Graded geometry in gauge theories

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We study some graded geometric constructions appearing naturally in the context of gauge theories. In terms of Q-bundles we describe the symmetries of the (twisted) Poisson and Dirac sigma models (PSM and DSM). Inspired by a known relation of gauging with equivariant cohomology we generalize the latter notion to the case of arbitrary Q-manifolds. This permits to obtain the mentioned sigma models by gauging essentially infinite dimensional groups and describe their symmetries in terms of classical differential geometry. We also show that the Dirac sigma model is universal in space-time dimension 2. The approach can also be useful to study supersymmetric theories and to address the quantization problems.

Preliminaries: Q-structures

Q-manifold – *graded manifold* equipped with a Q-structure – vector field Q satisfying $\deg Q = 1$, $[Q,Q] \equiv 2Q^2 = 0$.
Examples: • Tangent bundle: $T[1]M$ with $\mathrm{d}_{\text{de Rham}}$;
• Cotangent bundle to a Poisson manifold: $T^*[1]M$, $Q_\pi = \{\frac{1}{2}\pi^{ij}p_i p_j, \cdot\}$;
• Lie algebra: $\mathcal{G}[1]$ with Chevalley-Eilenberg differential $d_{ce} = C^a_{bc}\xi^b\xi^c\frac{\partial}{\partial\xi^a}$;
• Dirac structure; • Lie algebroid; • Courant algebroid sympl. realization.

Given two Q-manifolds $(\mathcal{M}_1, Q_1)$, $(\mathcal{M}_2, Q_2)$, a degree preserving map $f\colon \mathcal{M}_1 \to \mathcal{M}_2$, is a Q-morphism iff $Q_1 f^* - f^* Q_2 = 0$.

Proposition. Given a degree preserving map a between Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}_2, Q_2)$, there exists a Q-morphism f between the Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\tilde{\mathcal{M}}, \tilde{Q}) = (T[1]\mathcal{M}_2, \mathrm{d}_{DR} + \mathcal{L}_{Q_2})$, covering a.

Gauging problem, Wess-Zumino terms

Gauging example. Consider $X\colon \Sigma^d \to M^n$, and $B \in \Omega^d(M)$ Assume that a Lie group G acts on M and leaves B invariant. This induces a G-action on M^Σ, which leaves invariant the functional $S[X] := \int_\Sigma X^*B$. The invariance w.r.t. G is called a (global) rigid invariance. The functional is called (locally) *gauge* invariant, if it is invariant with respect to the group $G^\Sigma \equiv C^\infty(\Sigma, G)$. The gauging problem can be solved in this case by extending S to a functional $\widetilde{S}$ defined on $(X, A) \in M^\Sigma \times \Omega^1(\Sigma, \mathfrak{g})$ by means of so-called minimal coupling. E.g. for $d = 2$:

$$\widetilde{S}[X,A] := \int_\Sigma \left(X^*B - A^a X^* \iota_{v_a} B + \frac{1}{2} A^a A^b X^* \iota_{v_a}\iota_{v_b} B \right).$$

Wess-Zumino term: $S[X] := \int_{\Sigma^{d+1}} X^*H$, where $\partial\Sigma^{d+1} = \Sigma^d$, $\mathrm{d}H = 0$.
Gauging of such a WZ-term can be obstructed. J.M. Figueroa-O'Farrill and S. Stanciu '94: gauging is possible, iff H permits an equivariantly closed extension. Serious limitation: $\dim G < \infty$

Equivariant Q-cohomology

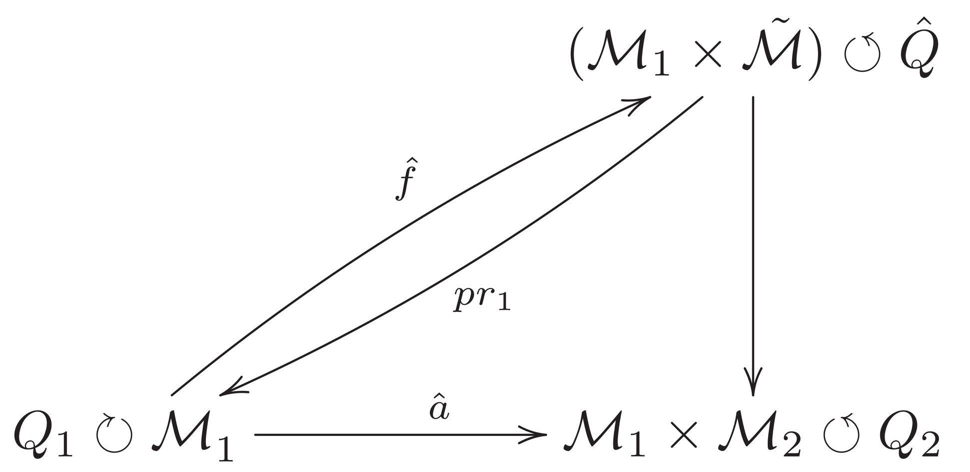
Let $(\mathcal{M}, Q)$ be a Q-manifold, consider an algebra $\mathcal{G}$ of degree -1 vector fields ε on $\mathcal{M}$ with the Q-derived bracket. Consider the action of Q on superfunctions on $\mathcal{M}$ as a *generalized differential* (degree $+1$) and the action of ε as a *generalized contraction* (degree -1). [1, 2]
Def. A superfunction ω on $\mathcal{M}$ is $\mathcal{G}$-horizontal iff $\varepsilon\omega = 0$, $\forall \varepsilon \in \mathcal{G}$.
Def. A superfunction ω on $\mathcal{M}$ is $\mathcal{G}$-equivariant iff $[Q,\varepsilon]\omega = 0$, $\forall\varepsilon \in \mathcal{G}$.
Def. We call a superfunction (differential form) ω on $\mathcal{M}$ $\mathcal{G}$-basic iff it is $\mathcal{G}$-horizontal and $\mathcal{G}$-equivariant.
Remark 1. For Q-closed superfunctions $\mathcal{G}$-horizontal $\Leftrightarrow$ $\mathcal{G}$-basic
Remark 2. Standard equivariant cohomology is a particular case.

Gauge invariance and Q-bundles

Associate Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}_2, Q_2)$ respectively to the world-sheet and the target of a gauge theory.

$\tilde{\mathcal{M}} := T[1]\mathcal{M}_2$, $\tilde{Q} := \mathrm{d} + \mathcal{L}_{Q_2}$;
$\hat{Q} = Q_1 + \tilde{Q}$ acts on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$;

$\hat{a}$ and $\hat{f}$ – trivial extensions of a and f described above, $\hat{f}$ is a Q-morphism.



Key idea. Gauge transformations can be parametrized by $\hat{\varepsilon}$ – vector fields on $\mathcal{M}_1 \times \mathcal{M}_2$ of total degree -1, vertical w.r.t. $\mathcal{M}_1$:

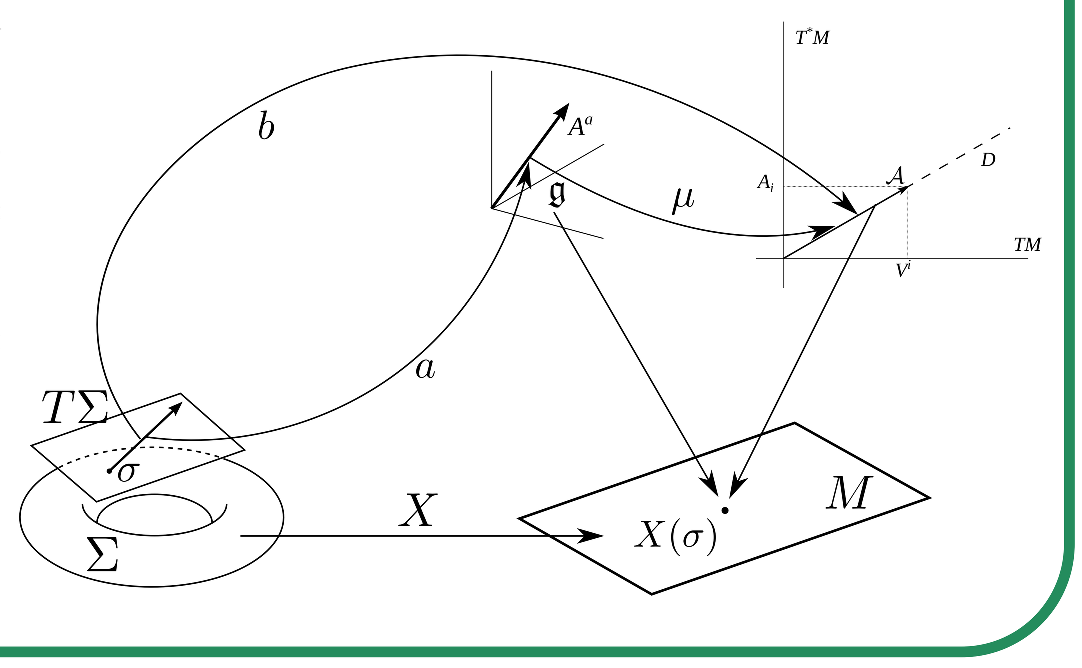
$$\delta_\varepsilon(\hat{f}^*\cdot) = \hat{f}^*([\hat{Q}, \hat{\varepsilon}]\cdot). \qquad (1)$$

Important remark. $(\mathcal{M}_2, Q_2)$ exists in a generic situation when field equations satisfy Bianchi identities and $d \geq 4$ (M. Grützmann, T. Strobl).
Remark. Extension of A. Kotov, T. Strobl '07.

Gauging via equivariant Q-cohomology

The gauge invariance condition $\delta_\varepsilon\omega = 0$ from (1) amounts to ω being $\mathcal{G}$-basic in the context of equivariant Q-cohomology on $(\tilde{\mathcal{M}}, \tilde{Q})$.
Gauging = finding a $\mathcal{G}$-equivariant $\tilde{Q}$-closed extension of a given WZ-term.
Necessary data for this procedure: Q-structures on world-sheet and target; morphism from the algebra of rigid symmetries to be gauged to degree -1 vector fields with a $\tilde{Q}$-derived bracket.

Remark 1. The functional with an infinite dimensional algebra of rigid symmetries can be gauged with a finite number of gauge fields [3].
Remark 2. The procedure applies in any dimension.
If $\dim\Sigma = 2$ it gives a morphism to a Dirac-Rinehart sheaf, so DSM is universal [3].



Symmetries of PSM and DSM

Twisted Poisson sigma model. World-sheet: Σ (closed, orientable, with no boundary, $dim = 2$). Target: (twisted) Poisson manifold (M, π, H). The functional over the space of vector bundle morphisms $T\Sigma \to T^*M$
$S = \int_\Sigma A_i \wedge \mathrm{d}X^i + \frac{1}{2}\pi^{ij} A_i \wedge A_j + \int_{\Sigma^3} H$, where $X^i : \Sigma \to M$ – scalar fields and $A_i \in \Omega^1(\Sigma, X^*T^*M)$ 1-form valued ("vector") fields.
Dirac sigma model. Functional on vector bundle morphisms from $T\Sigma$ to a Dirac structure $\mathcal{D}$. It specializes to twisted PSM if $\mathcal{D} = graph(\pi^\#)$.
Theorem 1 [1, 2] Any smooth map from Σ to $\Gamma(\mathcal{D})$ of sections defines an infinitesimal gauge transformation of the (metric independent part of the) Dirac sigma model governed by $\mathcal{D}$ in the above sense, iff for any point $\sigma \in \Sigma$ the section $v \oplus \eta \in \Gamma(\mathcal{D})$ satisfies $\iota_v H = \mathrm{d}\eta$.
Remark 1. Here degree -1 vector fields on $\tilde{\mathcal{M}}$ are lifted from $\mathcal{M}_2$
Remark 2. The theorem specializes to twisted PSM by setting $v = \pi^\#\eta$; and to PSM (cf. M. Bojowald, A. Kotov, T. Strobl '04) with $H = 0$.

Sigma models from gauging

For $\mathcal{G}$ from theorem 1, equivariant $\tilde{Q}$-closed extension can be ambiguous.
Way out: consider the full algebra $\tilde{\mathcal{G}}$ of degree -1 vector fields on $\tilde{\mathcal{M}}$.
Proposition [2] Consider $\tilde{\mathcal{M}} = T[1]T^*[1]M$ $\tilde{Q} = d + \mathcal{L}_{Q_\pi}$, governed by (π, H). The Lie algebra $(\tilde{\mathcal{G}}, [,]_{\tilde{Q}})$ is isomorphic to $\mathcal{G} \Subset \mathcal{A}$, where $\mathcal{G}$ is a Lie algebra of 1-forms $T^*[1]M$ with the Lie algebroid bracket

$$[\varepsilon^1, \varepsilon^2] = \mathcal{L}_{\pi^\#\varepsilon^1}\varepsilon^2 - \mathcal{L}_{\pi^\#\varepsilon^2}\varepsilon^1 - \mathrm{d}(\pi(\varepsilon^1, \varepsilon^2)) - \iota_{\pi^\#\varepsilon^1}\iota_{\pi^\#\varepsilon^2}H,$$

$\mathcal{A}$ is a Lie algebra of covariant 2-tensors on M with a bracket given by $[\alpha, \beta] =< \pi^{23}, \alpha\otimes\beta - \beta\otimes\alpha >$; $\mathcal{G}$ acts on $\mathcal{A}$:

$$\rho(\varepsilon)(\alpha) = \mathcal{L}_{\pi^\#\varepsilon}(\alpha) - < \pi^{23}, \mathrm{d}\varepsilon - \iota_{\pi^\#\varepsilon}H \otimes \alpha > .$$

Theorem 2 [2] In the above notations consider a couple (π, H) such that the pull-back of H to a dense set of orbits of π is non-degenerate. Consider a subalgebra $\mathcal{GT} \subset \tilde{\mathcal{G}}$ defined by $\iota_{\pi^\#\varepsilon}H = \mathrm{d}\varepsilon$, $\quad \alpha^A = 0$.
The $\mathcal{GT}$ equivariantly closed extension of a given 3-form H defines uniquely the functional of the twisted Poisson sigma model.
Remark 1. Non-degeneracy condition is a sufficient one, not necessary.
Remark 2. A likewise theorem holds true ([1]) for the (metric independent part of the) Dirac sigma model: consider $\tilde{\mathcal{M}} = T[1]\mathcal{D}[1]$.

Related works

[1] V.S., T.Strobl, Dirac Sigma Models from Gauging, JHEP, 11/2013
[2] V.S., Graded geometry in gauge theories and beyond, J.Geom.Phys, Vol.87, 2015
[3] A. Kotov, V.S., T. Strobl, 2d Gauge Theories and Generalized Geometry, JHEP 2014:21

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