell$, ρ_ℓ is unramified at v and $P_{v,\rho_\ell}(T)$ has rational coefficients.
  (b) $P_{v,\rho_\ell}(T) = P_{v,\rho_{\ell'}}(T)$ if $v \notin S \cup S_\ell \cup S_{\ell'}$.

(**3.6**) Serre associates to every number field K a projective family $\{S_\mathfrak{m}\}$ of commutative algebraic groups over $\mathbb{Q}$ and shows that each $S_\mathfrak{m}$ gives rise to a compatible system of rational ℓ-adic representation of K ([19, Chap. 2, §1, 2]). We give a brief introduction to the algebraic groups $S_\mathfrak{m}$ associated to the number field K. Let S be a finite subset of $\sum_K$. Then by a *modulus of support S* we mean a family $\mathfrak{m} = (m_v)_{v \in S}$ where the m_v are integers ≥ 1. If E is the group of units of K,

$$E_\mathfrak{m} := \{u \in E : v(1 - u) \geq m_v \ for \ all \ v \in S\}.$$

Then $S_\mathfrak{m}$ is an algebraic group over $\mathbb{Q}$ whose connected component is

$$T_\mathfrak{m} := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m/K})/\overline{E_\mathfrak{m}},$$

where $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m/K})$ is obtained from the multiplicative group $\mathbb{G}_m$ by restriction of scalars from K to $\mathbb{Q}$ and $\overline{E_\mathfrak{m}}$ is the Zariski closure of $E_\mathfrak{m}$ in the algebraic torus $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m/K})$. $S_\mathfrak{m}$

are called the *Serre groups* associated to K.

Definition 3.7. Let H be a linear algebraic group over $\mathbb{Q}$, and let K be a number field. A continuous homomorphism $\rho : G_K \to H(\mathbb{Q}_\ell)$ is called an *ℓ-adic representation of K with values in H*.

One defines in an analogous way what it means for ρ to be unramified at a place or rational and for a system $\{\rho_\ell\}$ to be compatible or strictly compatible. Using class field theory, Serre defines for each prime ℓ and for each modulus $\mathfrak{m}$ an ℓ-adic representations ϵ_ℓ with values in $S_\mathfrak{m}$ (Definition 3.7)

$$\epsilon_\ell : G_K^{\mathrm{ab}} \longrightarrow S_\mathfrak{m}(\mathbb{Q}_\ell).$$

Theorem 3.8.
   (1) The dimensions of the Serre groups associated to K only depend on K. Denote the common dimension by d_K.
   (2) [19, Chap. 2.2.3] The image of ϵ_ℓ is an open subgroup of $S_\mathfrak{m}(\mathbb{Q}_\ell)$ and is Zariski dense in $S_\mathfrak{m}$.
   (3) [19, Chap. 2.2.5] Let $\phi : S_\mathfrak{m} \to \mathrm{GL}_{n,\mathbb{Q}}$ be a $\mathbb{Q}$-morphism, then the representation

$$\phi \circ \epsilon_\ell : G_K^{\mathrm{ab}} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$$

   is semisimple and $\{\epsilon_\ell\}_{\ell \in \mathscr{P}}$ is strictly compatible.
   (4) The dimension of $S_\mathfrak{m}(\mathbb{Q}_\ell)$ as an ℓ-adic Lie group is equal to d_K for all ℓ.

Proof. We have provided references for (2) and (3), so we just need to prove (1) and (4). Since $E_\mathfrak{m}$ is of finite index in E [19, Chap. 2.2.1], Serre groups have the same dimension. By taking an $\mathbb{Q}$-embedding ϕ of $S_\mathfrak{m}$ to some $\mathrm{GL}_{n,\mathbb{Q}}$, one obtains (4) by the algebraicity of the Lie algebra of $\phi \circ \epsilon_\ell(G_K)$ [7, Thm. 4], (2) and (3) of this theorem. $\square$

(**3.9**) We are going to define local algebraicity for an abelian ℓ-adic representation of a number field K. We first need to define local algebraicity for K, a finite extension of $\mathbb{Q}_\ell$. Let $T = \mathrm{Res}_{K/\mathbb{Q}_\ell}(\mathbb{G}_{m/K})$ be the algebraic torus over $\mathbb{Q}_\ell$ and V a finite dimensional vector space over $\mathbb{Q}_\ell$. If $i : K^* \to G_K^{\mathrm{ab}}$ is the canonical homomorphism of local class field theory and $\rho : G_K^{\mathrm{ab}} \to \mathrm{Aut}(V)$ is an abelian ℓ-adic representation of K, we then get a continuous homomorphism $\rho \circ i$ of $K^* = T(\mathbb{Q}_\ell)$ into $\mathrm{Aut}(V)$.

Definition 3.10. The representation ρ is said to be *locally algebraic* if there is an algebraic morphism $r : T \to \mathrm{GL}_V$ such that $\rho \circ i(x) = r(x^{-1})$ for all $x \in K^*$ close enough to 1.

(**3.11**) Now if K is a number field and V_ℓ is a finite dimensional vector space over $\mathbb{Q}_\ell$. Let

$$\rho : G_K^{\mathrm{ab}} \longrightarrow \mathrm{Aut}(V_\ell)$$

be an abelian ℓ-adic representation of K. Let $v \in \sum_K$ be a place of K of residue characteristic ℓ and let $D_v \subset G_K^{\mathrm{ab}}$ be the corresponding decomposition group which is isomorphic to $\mathrm{Gal}(\overline{K_v}/K_v)^{\mathrm{ab}}$. Hence, we get an ℓ-adic representation of K_v by composition

$$\rho_v : \mathrm{Gal}(\overline{K_v}/K_v)^{\mathrm{ab}} \to D_v \to \mathrm{Aut}(V_\ell).$$

Definition 3.12. The representation ρ is said to be *locally algebraic* if all the local representations ρ_v, with $p_v = \ell$, are locally algebraic in the sense of Definition 3.10.

The following two theorems are crucial for this section. Let $\rho : G_K^{\mathrm{ab}} \to \mathrm{Aut}(V_\ell)$ be an abelian ℓ-adic representation of the number field K.

Theorem 3.13. [19, Chap. 3, Thm. 2] If ρ is locally algebraic, then there exists a modulus $\mathfrak{m}$, an abelian ℓ-adic representation $\epsilon_\ell : G_K^{\mathrm{ab}} \to S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ (**3.8**) and a morphism of algebraic groups $\phi : S_{\mathfrak{m}} \times_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathrm{GL}_{V_\ell}$ over $\mathbb{Q}_\ell$ such that $\rho = \phi \circ \epsilon_\ell$.

Theorem 3.14. [19, Chap. 3 §3], [23], [7, §5] If ρ is rational and semisimple, then ρ is locally algebraic.

Remark 3.15. An abelian ℓ-adic representation ρ of K is always unramified at all $v \in \sum_K \backslash S$, for some finite set S ([19, Chap.3 §2.2]). The proof of Theorem 3.14 [7, §5] consists of two parts. Part one proves that ρ is almost locally algebraic, i.e. there exists an integer N such that ρ^N is locally algebraic. Part two proves that if ρ is almost locally algebraic, then ρ is locally algebraic. The crucial observation is that part one does not use the full rational condition, it only needs $\rho(F_w)$'s eigenvalues are algebraic for all $w|v$ when $v \notin S$. We will use this in the proof of Proposition 3.18.

(**3.16**) Suppose $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ is a semisimple, rational ℓ-adic representation of K. Let G_ℓ be the Zariski closure of $\rho(G_K)$ in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$. It is defined over $\mathbb{Q}_\ell$. Semisimplicity of ρ implies G_ℓ is a reductive group. Indeed, G_ℓ acts on an n-dimensional vector space V over $\mathbb{Q}_\ell$. Semisimplicity implies V decomposes into a direct sum of irreducibles $V_1 \oplus V_2 \oplus \cdots \oplus V_m$. If U is the unipotent radical of G_ℓ°, then the eigenspace W (of eigenvalue 1, the only eigenvalue) of U decomposes as $W_1 \oplus W_2 \oplus \cdots \oplus W_m$ such that $W_i \subset V_i$ and W_i is non-trivial for all $1 \leq i \leq m$. Since U is normal in G_ℓ, W is an invariant subspace of V. Therefore, $W = V$, and G_ℓ is a reductive algebraic group.

(**3.17**) We may now assume G_ℓ is connected, reductive which corresponds to restricting ρ to the open subgroup $G_{K^{\mathrm{conn}}}$ of G_K. The quotient group $G_\ell/[G_\ell, G_\ell]$ is a $\mathbb{Q}_\ell$-torus. Let

$$j : G_\ell/[G_\ell, G_\ell] \to \mathrm{GL}_{m,\mathbb{Q}_\ell}$$

be an embedding defined over $\mathbb{Q}_\ell$ and introduce the map

$$\theta : G_\ell \to G_\ell/[G_\ell, G_\ell] \xrightarrow{j} \mathrm{GL}_{m,\mathbb{Q}_\ell}.$$

If ρ is unramified outside a finite subset S of $\sum_K$ and w is a valuation extending $v \in \sum_K \backslash S$, then $\rho(F_w)$ is well defined. Since θ is algebraic, the eigenvalues of $\theta(\rho(F_w))$ in $\mathrm{GL}_m(\mathbb{Q}_\ell)$ are also algebraic numbers. Indeed, if we write $\rho(F_w) = g_{ss}g_u$ by Jordan decomposition, then the eigenvalues of $\rho(F_w)$ are the same as the eigenvalues of g_{ss} and $\theta(\rho(F_w)) = \theta(g_{ss})$ because $G_\ell/[G_\ell, G_\ell]$ is a torus. g_{ss} is contained in some maximal torus in G_ℓ and we see that the eigenvalues of $\theta(g_{ss})$ are products of integral powers of eigenvalues of g_{ss}; therefore the

eigenvalues of $\theta(\rho(F_w))$ are algebraic.

Proposition 3.18. Let $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ be a semisimple, rational ℓ-adic representation of K, G_ℓ the Zariski closure of $\rho(G_K)$ in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ and K^{conn} the field corresponding to G_ℓ°. Then G_ℓ is reductive, the dimension of the center of G_ℓ° is less than or equal to $d_{K^{\mathrm{conn}}}$.

Proof. The algebraic group G_ℓ is reductive with Lie algebra $\mathfrak{g}_\ell$. We just need to estimate the dimension of the center of G_ℓ°. We may assume G_ℓ is connected by Theorem 1.3. Consider the composition of maps where θ is defined above,

$$\theta \circ \rho : G_K \to G_\ell(\mathbb{Q}_\ell) \to G_\ell/[G_\ell, G_\ell](\mathbb{Q}_\ell) \to \mathrm{GL}_m(\mathbb{Q}_\ell).$$

The quotient map $G_K \to G_K^{\mathrm{ab}}$ factors through the composition, hence induces an abelian ℓ-adic representation

$$\Psi : G_K^{\mathrm{ab}} \longrightarrow \mathrm{GL}_m(\mathbb{Q}_\ell).$$

The discussion above implies the eigenvalues of $\Psi(F_w)$ are algebraic. Therefore, there exists some integer N such that $(\Psi)^N$ is locally algebraic by Remark 3.15.

Apply Theorem 3.13 to $(\Psi)^N : G_K^{\mathrm{ab}} \to \mathrm{GL}_m(\mathbb{Q}_\ell)$, we have

$$(\Psi)^N = \phi \circ \epsilon_\ell.$$

We see that $\phi(S_{\mathfrak{m}}(\mathbb{Q}_\ell))$ contains the image of $(\Psi)^N$. Since $\Psi(G_K^{\mathrm{ab}})$ is abelian, the dimensions of $(\Psi)^N(G_K^{\mathrm{ab}})$ and $\Psi(G_K^{\mathrm{ab}})$ as ℓ-adic Lie groups are the same. This implies that $(\Psi)^N(G_K^{\mathrm{ab}})$ is also Zariski dense in $j(G_\ell/[G_\ell, G_\ell])$. Since $S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ is Zariski dense in $S_{\mathfrak{m}}$ by Theorem 3.8(2), we obtain

$$j(G_\ell/[G_\ell, G_\ell]) \subset \phi(S_{\mathfrak{m}})$$

and conclude that the dimension of the center of G_ℓ is less than or equal to the dimension of $S_{\mathfrak{m}}$ which is d_K by Theorem 3.8(4). $\square$

Theorem 3.19. Let $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ be a semisimple, compatible system of ℓ-adic representations of a number field K.

$$\rho_\ell : G_K \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell).$$

Let T_ℓ be a maximal torus of G_ℓ° and denote the embedding of T_ℓ into $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ by Φ_ℓ. Then the triple $(([G_\ell^\circ, G_\ell^\circ] \cap (T_\ell))^\circ, T_\ell, \Phi_\ell)$ is independent of ℓ. Therefore, the formal character of the tautological representation $(G_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ and hence the semisimple rank of G_ℓ° are independent of ℓ.

Proof. Assume $K = K^{\mathrm{conn}}$. Choose some Serre group $S_{\mathfrak{m}}$ associated to K, then it induces an abelian, semisimple, compatible system $\{\epsilon_\ell\}_{\ell \in \mathscr{P}}$ of ℓ-adic representations with values in $S_{\mathfrak{m}}$.

$$\epsilon_\ell : G_K^{\mathrm{ab}} \longrightarrow S_{\mathfrak{m}}(\mathbb{Q}_\ell).$$

The image of ϵ_ℓ is an open subgroup of $S_{\mathfrak{m}}(\mathbb{Q}_\ell)$. Now we choose some faithful representation $i : S_{\mathfrak{m}} \to \mathrm{GL}_{m,\mathbb{Q}}$ over $\mathbb{Q}$, then by base change with $\mathbb{Q}_\ell$ and composing ϵ_ℓ with i we get an

abelian, semisimple, compatible system of ℓ-adic representations of K. Still denote it by $\{\epsilon_\ell\}_{\ell \in \mathscr{P}}$:

$$\epsilon_\ell : G_K^{\mathrm{ab}} \longrightarrow \mathrm{GL}_m(\mathbb{Q}_\ell).$$

Now consider the system of ℓ-adic representations $\{\beta_\ell := \rho_\ell \oplus \epsilon_\ell\}_{\ell \in \mathscr{P}}$,

$$\beta_\ell : G_K \longrightarrow \mathrm{GL}_{n+m}(\mathbb{Q}_\ell),$$

which is semisimple and compatible and denote the Zariski closure of the image by B_ℓ. Since $[\beta_\ell(G_K), \beta_\ell(G_K)] = [\rho_\ell(G_K), \rho_\ell(G_K)] \times \{\mathrm{Id}_m\}$, the Zariski closure of $[\beta_\ell(G_K), \beta_\ell(G_K)]$ is isomorphic to the Zariski closure of $[\rho_\ell(G_K), \rho_\ell(G_K)]$ in G_ℓ. On the other hand, $\rho_\ell(G_K) \times \rho_\ell(G_K)$ is dense in $G_\ell \times G_\ell$ and the image of the commutator morphism

$$[\ ,\ ] : G_\ell \times G_\ell \to G_\ell$$

is closed because G_ℓ is connected [2, Chap. 1 §2.3]. We conclude that the Zariski closure of $[\beta_\ell(G_K), \beta_\ell(G_K)]$ is $[G_\ell, G_\ell] \times \{\mathrm{Id}_m\}$. The group $[G_\ell, G_\ell]$ is embedded as a closed normal subgroup of $B_\ell \subset G_\ell \times S_{\mathfrak{m}}$. So we have

$$B_\ell/[G_\ell, G_\ell] \subset G_\ell/[G_\ell, G_\ell] \times S_{\mathfrak{m}}.$$

Choose an embedding $G_\ell/[G_\ell, G_\ell] \hookrightarrow \mathrm{GL}_{n', \mathbb{Q}_\ell}$. Denote the following map by α_ℓ,

$$\alpha_\ell : G_K \to B_\ell(\mathbb{Q}_\ell) \to (B_\ell/[G_\ell, G_\ell])(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_{n'}(\mathbb{Q}_\ell) \times \mathrm{GL}_m(\mathbb{Q}_\ell) \subset \mathrm{GL}_{n'+m}(\mathbb{Q}_\ell).$$

This map is semisimple and factors through G_K^{ab}. Denote the Zariski closure of $\alpha_\ell(G_K)$ by C_ℓ which is diagonalizable. The eigenvalues of the Frobenius elements whenever defined are algebraic. Therefore α_ℓ is almost locally algebraic by Remark 3.15, i.e. α_ℓ^N is locally algebraic for some positive integer N. Since the dimensions of the abelian image of α_ℓ and α_ℓ^N are equal, we have

$$\dim(C_\ell) \leq d_K$$

by the same argument in the last paragraph of the proof of Proposition 3.18. Observe that the projection of C_ℓ to the second factor is $i(S_{\mathfrak{m}})$, we obtain

$$\dim(C_\ell) \geq \dim(S_{\mathfrak{m}}) = d_K.$$

We conclude that $\dim(C_\ell) = d_K$ is independent of ℓ by Theorem 3.8(4). Since C_ℓ is isomorphic to $B_\ell/[G_\ell, G_\ell]$, we have an exact sequence of reductive algebraic groups for each ℓ

$$0 \to [G_\ell, G_\ell] \to B_\ell \to C_\ell \to 0.$$

Finally, since $\{\beta_\ell\}_{\ell \in P}$ is a semisimple, compatible system of ℓ-adic representations of K, the rank of B_ℓ is independent of ℓ by Theorem 1.4. Together with the ℓ-independence of $\dim(C_\ell)$ and the exact sequence above, we obtain ℓ-independence of the rank of $[G_\ell, G_\ell]$. Therefore, the dimension of the center of G_ℓ is also independent of ℓ by Theorem 1.4 again.

Let π_1 and π_2 be the projection to the first n coordinates and the last m coordinates respectively. By base change with $\mathbb{C}$, we assume T_ℓ and $\epsilon_\ell(G_K)$ are diagonalized. Also denote the Zariski closure of $\beta_\ell(G_K)$ in $\mathrm{GL}_{m+n, \mathbb{C}}$ by B_ℓ. We know that the semisimple ranks of G_ℓ and B_ℓ° are equal by the previous paragraph. Let D^{n+m} be the group of diagonal matrices in $\mathrm{GL}_{n+m, \mathbb{C}}$. Let D_ℓ be the connected component of $B_\ell \cap D^{n+m}$. It is a maximal torus of B_ℓ. Indeed, $\pi_2(D_\ell)$ has dimension equal to $\dim(S_{\mathfrak{m}})$ and D_ℓ contains the connected component

of $[B_\ell^\circ, B_\ell^\circ] \cap D^{n+m}$ which is equal to the connected component of $([G_\ell, G_\ell] \times \{\mathrm{Id}_m\}) \cap D^{n+m}$ having dimension equal to the dimension of $[G_\ell, G_\ell] \cap T_\ell$. Since $\pi_2(([G_\ell, G_\ell] \times \{\mathrm{Id}_m\}) \cap D^{n+m})$ is trivial, this implies the dimension of D_ℓ is equal to the rank of B_ℓ. In other words, we could pick for each ℓ a diagonalized maximal torus D_ℓ of B_ℓ such that $\pi_1(D_\ell) = T_\ell$. Since the systems $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ and $\{\epsilon_\ell\}_{\ell \in \mathscr{P}}$ are both compatible, every D_ℓ is conjugate by a permutation which permutes the first n coordinates and the last m coordinates. Therefore, we may assume $\Theta_\ell : D_\ell \hookrightarrow \mathrm{GL}_{n+m,\mathbb{C}}$ is independent of ℓ and write $\Theta_\ell := \Theta$.

$$\Theta := (\phi_1, ..., \phi_n, \phi_{n+1}, ..., \phi_{n+m}).$$

Since $[B_\ell^\circ, B_\ell^\circ] \cap D_\ell = ([G_\ell, G_\ell] \times \{\mathrm{Id}_m\}) \cap D_\ell \subset \mathrm{Ker}(\pi_2)$ and π_2 is an isogeny of the center of B_ℓ° to its image, we have

$$([B_\ell^\circ, B_\ell^\circ] \cap D_\ell)^\circ = \mathrm{Ker}(\pi_2|_{D_\ell})^\circ$$

which is independent of ℓ. By projecting on the first n coordinates, we have

$$([G_\ell, G_\ell] \cap T_\ell)^\circ = \pi_1(([B_\ell^\circ, B_\ell^\circ] \cap D_\ell)^\circ) = \pi_1(\mathrm{Ker}(\pi_2|_{D_\ell})^\circ)$$

is also independent of ℓ. Hence if we denote $(\phi_1, ..., \phi_n)$ by Φ_ℓ, then the triple

$$(([G_\ell, G_\ell] \cap T_\ell)^\circ, T_\ell, \Phi_\ell)$$

is independent of ℓ. It follows that the formal character of $(G_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is independent of ℓ, where $(G_\ell^\circ)^{\mathrm{der}} := [G_\ell^\circ, G_\ell^\circ]$. $\qquad\qquad\square$

(**3.20**) Let's focus on the level of Lie algebra. By Theorem 3.19, we have a pair $((\mathfrak{g}_\ell)_{ss} \otimes \mathbb{C}, \Phi_\ell)$ for each ℓ, where $(\mathfrak{g}_\ell)_{ss} \otimes \mathbb{C}$ is a complex semisimple Lie algebra and Φ_ℓ is a faithful representation of $(\mathfrak{g}_\ell)_{ss} \otimes \mathbb{C}$ to an n-dimensional complex vector space such that

(1) The rank of $(\mathfrak{g}_\ell)_{ss} \otimes \mathbb{C}$ is independent of ℓ.
(2) The formal character of Φ_ℓ is independent of ℓ (see (**2.1**)).

By Theorem 2.14, 2.17, 2.19, and Proposition 2.20, we obtain the following theorem.

Theorem 3.21 Let $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ be a semisimple, compatible system of ℓ-adic representations of a number field K.

$$\rho_\ell : G_K \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell).$$

Then the semisimple parts of $\mathfrak{g}_\ell \otimes \mathbb{C}$ satisfy equal-rank subalgebra equivalence (Definition 2.18) for all ℓ which is equivalent to the number of A_n factors for $n \in \{6, 9, 10, ...\}$ and the parity of A_4 factors of $\mathfrak{g}_\ell \otimes \mathbb{C}$ are independent of ℓ.

Remark 3.22 Let $\{\rho_\lambda\}$ be a compatible system of semisimple, E_λ-adic representations of G_K [19, Chap. 1 §2.3]. Locally algebraicity is also defined for abelian λ-adic representation [7, §2]. Since Theorem 1.3, 1.4, 3.13, 3.14 (see [7, Thm. 2]), and Remark 3.15 still hold analogously for E_λ-adic representations (the morphisms are then defined over E_λ), one can prove that Proposition 3.18, Theorem 3.19, and Theorem 3.21 are also true in E_λ-adic case using identical arguments.

§4 Abelian varieties and Galois representations

Let A be an abelian variety of dimension g over a field K, finitely generated over $\mathbb{Q}$. Let ρ_ℓ denote the action of G_K on

$$V_\ell(A) := (\varprojlim A[\ell^n]) \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{2g}.$$

The image of ρ_ℓ is an ℓ-adic Lie group; denote its Lie algebra by $\mathfrak{g}_\ell$.

There exists an abelian scheme E over a smooth variety X defined over a number field k such that the function field of X is K and $E_\eta = A$ where η is the generic point of X (see, e.g. Milne [14, §20]). Every closed point x of X induces an ℓ-adic representation of $\mathbf{k}(x)$ given by the Galois action of $G_{\mathbf{k}(x)}$ on the ℓ-adic Tate module of E_x, here $\mathbf{k}(x)$ is the residue field of x which is a finite extension of k.

$$(\rho_\ell)_x : G_{\mathbf{k}(x)} \longrightarrow \mathrm{GL}(V_\ell(E_x))$$

Denote the Lie algebra of the image of $(\rho_\ell)_x$ by $(\mathfrak{g}_\ell)_x$, we have $(\mathfrak{g}_\ell)_x \subset \mathfrak{g}_\ell$ by specialization (see Hui [8, §1]). By [16, §1], there always exists a closed point $x \in X$ such that $(\mathfrak{g}_\ell)_x = \mathfrak{g}_\ell$. Therefore, we have $(\mathfrak{g}_\ell)_x = \mathfrak{g}_\ell$ for any prime ℓ by [8, Thm. 2.5]. Since the system $\{(\rho_\ell)_x\}_{\ell \in P}$ of ℓ-adic representation of $\mathbf{k}(x)$ is compatible ($\mathbf{k}(x)$ is a number field) and semisimple (Faltings [4]), the semisimple parts of $(\mathfrak{g}_\ell)_x \otimes \mathbb{C} = \mathfrak{g}_\ell \otimes \mathbb{C}$ satisfy equal-rank subalgebra equivalence in Definition 2.18 by Theorem 3.21. Thus, we obtain the following result.

Theorem 4.1 Let A be an abelian variety of dimension g over a field K, finitely generated over $\mathbb{Q}$. Then we have the following system $\{\rho_\ell\}_{\ell \in \mathscr{P}}$ of ℓ-adic representation of K

$$\rho_\ell : G_K \longrightarrow \mathrm{GL}(V_\ell(A)).$$

Then the semisimple parts of $\mathfrak{g}_\ell \otimes \mathbb{C}$ satisfy equal-rank subalgebra equivalence (Definition 2.18) for all ℓ which is equivalent to the number of A_n factors for $n \in \{6, 9, 10, ...\}$ and the parity of A_4 factors of $\mathfrak{g}_\ell \otimes \mathbb{C}$ are independent of ℓ.

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Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel

E-mail address: chhui@indiana.edu, pslnfq@gmail.com