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Graded geometry in gauge theories: above and beyond

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UNIVERSITÉ DU LUXEMBOURG
Mathematics Research Unit (RMATH)

Lyon 1

Graded geometry in gauge theories (+ above and beyond)

- Graded geometry, Q-manifolds:
  (twisted) Poisson manifolds, Dirac structures

- Gauging problem, its relation to equivariant (Q-)cohomology
- Gauge transformations via Q-language
- Gauged sigma models: (twisted) Poisson, Dirac

- Generality of the Dirac sigma model
- Multigraded geometry and supersymmetric theories
- Courant algebroids – equivariant cohomology

Graded manifolds – example

Consider functions on $T[1]\Sigma$.

$\sigma^1, \dots, \sigma^d$ – coordinates on Σ:
$deg(\sigma^\mu) = 0$, $\sigma^{\mu_1}\sigma^{\mu_2} = \sigma^{\mu_2}\sigma^{\mu_1}$.
$deg(h(\sigma^1, \dots, \sigma^d)) = 0$.

$\theta^1, \dots, \theta^d$ – fiber linear coordinates:
$deg(\theta^\mu) := 1$, $\theta^{\mu_1}\theta^{\mu_2} = -\theta^{\mu_2}\theta^{\mu_1}$

Arbitrary homogeneous function on $T[1]\Sigma$ of $deg = p$:
$f = f_{\mu_1 \dots \mu_p}(\sigma^1, \dots, \sigma^d)\theta^{\mu_1} \dots \theta^{\mu_p}$.

Graded commutative product: $f \cdot g = (-1)^{deg(f)deg(g)} g \cdot f$

$f \leftrightarrow \omega = f_{\mu_1 \dots \mu_p} \mathrm{d}\sigma^{\mu_1} \wedge \dots \wedge \mathrm{d}\sigma^{\mu_p} \in \Omega(\Sigma)$

$\Rightarrow$ "Definition" of a graded manifold
– manifold with a ($\mathbb{Z}$-)grading defined on the sheaf of functions.

Graded manifolds/Q-manifolds (DG-manifolds)

D. Roytenberg: "...graded manifolds are just manifolds with a few bells and whistles..."

$T[1]\Sigma$, $deg(\sigma^\mu) = 0$, $deg(\theta^\mu) = 1$, $f_{\mu_1 \dots \mu_p}(\sigma^1, \dots, \sigma^d)\theta^{\mu_1} \dots \theta^{\mu_p}$

Remark. The grading can be encoded in the Euler vector field $\epsilon = deg(q^\alpha) q^\alpha \frac{\partial}{\partial q^\alpha}$ (can be a "definition").

Consider a vector field $Q = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

$$\left.\begin{array}{l} \deg Q = 1 \\ Q(f \cdot g) = (Qf) \cdot g + (-1)^{1 \cdot deg(f)} f \cdot (Qg) \\ [Q, Q] \equiv 2Q^2 = 0 \end{array}\right\} \leftarrow \mathrm{d}_{\text{de Rham}}$$

$\Rightarrow$ Definition of a Q-structure: a vector field on a graded manifold, which is of degree 1 and squaring to zero.

Poisson manifold $\to (T^*[1]M, Q_\pi)$

Consider a Poisson manifold M,
$\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$.

A Poisson bracket can be written as $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$, where $\pi \in \Gamma(\Lambda^2 TM)$ is a bivector field. $\pi^{ij}(x) = \{x^i, x^j\}$.

Consider $T^*[1]M$ (coords. $x^i(0), p_i(1)$), with a degree 1 vector field

$$Q_\pi = \left\{\frac{1}{2}\pi^{ij} p_i p_j, \cdot\right\}_{T^*M} = \pi^{ij}(x) p_j \frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial \pi^{jk}}{\partial x^i} p_j p_k \frac{\partial}{\partial p_i}$$

Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$
$$\Leftrightarrow [\pi, \pi]_{SN} = 0 \quad \Leftrightarrow Q_\pi^2 = 0$$

Twisted Poisson structure

Consider a bivector field $\pi \in \Gamma(\Lambda^2 TM)$.
It defines an antisymmetric bracket $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$.
Let $H \in \Omega^3_{cl}(M)$.
A couple (π, H) defines a twisted Poisson structure if it satisfies the twisted Jacobi identity:

$$[\pi, \pi]_{SN} =< H, \pi \otimes \pi \otimes \pi >$$

Consider $T^*[1]M$ (coords. $x^i(0), p_i(1)$), with a degree 1 vector field

$$Q_{\pi,H} = \pi^{ij}(x) p_j \frac{\partial}{\partial x^i} - \frac{1}{2} C_i^{jk}(x) p_j p_k \frac{\partial}{\partial p_i},$$

where $C_i^{jk} = \frac{\partial \pi^{jk}}{\partial x^i} + H_{ij'k'} \pi^{jj'} \pi^{kk'}$.

Twisted Jacobi identity $\Leftrightarrow Q^2_{\pi,H} = 0$

Courant algebroids, Dirac structures

Let us construct on $E = TM \oplus T^*M$ a twisted exact Courant algebroid structure, governed by a closed 3-form H on M.
The symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
the anchor: $\rho(v \oplus \eta) = v$
the H-twisted bracket (Dorfman):

$$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \iota_{v'} \mathrm{d}\eta + \iota_v \iota_{v'} H).$$

Courant algebroids, Dirac structures

A Courant algebroid is a vector bundle $E \to M$ equipped with the following operations: a symmetric non-degenerate pairing $< \cdot, \cdot >$ on E, an $\mathbb{R}$-bilinear bracket $[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ on sections of E, and an anchor ρ which is a bundle map $\rho : E \to TM$, satisfying the axioms:

$$\begin{aligned} \rho(\varphi) < \psi, \psi > &= 2 < [\varphi, \psi], \psi >, \\ [\varphi, [\psi_1, \psi_2]] &= [[\varphi, \psi_1], \psi_2] + [\psi_1, [\varphi, \psi_2]], \\ 2\,[\varphi, \varphi] &= \rho^*(d < \varphi, \varphi >), \end{aligned}$$

where $\rho^* : T^*M \to E$ (identifying E and E^* by $< \cdot, \cdot >$).

Theorem. (D. Roytenberg) Courant algebroids $\leftrightarrow$ degree 2 symplectic manifolds with compatible Q-structures.

Theorem. (P. Ševera) Exact (ρ surjective, $rkE = 2\, dimM$) Courant algebroids are classified by $H^3_{dR}(M)$.

Courant algebroids, Dirac structures

Let us construct on $E = TM \oplus T^*M$ a twisted exact Courant algebroid structure, governed by a closed 3-form H on M.
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$$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \iota_{v'} \mathrm{d}\eta + \iota_v \iota_{v'} H). \qquad (1)$$

A Dirac structure $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of an exact Courant algebroid E closed with respect to the bracket (1).

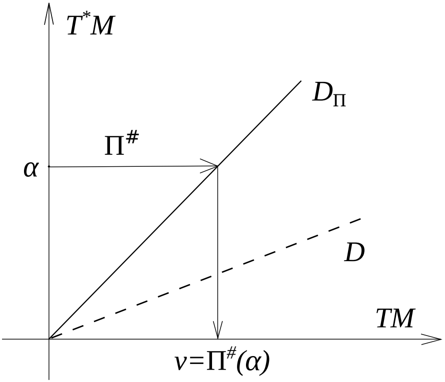
Trivial example: $\mathcal{D} = TM$ for $H = 0$.

Dirac structures: example.

Example. $\mathcal{D} = graph(\Pi^\sharp)$

Isotropy $\Leftrightarrow$
π^{ij} antisymmetric.

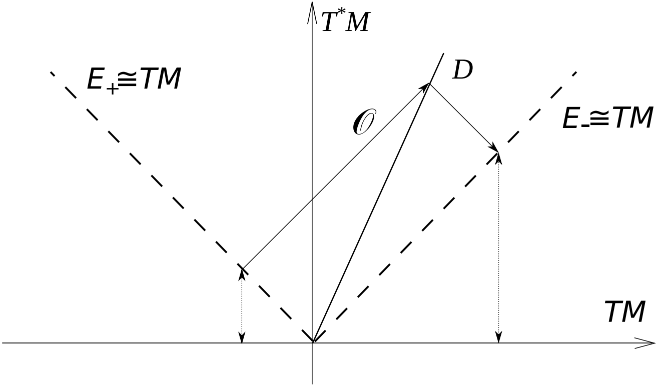
Involutivity $\Leftrightarrow$
Π (twisted) Poisson.



$$\mathcal{D}_\Pi = \{(\Pi^\sharp\alpha, \alpha)\}$$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.



(Almost) Dirac structure – a graph of an orthogonal operator
$\mathcal{O} \in \Gamma(\mathrm{End}(TM))$: $(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$

Dirac structures: general

Dirac structure – a graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$ subject to the (twisted Jacobi-type) integrability condition

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = \\ = \frac{1}{2}H((\mathrm{id}-\mathcal{O})\xi_1, (\mathrm{id}-\mathcal{O})\xi_2, (\mathrm{id}-\mathcal{O})\xi_3).$$

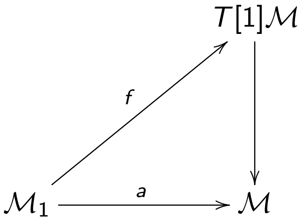
Remark 1. If the operator $(\mathrm{id}+\mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \frac{\mathrm{id}-\mathcal{O}}{\mathrm{id}+\mathcal{O}}$ (Cayley transform), and integrability reduces to $[\Pi,\Pi]_{SN} = \langle H, \Pi^{\otimes 3}\rangle$.

Remark 2. Any $\mathcal{D}[1]$ can be equipped with a Q-structure $\Leftrightarrow$ Lie algebroid structure on TM with $\rho = (\mathrm{id}-\mathcal{O})$,
$C^i_{jk} = (1-O)^m_j \Gamma^i_{mk} - (j \leftrightarrow k) + O^{m;i}_k O_{mj} + \frac{1}{2} H^i_{j'k'} (1-O)^{j'}_j (1-O)^{k'}_k$

Q-morphisms

Given two Q-manifolds $(\mathcal{M}_1, Q_1)$, $(\mathcal{M}_2, Q_2)$, a degree preserving map $f : \mathcal{M}_1 \to \mathcal{M}_2$, is a <u>$Q$-morphism</u> iff $Q_1 f^* - f^* Q_2 = 0$.

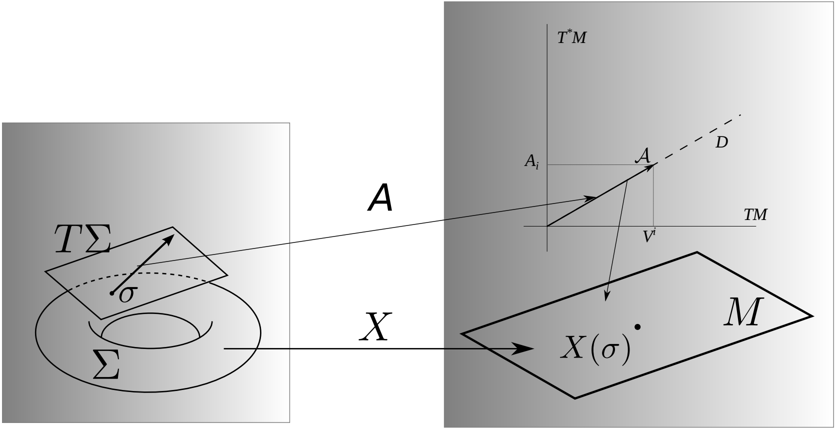
Proposition Given a degree preserving map between Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}, Q)$, there exists a Q-morphism between the Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}_2, Q_2) = (T[1]\mathcal{M}, \mathrm{d}_{DR} + \mathcal{L}_Q)$



Warning: Bernstein – Leites sign convention

$$x \cdot y = (-1)^{(deg_1 x + deg_2 x)(deg_1 y + deg_2 y)} y \cdot x$$

Sigma models



World-sheet (space-time) $\mapsto$ Target

Sigma model example – gauging problem

$$S[X] := \int_\Sigma X^* B$$

$B \in \Omega(M)$, $dim(\Sigma) = d = deg(B)$.
Assume that there is a Lie group G acting on M that leaves B invariant. It induces a G-action on M^Σ, which leaves S invariant.

The functional S is called (locally) gauge invariant, if it is invariant even with respect to the group $G^\Sigma \equiv C^\infty(\Sigma, G)$;
the invariance w.r.t. G is called a rigid (global) invariance.

Extending the functional S to a functional $\widetilde{S}$ defined on $(X, A) \in M^\Sigma \times \Omega^1(\Sigma, \mathfrak{g})$ by means of so-called minimal coupling,

$$\widetilde{S}_{2D}[X, A] := \int_\Sigma \left(X^* B - A^a X^* \iota_{v_a} B + \frac{1}{2} A^a A^b X^* \iota_{v_a} \iota_{v_b} B \right) .$$

Example. Standard Model: gauging of $SU(3)$ symmetry between three quarks $dim(SU(3)) = 8$ connection 1-forms – gluons.

Gauging the Wess–Zumino term

WZ-term: $H \in \Omega^{dim\Sigma+1}(M), \quad \mathrm{d}H = 0, \quad \partial\tilde{\Sigma} = \Sigma,$

$$S[X] := \int_{\tilde{\Sigma}} X^* H$$

Obstructions to gauging:
B. de Wit, C. Hull, M. Rocek "New topological terms in gauge invariant action" ('87)
C. M. Hull and B. J. Spence, "The Gauged Nonlinear σ Model With Wess-Zumino Term" ('89)
C. M. Hull and B. J. Spence, "The Geometry of the gauged sigma model with Wess-Zumino term" ('91)

Upshot: Gauging of such a WZ-term is possible, if and only if H permits an equivariantly closed extension. (J.M. Figueroa-O'Farrill and S. Stanciu, '94)

Limitation: number of gauge fields $= dimG$.

Equivariant cohomology

A Lie group G acting on a smooth manifold M. $\Omega^\bullet(M/G)$ – ?

First assume that G acts freely on M, i.e. M/G is a topological space. Consider $p\colon M \to M/G$, $\omega_0 \in \Omega^\bullet(M/G)$. $\omega = p^*(\omega_0)$ is well defined, ω is called basic.

Property (defining) of a basic form: $\iota_v \omega = 0$, $\mathcal{L}_v \omega = 0$, $v \in \mathcal{G}$.

Equivariant differential(s): $\tilde{d} = (d + \iota_v)$. $\tilde{d}^2|_{basic} = 0$

If the group does not act freely one can still perform a similar construction but modifying the manifold $M \to M \times EG \to$ huge space of differential forms, but not in cohomology. Instead one considers the Weil model or the Cartan model of equivariant cohomology. by defining the action on the Lie algebra valued connections (of degree 1) and curvatures (of degree 2) with some compatibility conditions.

Remark. d increases the form degree, ι_v decreases.

Equivariant cohomology for Q-manifolds

Let $(\mathcal{M}, Q)$ be a Q-manifold, and let $\mathcal{G}$ be a subalgebra of degree -1 vector fields ε on $\mathcal{M}$ closed w.r.t. the Q-derived bracket: $[\varepsilon, \varepsilon']_Q = [\varepsilon, [Q, \varepsilon']]$.

Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\mathcal{G}$-horizontal if $\varepsilon\omega = 0$, for any $\varepsilon \in \mathcal{G}$.

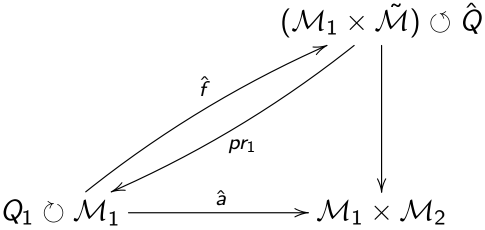
Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\mathcal{G}$-equivariant if $(ad_Q\varepsilon)\omega := [Q, \varepsilon]\omega = 0$, for any $\varepsilon \in \mathcal{G}$.

Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\mathcal{G}$-basic if it is $\mathcal{G}$-horizontal and $\mathcal{G}$-equivariant.

Remark. For Q-closed superfunctions $\mathcal{G}$-horizontal $\Leftrightarrow$ $\mathcal{G}$-basic

Key idea to apply to gauge theories:
Replace "gauge invariant" by "equivariantly Q-closed".

Q-morphisms for sigma models ("Above...")



where $\tilde{\mathcal{M}} = T[1]\mathcal{M}_2$,
$\hat{Q} = Q_1 + \tilde{Q} = Q_1 + \mathrm{d} + \mathcal{L}_{Q_2}$ – Q-structure on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$

Gauge transformations: $\delta_\varepsilon \hat{f}^*(\cdot) = \hat{f}^*(V_\varepsilon \cdot) = \hat{f}^*([\hat{Q}, \hat{\varepsilon}]\cdot)$,
where $\hat{\varepsilon}$ – degree -1 vector field on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$, vertical w.r.t. pr_1.

Gauge invariance of $S = \int_{\tilde{\Sigma}} \hat{f}^*(\bullet) \Leftrightarrow [\hat{Q}, \hat{\varepsilon}]\bullet = 0$
$\Leftrightarrow \bullet$ is equivariantly $\tilde{Q}$-closed

Poisson sigma model

World-sheet: Σ (closed, orientable, with no boundary, *dim* $= 2$).
Target: Poisson manifold (M, π). The functional is defined over the space of vector bundle morphisms $T\Sigma \to T^*M$

Field content: scalar fields $X^i : \Sigma \to M$ and 1-form valued ("vector") fields: $A_i \in \Omega^1(\Sigma, X^*T^*M)$.

The action functional: $S = \int_\Sigma A_i \wedge \mathrm{d}X^i + \frac{1}{2}\pi^{ij} A_i \wedge A_j$,

Equations of motion:

$$\begin{aligned} \mathrm{d}X^i &+ \pi^{ij} A_j = 0, \\ \mathrm{d}A_i &+ \pi^{jk}_{,i} A_j A_k = 0. \end{aligned}$$

Gauge transformations (complicated !):

$$\begin{aligned} \delta_\varepsilon X^i &= \pi^{ji} \varepsilon_j, \\ \delta_\varepsilon A_i &= \mathrm{d}\varepsilon_i + \pi^{jk}_{,i} A_j \varepsilon_k \end{aligned}$$

where $\varepsilon = \varepsilon_i \mathrm{d}X^i \in \Gamma(X^*T^*M)$ a 1-form.

(Twisted) Poisson sigma model

PSM (P.Schaller and T.Strobl, N.Ikeda – 1994)
Functional on vector bundle morphisms from $T\Sigma$ to T^*M, where (M, π) Poisson.

Twisted PSM, PSM with background
(C.Klimcik and T.Strobl, J.-S.Park – 2002)
Functional on vector bundle morphisms from $T\Sigma$ to T^*M, where $(M, (\pi, H))$ twisted Poisson.

$$S_{HPSM} = S_{PSM} + \int_{\Sigma^{(3)}} X^*(H)$$

where $\partial\Sigma^{(3)} = \Sigma$, $H \neq 0 \Rightarrow$ Wess-Zumino term.

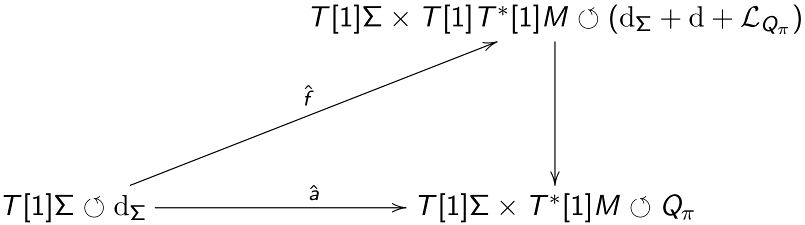
Dirac sigma model

Dirac sigma model (A.Kotov, P.Schaller, T.Strobl – 2005)
Functional on vector bundle morphisms from $T\Sigma$ to $\mathcal{D}$.
(Generalizes twisted PSM and G/G WZW model).

$$S^0_{DSM} = \int_\Sigma g(\mathrm{d}X, \stackrel{\wedge}{,} (1+\mathcal{O})A) + g(A, \stackrel{\wedge}{,} \mathcal{O}A) + \int_{\Sigma_3} H.$$

Important remark: Vector bundle morphisms $\to$ degree preserving maps between graded (Q-) manifolds
$\Rightarrow$ sigma models in the Q-language.

Gauge transformations of the PSM



Functional: $S_{PSM} = \int_{\Sigma^3} \hat{f}^*(\mathrm{d}p_i \mathrm{d}x^i)$.

E.o.m.: $\hat{f}^*(\mathrm{d}x^i) = 0$, $\hat{f}^*(\mathrm{d}p_i) = 0$.

Degree -1 vector field on $T^*[1]M$: $\varepsilon = \varepsilon_i \frac{\partial}{\partial p_i}$
$\rightarrow$ 1-form on M: $\varepsilon = \varepsilon_i \mathrm{d}x^i$
Canonical lift to $T[1]T^*[1]M$: $\tilde{\varepsilon} = \mathcal{L}_\varepsilon$.

Gauge transformations: $\hat{f}^*([\hat{Q}, \hat{\varepsilon}]\cdot)$. Invariance $\Leftrightarrow$ $\mathrm{d}\varepsilon = 0$

(cf. M.Bojowald, A.Kotov, T.Strobl '04; A.Kotov, T.Strobl '07).

Gauge transformations of the twisted PSM

Theorem (V.S., T.Strobl) Any smooth map from Σ to the space $\Gamma(T^*M)$ of sections of the cotangent bundle to a twisted Poisson manifold M defines an infinitesimal gauge transformation of the twisted PSM governed by (π, H) in the above sense, if and only if for any point $\sigma \in \Sigma$ the section $\varepsilon \in \Gamma(T^*M)$ satisfies

$$\mathrm{d}\varepsilon - \iota_{\pi^\sharp \varepsilon} H = 0,$$

where d is the de Rham differential on M.

Gauge transformations of the DSM

Theorem (V.S., T.Strobl) Any smooth map from Σ to the space $\Gamma(\mathcal{D})$ of sections of the Dirac structure $\mathcal{D} \subset TM \oplus T^*M$ defines an infinitesimal gauge transformation of the (metric independent part of the) Dirac sigma model governed by $\mathcal{D}$ in the above sense, if and only if for any point $\sigma \in \Sigma$ the section $v \oplus \eta \in \Gamma(\mathcal{D})$ satisfies

$$\mathrm{d}\eta - \iota_v H = 0,$$

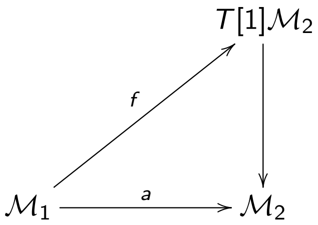
where d is the de Rham differential on M.

Remark 1. H non-degenerate – 2-plectic geometry.

Remark 2. Hydrodynamics (stationary Lamb equation).

Gauging via equivariant cohomology

KEY IDEA: Having $\mathfrak{g}$ isomorphic to $\mathcal{G}$ with a Q-derived bracket, gauging $\Leftrightarrow$ finding a (Q-)equivariantly (Q-)closed extension (everything on the target).



Vector fields on $\mathcal{M}_2$ – could be ambiguous.
Way out: consider vector fields on $T[1]\mathcal{M}_2 \Leftrightarrow$ extend the gauge algebra $\tilde{\mathcal{G}}$.

Can interpret for $\mathcal{M}_2 = T[1]T^*[1]M$ or $\mathcal{M}_2 = T[1]\mathcal{D}[1]M$.

Extension of the gauge algebra

Proposition. The Lie algebra $(\tilde{\mathcal{G}}, [,]_{\tilde{Q}})$ of degree -1 commuting vector fields, generalizing the $\mathcal{L}$. lift, is isomorphic to the semi-direct product of Lie algebras $\mathcal{G} \Subset \mathcal{A}$, where
$\mathcal{G}$ is a Lie algebra of 1-forms $T^*[1]M$ with the bracket

$$[\varepsilon^1, \varepsilon^2] = \mathcal{L}_{\pi^\# \varepsilon^1} \varepsilon^2 - \mathcal{L}_{\pi^\# \varepsilon^2} \varepsilon^1 - \mathrm{d}(\pi(\varepsilon^1, \varepsilon^2)) + \iota_{\pi^\# \varepsilon^1} \iota_{\pi^\# \varepsilon^2} H,$$

obtained from the (twisted) Lie algebroid of T^*M (anchor $= \pi^\#$);
$\mathcal{A}$ is a Lie algebra of covariant 2-tensors on M with a bracket

$$[\bar{\alpha}, \bar{\beta}] = < \pi^{23}, \bar{\alpha} \otimes \bar{\beta} - \bar{\beta} \otimes \bar{\alpha} >,$$

(the upper indeces "23" of π stand for the contraction on the 2d and 3rd entry of the tensor product);
$\mathcal{G}$ acts on $\mathcal{A}$ by

$$\rho(\varepsilon)(\bar{\alpha}) = \mathcal{L}_{\pi^\# \varepsilon}(\bar{\alpha}) - < \pi^{23}, (\mathrm{d}\varepsilon - \iota_{\pi^\# \varepsilon} H) \otimes \bar{\alpha} > .$$

Extension of the gauge algebra

Consider a subalgebra $\mathcal{GT} \subset \tilde{\mathcal{G}}$ defined by

$$\begin{aligned} \mathrm{d}\varepsilon - \iota_{\pi^{\#}\varepsilon} H = 0 \\ \bar{\alpha}^A = 0. \end{aligned}$$

Theorem (V.S., T.Strobl). Consider the graded manifold $\mathcal{M} = T[1]T^*[1]M$, equipped with the Q-structure $\tilde{Q} = \tilde{Q}_\pi$, governed by an H-twisted Poisson bivector Π, such that the pull-back of H to a dense set of orbits of Π is non-vanishing. The $\mathcal{GT}$ equivariantly closed extension of the given 3-form H defines uniquely the functional of the twisted Poisson sigma model.

Remark. Non-degeneracy of the pull-back of H is a sufficient condition.

Extension of the gauge algebra

Consider a subalgebra $\tilde{\mathcal{GT}} \subset \tilde{\mathcal{G}}$ defined by

$$\mathrm{d}\eta - \iota_v H = 0$$
$$\tilde{\alpha}^A = 0.$$

Theorem (V.S., T.Strobl).

Let H be a closed 3-form on M and D a Dirac structure on $(TM \oplus T^*M)_H$ such that the pullback of H to a dense set of orbits of D is non-zero. Then the $\widetilde{\mathcal{GT}}$-equivariantly closed extension $\tilde{H}$ of H is unique and $\int_{\Sigma^3} \widetilde{f}^*(\tilde{H})$ yields the (metric-independent part of) the Dirac sigma model on $\Sigma = \partial\Sigma^3$.

V.S., T.Strobl, "Dirac Sigma Models from Gauging", Journal of High Energy Physics, 11(2013)110.
V.S. "Graded geometry in gauge theories and beyond", Journal of Geometry and Physics, Volume 87, 2015.

Q-structures in gauge theories

Consider a gauge theory with some p-form gauge fields (e.g. $p \leq 2$). The most general form of the field strength for 0-form fields X^i, 1-form fields A^a and 2-form fields B^B respectively:

$$
\begin{aligned}
F^i &= \mathrm{d}X^i - \rho^i_a A^a, \\
F^a &= \mathrm{d}A^a + \frac{1}{2} C^a_{bc} A^b A^c - t^a_B B^B, \\
F^B &= \mathrm{d}B^B + \Gamma^B_{aC} A^a B^C - \frac{1}{6} H^B_{abc} A^a A^b A^c
\end{aligned}
$$

with arbitrary X-dependent coefficients, subject to Bianchi identities:

$$\mathrm{d}F^I|_{F^J=0} \equiv 0 \tag{2}$$

where $I, J = \{i, a, B\}$. $\Leftrightarrow$ In the algebra generated by A^I and $\mathrm{d}A^I$ the ideal $\mathcal{I}$ generated by F^I is differential: $\mathrm{d}\mathcal{I} \subset \mathcal{I}$.

Proposition (M.Grützmann, T.Strobl). (2) is equivalent to existence of a Q-structure on the target for a sufficiently large ($\geq$ **4**) dimension of the world-sheet.

Generality of the DSM in dim = 2.

A.Kotov, V.S., T.Strobl, "2d Gauge theories and generalized geometry", Journal of High Energy Physics, 08(2014)021.

Standard gauging: introduce Lie algebra valued 1-forms $A^a e_a \in \Omega^1(\Sigma, \mathfrak{g})$.

Lie algebra acting on the target $\Rightarrow e_a \mapsto v_a \in \Gamma(TM)$.
Rigid invariance $\Rightarrow e_a \mapsto \alpha_a \in \Gamma(T^*M)$

Composite gauge fields:
$(V, A) = (v_a A^a, \alpha_a A^a) \in \Omega^1(\Sigma, X^*(TM \oplus T^*M))$.

V and A are dependent $\Rightarrow$ isotropy condition.

Generality of the DSM in dim $= 2$.

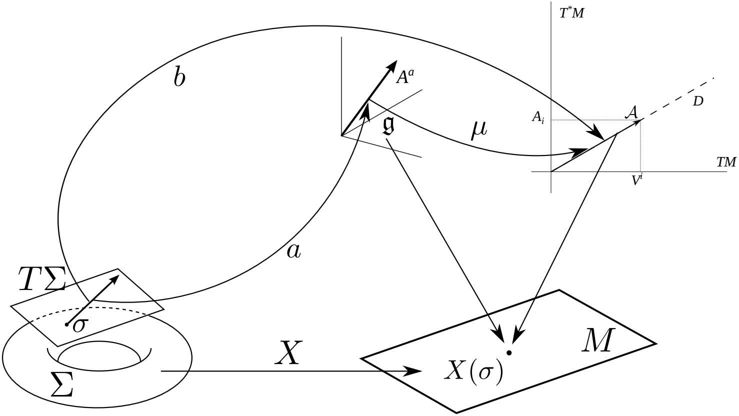
Courant algebroid: $E = TM \oplus T^*M$
Pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
Anchor: $\rho(v \oplus \eta) = v$
$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \iota_{v'} \mathrm{d}\eta + \iota_v \iota_{v'} H). \quad (*)$

Q–symplectic realization: $T^*[2]T[1]M$ $(p_i(1), \psi_i(2), \theta^i(1), x^i(0))$.
(cf. Roytenberg)
$Q = \{\mathcal{Q}, \cdot\}$, where $\mathcal{Q} = \theta^i \psi_i + \frac{1}{6} H_{ijk} \theta^i \theta^j \theta^k$ and
$\epsilon = \eta_i \theta^i + v^i p_i$, $\varepsilon = \{\epsilon, \cdot\}$. $[\varepsilon, \varepsilon']_Q \Leftrightarrow (*)$

To recover (A, V) gauge transformations, consider
$T[1]E, \tilde{Q} = \mathrm{d} + \mathcal{L}_Q, \tilde{\varepsilon} = \mathcal{L}_\varepsilon$.

$A_i = f^*(p_i), V^i = f^*(\theta^i) \Rightarrow \cdots \Rightarrow \cdots \Rightarrow$ DSM

Generality of the DSM in dim = 2.



Multigraded geometry

Joint work (in progress) with J.-P. Michel, T. Strobl, A. Kotov, ...

- Super Poisson sigma model, super Chern-Simons theory.
- AKSZ procedure
- Multigraded geometry $\rightarrow$ AKSZ
- Super sigma models – examples
- Supersymmetrization
- Applications to physical theories

… and beyond. (Commercial)[1]

- Poisson/symplectic/n-plectic/… geometry
- Dirac geometry (for dynamical systems or not)
- Equivariant theory/localization (e.g. Lie/Courant algebroids)
- Quantization
- Integrable systems
- Contact manifolds, jet spaces/bundles (e.g. for PDEs)
- …

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Thank you for your attention!