KRICHEVER–NOVIKOV TYPE ALGEBRAS AND WESS–ZUMINO–NOVIKOV–WITTEN MODELS

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Abstract. Krichever–Novikov type algebras are generalizations of the Witt, Virasoro, affine Lie algebras, and their relatives to Riemann surfaces of arbitrary genus and/or the multi-point situation. They play a very important role in the context of quantization of Conformal Field Theory. In this review we give the most important results about their structure, almost-grading and central extensions. Furthermore, we explain how they are used in the context of Wess–Zumino–Novikov–Witten models, respectively Knizhnik-Zamolodchikov connections. There they play a role as gauge algebras, as tangent directions to the moduli spaces, and as Sugawara operators.

1. Introduction

Krichever–Novikov (KN) type algebras are higher genus and/or multi-point generalizations of the nowadays well-known Virasoro algebra and its relatives. These classical algebras appear in Conformal Field Theory (CFT) [3], [51]. But this is not their only application. At many other places, in- and outside of mathematics, they play an important role. The algebras can be given by meromorphic objects on the Riemann sphere (genus zero) with possible poles only at $\{0, \infty\}$. For the Witt algebra these objects are vector fields. More generally, one obtains its central extension the Virasoro algebra, the current algebras and their central extensions the affine Kac-Moody algebras. For Riemann surfaces of higher genus, but still only for two points where poles are allowed, they were generalized by Krichever and Novikov [24], [25], [26] in 1986. In 1990 the author [31], [32], [33], [34] extended the approach further to the general multi-point case.

These extensions were not at all straight-forward. The main point is to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that an almost-grading, see Definition 4.1 below, will be enough to allow for the standard constructions in representation theory. In [33], [34] it was realized that a splitting of the set A of points where poles are allowed, into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading. The corresponding almost-grading was explicitly given. A Krichever-Novikov (KN) type algebra is an algebra of meromorphic objects with an almost-grading coming from such a splitting. In the classical situation there is only one such splitting possible (up to inversion), Hence, there is only one almost-grading, which is indeed a grading.

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An important feature forced by quantization is the construction and classification of central extensions. In the classical situation (i.e. the Virasoro case) there is only one non-trivial central extension. This is not true anymore in the higher genus and/or multi-point situation. The author classified [37], [38] bounded cocycles and showed uniqueness of non-trivial almost-graded central extensions.

Quite recently the book *Krichever–Novikov type algebras. Theory and applications* [43] written by the current author appeared. It gives a more or less complete treatment of the state of the art of the theory of KN type algebras including some applications. For more applications in direction of integrable systems and description of the Lax operator algebras see also the recent book *Current algebras on Riemann surfaces* [50] by Sheinman.

The goal of this review is to give an introduction to the KN type algebras in the multi-point setting, their definitions and main properties. Proofs are mostly omitted. They all can be found in [43], beside in the original works. KN type algebras carry a very rich representation theory. We have Verma modules, highest weight representations, Fermionic and Bosonic Fock representations, semi-infinite wedge forms, $b-c$ systems, Sugawara representations and vertex algebras. As much as these representations are concerned we are very short here and have to refer to [43] too. There also a quite extensive list of references can be found, including articles published by physicists on applications in the field-theoretical context. As a special application we consider in the current review the use of KN type algebras in the context of Wess–Zumino–Novikov–Witten (WZNW) models and Knizhnik–Zamolodchikov (KZ) equations, [47], [48], respectively [43]. There they play a role as gauge algebras, as tangent directions to the moduli spaces, and as Sugawara operators.

There are other applications which we cannot touch here. I only recall the fact that they supply examples of non-trivial deformations of the Witt and Virasoro algebras [11], [12], [13] despite the fact that they are formally rigid, see e.g. [40].

This review extends and updates in certain respects the previous reviews [42], [44] and has consequently some overlap with them.

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2. The Geometric Set-up for our algebras

2.1. **The geometric data.** In the following let $\Sigma = \Sigma_g$ be a compact Riemann surface without any restriction on its genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of points on Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_M) \tag{2.1}$$

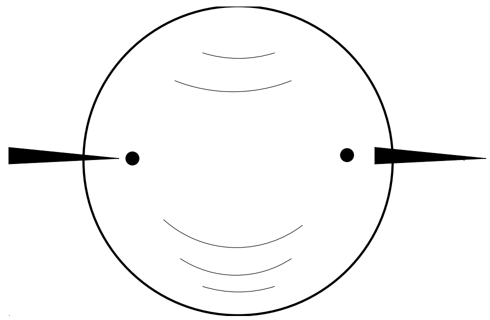

FIGURE 1. Riemann surface of genus zero with one incoming and one outgoing point.

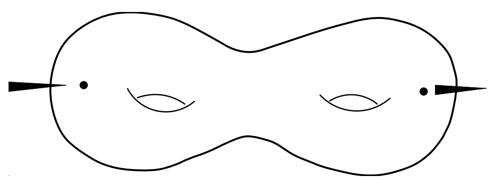

FIGURE 2. Riemann surface of genus two with one incoming and one outgoing point.

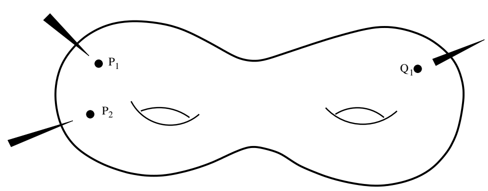

FIGURE 3. Riemann surface of genus two with two incoming points and one outgoing point.

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Occasionally, we consider I and O simply as sets[1].

Sometimes we refer to the classical situation. By this we understand

$$\Sigma_0 = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{z = 0\}, \quad O = \{z = \infty\}. \tag{2.2}$$

The figures should indicate the geometric picture. Figure 1 shows the classical situation. Figure 2 is the genus 2 two-point situation. Finally, in Figure 3 the case of a Riemann surface of genus 2 with two incoming points and one outgoing point is visualized.

[1]In the interpretation of CFT, the points in A correspond to points where free fields either are entering or leaving the region of interaction. In particular, there is from the very beginning a natural decomposition of the set of points A into two disjoint subsets I and O.

2.2. **Meromorphic forms.** To introduce the elements of our KN type algebras we first have to discuss forms of certain (conformal) weights. Recall that Σ is a compact Riemann surface of genus $g \geq 0$ and A a fixed finite subset of Σ. In fact we could allow for this and the following sections (as long as we do not talk about almost-grading) that A is an arbitrary subset. This includes the extremal cases $A = \emptyset$ or $A = \Sigma$.

Let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ. A holomorphic (meromorphic) section of $\mathcal{K}$, i.e. a *holomorphic (meromorphic) differential* is given as a collection of local meromorphic functions $(h_i)_{i \in J}$ with respect to a coordinate covering for which the transformation law

$$h_j = h_i \cdot c_1^1, \quad \text{with} \quad c_1 := \left(\frac{dz_j}{dz_i}\right)^{-1} = \frac{dz_i}{dz_j}. \tag{2.3}$$

is true. We will not make any distinction between the canonical bundle and its sheaf of sections, which is a locally free sheaf of rank 1.

In the following λ is either an integer or a half-integer. If λ is an integer then
(1) $\mathcal{K}^\lambda = \mathcal{K}^{\otimes \lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.
Here $\mathcal{K}^*$ denotes the dual line bundle of the canonical line bundle. This is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$.

If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle, sometimes called a *theta characteristics*. This means we fix a line bundle L for which $L^{\otimes 2} = \mathcal{K}$. After such a choice of L is done we set $\mathcal{K}^\lambda := \mathcal{K}_L^\lambda := L^{\otimes 2\lambda}$. In most cases we will drop the mentioning of L, but we have to keep the choice in mind. The fine-structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark 2.1. A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. For $g = 0$ we have $\mathcal{K} = \mathcal{O}(-2)$, and $L = \mathcal{O}(-1)$, the tautological bundle, is the unique square root. Already for $g = 1$ we have four non-isomorphic ones. As in this case $\mathcal{K} = \mathcal{O}$ one solution is $L_0 = \mathcal{O}$. But we have also other bundles L_i, $i = 1, 2, 3$. Note that L_0 has a nonvanishing global holomorphic section, whereas this is not the case for L_1, L_2 and L_3. In general, depending on the parity of the dimension of the space of globally holomorphic sections, i.e. of $\dim \mathrm{H}^0(\Sigma, L)$, one distinguishes even and odd theta characteristics L. For $g = 1$ the bundle $\mathcal{O}$ is an odd, the others are even theta characteristics. The choice of a theta characteristics is also called a spin structure on Σ [2].

We set

$$\mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid \\ f \text{ is holomorphic on } \Sigma \setminus A\}. \tag{2.4}$$

Obviously this is a $\mathbb{C}$-vector space. To avoid cumbersome notation, we will often drop the set A in the notation if A is fixed and/or clear from the context. Recall that in the case of half-integer λ everything depends on the theta characteristics L.

Definition 2.2. The elements of the space $\mathcal{F}^\lambda(A)$ are called *meromorphic forms of weight* λ (with respect to the theta characteristics L).

Remark 2.3. In the two extremal cases for the set A we obtain $\mathcal{F}^\lambda(\emptyset)$ the global holomorphic forms, and $\mathcal{F}^\lambda(\Sigma)$ all meromorphic forms. By compactness each $f \in \mathcal{F}^\lambda(\Sigma)$ will have only finitely many poles. In the case that $f \not\equiv 0$ it will also have only finitely many zeros.

For sections of $\mathcal{K}^\lambda$ with $\lambda \in \mathbb{Z}$ the transition functions are $c_\lambda = (c_1)^\lambda$, with c_1 from (2.3). The corresponding is true also for half-integer λ. In this case the basic transition function of the chosen theta characteristics L is given as $c_{1/2}$ and all others are integer powers of it. Symbolically, we write $\sqrt{dz_i}$ or $(dz)^{1/2}$ for the local frame, keeping in mind that the signs for the square root is not uniquely defined but depends on the bundle L.

If f is a meromorphic λ-form it can be represented locally by meromorphic functions f_i. We define for $P \in \Sigma$ the *order*

$$\operatorname{ord}_P(f) := \operatorname{ord}_P(f_i), \tag{2.5}$$

where $\operatorname{ord}_P(f_i)$ is the lowest nonvanishing order in the Laurent series expansion of f_i in the variable z_i around P. The order $\operatorname{ord}_P(f)$ is (strictly) positive if and only if P is a zero of f. It is negative if and only if P is a pole of f. Its value gives the order of the zero and pole respectively. By compactness our Riemann surface Σ, f can only have finitely many zeros and poles. We define the *(sectional) degree* of f to be

$$\operatorname{sdeg}(f) := \sum_{P \in \Sigma} \operatorname{ord}_P(f). \tag{2.6}$$

Proposition 2.4. *Let $f \in \mathcal{F}^\lambda$, $f \not\equiv 0$ then*

$$\operatorname{sdeg}(f) = 2\lambda(g-1). \tag{2.7}$$

For this and related results see [39].

3. Algebraic Structures

Next we introduce algebraic operations on the vector space of meromorphic forms of arbitrary weights. This space is obtained by summing over all weights

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{3.1}$$

The basic operations will allow us to introduce finally the algebras we are heading for.

3.1. **Associative structure.**

In this section A is still allowed to be an arbitrary subset of points in Σ. We will drop the subset A in the notation. The natural map of the locally free sheaves of rang one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{3.2}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{3.3}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\; dz^{\lambda+\nu}. \tag{3.4}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

The following is obvious

Proposition 3.1. *The space $\mathcal{F}$ is an associative and commutative graded (over $\frac{1}{2}\mathbb{Z}$) algebra. Moreover, $\mathcal{A} = \mathcal{F}^0$ is a subalgebra and the $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.*

Of course, $\mathcal{A}$ is the algebra of those meromorphic functions on Σ which are holomorphic outside of A. In case that $A = \emptyset$, it is the algebra of global holomorphic functions. By compactness, these are only the constants, hence $\mathcal{A}(\emptyset) = \mathbb{C}$. In case that $A = \Sigma$ it is the field of all meromorphic functions $\mathcal{M}(\Sigma)$.

3.2. **Lie and Poisson algebra structure.** Next we define a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (e, f) \mapsto [e, f], \tag{3.5}$$

which is defined in local representatives of the sections by

$$(e\, dz^\lambda, f\, dz^\nu) \mapsto [e\, dz^\lambda, f\, dz^\nu] := \left((-\lambda) e \frac{df}{dz} + \nu f \frac{de}{dz}\right) dz^{\lambda+\nu+1}, \tag{3.6}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 3.2. [43, Prop. 2.6 and 2.7] *The prescription $[.,.]$ given by (3.6) is well-defined and defines a Lie algebra structure on the vector space $\mathcal{F}$.*

Proposition 3.3. [43, Prop. 2.8] *The subspace $\mathcal{L} = \mathcal{F}^{-1}$ is a Lie subalgebra, and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.*

Theorem 3.4. [43, Thm. 2.10] *The space $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra.*

Next we consider important substructures. We already encountered the subalgebras $\mathcal{A}$ and $\mathcal{L}$. But there are more structures around.

3.3. **The vector field algebra and the Lie derivative.** First we look again at the Lie subalgebra $\mathcal{L} = \mathcal{F}^{-1}$. Here the Lie action respect the homogeneous subspaces $\mathcal{F}^\lambda$. As forms of weight -1 are vector fields, it could also be defined as the Lie algebra of those meromorphic vector fields on the Riemann surface Σ which are holomorphic outside of A. For vector fields we have the usual Lie bracket and the usual Lie derivative for their actions on forms. For the vector fields $e, f \in \mathcal{L}$ we have (again naming the local functions with the same symbol as the section)

$$[e, f]_| = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{3.7}$$

For the Lie derivative we get

$$\nabla_e(f)_| = L_e(g)_| = e\,.\,g_| = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right) dz^\lambda\ . \tag{3.8}$$

Obviously, these definitions coincide with the definitions already given above. But now we obtained a geometric interpretation.

3.4. **The algebra of differential operators.** If we look at $\mathcal{F}$, considered as Lie algebra, more closely, we see that $\mathcal{F}^0$ is an abelian Lie subalgebra and the vector space sum $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L}$ is also a Lie subalgebra. In an equivalent way this can also be constructed as semidirect sum of $\mathcal{A}$ considered as abelian Lie algebra and $\mathcal{L}$ operating on $\mathcal{A}$ by taking the derivative. It is called *Lie algebra of differential operators of degree* ≤ 1 and denoted by $\mathcal{D}^1$. In terms of elements the Lie product is

$$[(g,e),(h,f)] = (e\,.\,h - f\,.\,g\,,\,[e,f]). \tag{3.9}$$

The projection onto the second factor $(g,e) \mapsto e$ is a Lie homomorphism and we obtain a short exact sequences of Lie algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{D}^1 \longrightarrow \mathcal{L} \longrightarrow 0\ . \tag{3.10}$$

Hence $\mathcal{A}$ is an (abelian) Lie ideal of $\mathcal{D}^1$ and $\mathcal{L}$ a quotient Lie algebra. Obviously, $\mathcal{L}$ is also a subalgebra of $\mathcal{D}^1$.

Proposition 3.5. *The vector space $\mathcal{F}^\lambda$ becomes a Lie module over $\mathcal{D}^1$ by the operation*

$$(g,e).f := g \cdot f + e.f, \quad (g,e) \in \mathcal{D}^1(A),\ f \in \mathcal{F}^\lambda(A). \tag{3.11}$$

3.5. **Differential operators of all degree.** Differential operators of arbitrary degree acting on $\mathcal{F}^\lambda$ can be obtained via universal constructions. See [43] for more details.

3.6. **Lie superalgebras of half forms.** With the help of our associative product (3.2) we will obtain examples of Lie superalgebras. First we consider

$$\cdot\ \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L}\ , \tag{3.12}$$

and introduce the vector space $\mathcal{S}$ with the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f] + \varphi \cdot \psi, e\,.\,\varphi - f\,.\,\psi). \tag{3.13}$$

The elements of $\mathcal{L}$ are denoted by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

The definition (3.13) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a super-bracket (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi,e] := e\,.\,\varphi = (e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2} \tag{3.14}$$

and

$$[\varphi,\psi] := \varphi \cdot \psi\ . \tag{3.15}$$

We call the elements of $\mathcal{L}$ elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ elements of odd parity.

The sum (3.13) can also be described as $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 3.6. [43, Prop. 2.15] *The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Remark 3.7. For the relation of the superalgebra to the geometry of graded Riemann surfaces see Bryant [7]. Also Jordan superalgebras can be constructed in the above setting, see e.g. Leidwanger and Morier-Genoux [28]. This superalgebra is the so-called Neveu-Schwarz superalgebra. In physics literature there is another superalgbra of importance, the Ramond superalgebra. To define it additional geometric data consisting of pairs of marked points and paths between them is needed. On the relation to moduli space problems see the recent preprint of Witten [52].

3.7. **Higher genus current algebras.** We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$.

Definition 3.8. The *higher genus current algebra* associated to the Lie algebra $\mathfrak{g}$ and the geometric data (Σ, A) is the Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A) = \overline{\mathfrak{g}}(\Sigma, A)$ given as vector space by $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \qquad x, y \in \mathfrak{g}, \quad f, g \in \mathcal{A}. \tag{3.16}$$

Proposition 3.9. *$\overline{\mathfrak{g}}$ is a Lie algebra.*

As usual we will suppress the mentioning of (Σ, A) if not needed. The elements of $\overline{\mathfrak{g}}$ can be interpreted as meromorphic functions $\Sigma \to \mathfrak{g}$ from the Riemann surface Σ to the Lie algebra $\mathfrak{g}$, which are holomorphic outside of A.

For some applications it is useful to extend the definition by considering differential operators (of degree ≤ 1) associated to $\overline{\mathfrak{g}}$. We define $\mathcal{D}^1_{\mathfrak{g}} := \overline{\mathfrak{g}} \oplus \mathcal{L}$ and take in the summands the Lie product defined there and set additionally

$$[e, x \otimes g] := -[x \otimes g, e] := x \otimes (e.g). \tag{3.17}$$

This operation can be described as semidirect sum of $\overline{\mathfrak{g}}$ with $\mathcal{L}$. We get

Proposition 3.10. [43, Prop. 2.15] *$\mathcal{D}^1_{\mathfrak{g}}$ is a Lie algebra.*

3.8. **Krichever–Novikov type algebras.** Above the set A of points where poles are allowed was arbitrary. In case that A is finite and moreover $\#A \ge 2$ the constructed algebras are called Krichever–Novikov (KN) type algebras. In this way we get the KN vector field algebra, the function algebra, the current algebra, the differential operator algebra, the Lie superalgebra, etc. The reader might ask what is so special about this situation so that these algebras deserve special names. In fact in this case we can endow the algebra with a (strong) almost-graded structure. This will be discussed in the next section. The almost-grading is a crucial tool for extending the classical result to higher genus. Recall that in the classical case we have genus zero and $\#A = 2$.

Strictly speaking, a KN type algebra should be considered to be one of the above algebras for $2 \le \#A < \infty$ together with a fixed chosen almost-grading induced by the splitting $A = I \cup O$ into two disjoint non-empty subset, see Definition 4.1.

3.9. **The classical algebras.** If we use the above definitions of the algebras for the classical setting $g = 0$, $A = \{0, \infty\}$ we obtain the well-known classical algebras. For further reference we will already introduce their classical central extensions.

3.9.1. *The Witt algebra.* In the classical situation the vector field algebra $\mathcal{L}$ is called *Witt algebra* $\mathcal{W}$, sometimes also called Virasoro algebra without central term. It is the Lie algebra generated as vector space over $\mathbb{C}$ by the basis elements
$\{e_n = z^{n+1}\frac{d}{dz} \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}. \tag{3.18}$$

Here z is the quasi-global coordinate on $\mathbb{P}^1$. By setting the degree $\deg(e_n) := n$ it will become a graded Lie algebra.

3.9.2. *The Virasoro algebra.* For the Witt algebra the universal central extension is the *Virasoro algebra* $\mathcal{V}$. As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its basis elements are $\hat{e}_n,\ n \in \mathbb{Z}$ and t with the Lie product [2].

$$[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m}\, t, \qquad [\hat{e}_n, t] = [t, t] = 0\,. \tag{3.19}$$

By setting $\deg t := 0$ we extend the grading of $\mathcal{W}$ to $\mathcal{V}$.

3.9.3. *The affine Lie algebra.* In the classical situation the algebra $\mathcal{A}$ of functions corresponds to the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. Given $\mathfrak{g}$ a finite-dimensional Lie algebra (i.e. a finite-dimensional simple Lie algebra) then the tensor product of $\mathfrak{g}$ with the associative algebra $\mathcal{A} = \mathbb{C}[z, z^{-1}]$ introduced above writes as

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}. \tag{3.20}$$

This algebra $\overline{\mathfrak{g}}$ is the classical *current algebra* or *loop algebra*. Also in this case we consider central extensions. For this let β be a symmetric, bilinear form for $\mathfrak{g}$ which is invariant (e.g. $\beta([x, y], z) = \beta(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$). Then a central extension is given by

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot m\, \delta_n^{-m} \cdot t. \tag{3.21}$$

This algebra is denoted by $\widehat{\mathfrak{g}}$ and called *affine Lie algebra*. With respect to the classification of Kac-Moody Lie algebras, in the case of a simple $\mathfrak{g}$ they are exactly the Kac-Moody algebras of affine type, [19], [20], [29].

3.9.4. *The Lie superalgebra.* Also the classical Lie superalgebra of Neveu-Schwarz type shows up by the above constructions. It has as basis elements

$$e_n = z^{n+1}(dz)^{-1},\ n \in \mathbb{Z}, \quad \varphi_m = z^{m+1/2}(dz)^{-1/2},\ m \in \mathbb{Z} + \frac{1}{2}. \tag{3.22}$$

To avoid doubling I give the structure equations already for the central extension. Here t is an additional central element.

$$\begin{aligned}
[e_n, e_m] &= (m-n)e_{m+n} + \frac{1}{12}(n^3 - n)\,\delta_n^{-m}\, t,\\
[e_n, \varphi_m] &= (m - \frac{n}{2})\,\varphi_{m+n},\\
[\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\,(n^2 - \frac{1}{4})\,\delta_n^{-m}\, t.
\end{aligned} \tag{3.23}$$

3.9.5. *Generalisations.* Our classical algebras had certain important features which we like to recover at least to a certain extend in our general KN type algebras. They are graded algebra. They have a certain system of basis elements which are homogeneous with respect to this grading. A generalisation of this is obtained by the almost-grading discussed in Section 4.

Furthermore, we have central extensions, which indeed are forced by the applications, e.g. by the quantization of field theories and their necessary regularisation. In the classical situation they are essentially unique (up to equivalence). For the generalised algebras we will discuss this in Section 5.

Let me point out here that a special central extension of the function algebra $\mathcal{A}$ will be the (infinite dimensional) Heisenberg algebra.

[2]Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero.

4. Almost-Graded Structure

4.1. **Definition of almost-gradedness.** In the classical situation discussed in Section 3.9 the algebras introduced in the last section are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [24] in the two point case there is a weaker concept, an almost-grading, which to a large extend is a valuable replacement of a honest grading.

Definition 4.1. Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,

(ii) There exists constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

If $\dim \mathcal{L}_n$ is bounded with a bound independent of n we call $\mathcal{L}$ *strongly almost-graded.* If we drop the condition that $\dim \mathcal{L}_n$ is finite we call $\mathcal{L}$ *weakly almost-graded.*

Note that in [24] the name *quasi-grading* was used instead.

In a similar manner almost-graded modules over almost-graded algebras are defined. We can extend in an obvious way the definition to superalgebras, respectively even to more general algebraic structures. This definition makes complete sense also for more general index sets $\mathbb{J}$. In fact, we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ in the case of superalgebras. The even elements (with respect to the super-grading) will have integer degree, the odd elements half-integer degree.

4.2. **Separating cycle and Krichever-Novikov pairing.** Before we give the almost-grading we introduce an important geometric structure. Let C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, K$ and C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$.

A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points. It might have more than one component. In the following we will integrate meromorphic differentials on Σ without poles in $\Sigma \setminus A$ over closed curves C. Hence, we might consider C and C' as equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma \setminus A, \mathbb{Z})$. In this sense we write for every separating cycle

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{4.1}$$

The minus sign appears due to the opposite orientation. Another way for giving such a C_S is via level lines of a "proper time evolution", for which I refer to [43, Section 3.9].

Given such a separating cycle C_S (respectively cycle class) we define a linear map

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega. \tag{4.2}$$

The map will not depend on the separating line C_S chosen, as two of such will be homologous and the poles of ω are only located in I and O.

Consequently, the integration of ω over C_S can also be described over the special cycles C_i or equivalently over C_j^*. This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}}\int_{C_S}\omega \;=\; \sum_{i=1}^{K}\mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M}\mathrm{res}_{Q_l}(\omega). \tag{4.3}$$

Definition 4.2. The pairing

$$\mathcal{F}^{\lambda}\times\mathcal{F}^{1-\lambda}\to\mathbb{C},\quad (f,g)\mapsto\langle f,g\rangle:=\frac{1}{2\pi\mathrm{i}}\int_{C_S} f\cdot g, \tag{4.4}$$

between λ and $1-\lambda$ forms is called *Krichever-Novikov (KN) pairing.*

Note that the pairing depends not only on A (as the $\mathcal{F}^{\lambda}$ depend on it) but also critically on the splitting of A into I and O as the integration path will depend on it. Once the splitting is fixed the pairing will be fixed too.

In fact, there exit dual basis elements (see (4.9)) hence the pairing is non-degenerate.

4.3. **The homogeneous subspaces.** Given the vector spaces $\mathcal{F}^{\lambda}$ of forms of weight $\mathcal{L}$ we will now single out subspaces $\mathcal{F}^{\lambda}_m$ of degree m by giving a basis of these subspaces. This has been done in the 2-point case by Krichever and Novikov [24] and in the multi-point case by the author [31], [32], [33], [34], see also Sadov [30]. See in particular [43, Chapters 3,4,5] for a complete treatment. All proofs of the statements to come can be found there. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O.

Depending on whether the weight λ is integer or half-integer we set $\mathbb{J}_\lambda=\mathbb{Z}$ or $\mathbb{J}_\lambda=\mathbb{Z}+1/2$. For $\mathcal{F}^{\lambda}$ we introduce for $m\in\mathbb{J}_\lambda$ subspaces $\mathcal{F}^{\lambda}_m$ of dimension K, where $K=\#I$, by exhibiting certain elements $f^{\lambda}_{m,p}\in\mathcal{F}^{\lambda}$, $p=1,\ldots,K$ which constitute a basis of $\mathcal{F}^{\lambda}_m$. Recall that the spaces $\mathcal{F}^{\lambda}$ for $\lambda\in\mathbb{Z}+1/2$ depend on the chosen square root L (the theta characteristic) of the bundle chosen. The elements are the elements of degree m. As explained in the following, the degree is in an essential way related to the zero orders of the elements at the points in I.

Let $I=\{P_1,P_2,\ldots,P_K\}$ then we have for the zero-order at the point $P_i\in I$ of the element $f^{\lambda}_{n,p}$

$$\mathrm{ord}_{P_i}(f^{\lambda}_{n,p})=(n+1-\lambda)-\delta_i^p,\quad i=1,\ldots,K\;. \tag{4.5}$$

The prescription at the points in O is made in such a way that the element $f^{\lambda}_{m,p}$ is essentially uniquely given. Essentially unique means up to multiplication with a constant[3]. After fixing as additional geometric data a system of coordinates z_l centered at P_l for $l=1,\ldots,K$ and requiring that

$$f^{\lambda}_{n,p}(z_p)=z_p^{n-\lambda}(1+O(z_p))(dz_p)^{\lambda} \tag{4.6}$$

the element $f_{n,p}$ is uniquely fixed. In fact, the element $f^{\lambda}_{n,p}$ only depends on the first jet of the coordinate z_p.

[3]Strictly speaking, there are some special cases where some constants have to be added such that the Krichever-Novikov duality (4.9) is valid.

Here we will not give the general recipe for the prescription at the points in O. Just to give an example, which is also the important special case appearing in the case of WZNW models discussed later, assume $O = \{Q\}$ is a one-element set. If either the genus $g = 0$, or $g \geq 2$, $\lambda \neq 0, 1/2, 1$ and the points in A are in generic position then we require

$$\operatorname{ord}_Q(f_{n,p}^{\lambda}) = -K \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1). \tag{4.7}$$

In the other cases (e.g. for $g = 1$) there are some modifications at the point in O necessary for finitely many m.

Theorem 4.3. [43, Thm. 3.6] *Set*

$$\mathcal{B}^{\lambda} := \{\, f_{n,p}^{\lambda} \mid n \in \mathbb{J}_{\lambda},\ p = 1, \ldots, K \,\}. \tag{4.8}$$

Then (a) $\mathcal{B}^{\lambda}$ is a basis of the vector space $\mathcal{F}^{\lambda}$.

(b) The introduced basis $\mathcal{B}^{\lambda}$ of $\mathcal{F}^{\lambda}$ and $\mathcal{B}^{1-\lambda}$ of $\mathcal{F}^{1-\lambda}$ are dual to each other with respect to the Krichever-Novikov pairing (4.4), i.e.

$$\langle f_{n,p}^{\lambda}, f_{-m,r}^{1-\lambda} \rangle = \delta_p^r\, \delta_n^m, \quad \forall n, m \in \mathbb{J}_{\lambda}, \quad r, p = 1, \ldots, K. \tag{4.9}$$

In particular, from part (b) of the theorem it follows that the Krichever-Novikov pairing is non-degenerate. Moreover, any element $v \in \mathcal{F}^{1-\lambda}$ acts as linear form on $\mathcal{F}^{\lambda}$ via

$$\Phi_v : \mathcal{F}^{\lambda} \mapsto \mathbb{C}, \quad w \mapsto \Phi_v(w) := \langle v, w \rangle. \tag{4.10}$$

Via this pairing $\mathcal{F}^{1-\lambda}$ can be considered as restricted dual of $\mathcal{F}^{\lambda}$. The identification depends on the splitting of A into I and O as the KN pairing depends on it. The full space $(\mathcal{F}^{\lambda})^*$ can even be described with the help of the pairing in a "distributional interpretation" via the distribution $\Phi_{\hat{v}}$ associated to the formal series

$$\hat{v} := \sum_{m \in \mathbb{J}_{\lambda}} \sum_{p=1}^{K} a_{m,p} f_{m,p}^{1-\lambda}, \quad a_{m,p} \in \mathbb{C}\ . \tag{4.11}$$

The elements of the dual space of vector fields $\mathcal{L}$ will be given by the formal series (4.11) with basis elements from $\mathcal{F}^2$, the quadratic differentials; the elements of the dual of $\mathcal{A}$ correspondingly from $\mathcal{F}^1$, the differentials; and the elements of the dual of $\mathcal{F}^{-1/2}$ correspondingly from $\mathcal{F}^{3/2}$.

It is quite convenient to use special notations for elements of some important weights:

$$\begin{gathered} e_{n,p} := f_{n,p}^{-1}, \quad \varphi_{n,p} := f_{n,p}^{-1/2}, \quad A_{n,p} := f_{n,p}^{0}, \\ \omega^{n,p} := f_{-n,p}^{1}, \quad \Omega^{n,p} := f_{-n,p}^{2}. \end{gathered} \tag{4.12}$$

In view of (4.9) for the forms of weight 1 and 2 we invert the index n and write it as a superscript.

Remark 4.4. It is also possible (and for certain applications necessary) to write explicitely down the basis elements $f_{n,p}^{\lambda}$ in terms of "usual" objects defined on the Riemann surface Σ. For genus zero they can be given with the help of rational functions in the quasi-global variable z. For genus one (i.e. the torus case) representations with the help of Weierstraß σ and Weierstraß $\wp$ functions exists. For genus ≥ 1 there exists expressions in terms of theta functions (with characteristics) and prime forms. Here the Riemann surface has first to be embedded into its Jacobian via the Jacobi map. See [43, Chapter 5], [32], [35] for more details.

4.4. **The almost-graded algebras.**

Theorem 4.5. [43, Thm. 3.8] *There exists constants R_1 and R_2 (depending on the number and splitting of the points in A and on the genus g) independent of λ and ν and independent of $n, m \in \mathbb{J}$ such that for the basis elements*

$$
\begin{aligned}
f^{\lambda}_{n,p} \cdot f^{\nu}_{m,r} = \quad & f^{\lambda+\nu}_{n+m,r}\delta^r_p \\
& + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu}_{h,s}, \qquad a^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C},
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
[f^{\lambda}_{n,p}, f^{\nu}_{m,r}] = \quad & (-\lambda m + \nu n)\, f^{\lambda+\nu+1}_{n+m,r}\delta^r_p \\
& + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu+1}_{h,s}, \quad b^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C}.
\end{aligned}
$$

This says in particular that with respect to both the associative and Lie structure the algebra $\mathcal{F}$ is weakly almost-graded. In generic situations and for $N = 2$ points one obtains $R_1 = g$ and $R_2 = 3g$.

The reason why we only have weakly almost-gradedness is that

$$
\mathcal{F}^{\lambda} = \bigoplus_{m \in \mathbb{J}_\lambda} \mathcal{F}^{\lambda}_m, \qquad \text{with} \quad \dim \mathcal{F}^{\lambda}_m = K,
\tag{4.14}
$$

and if we add up for a fixed m all λ we get that our homogeneous spaces are infinite dimensional.

In the definition of our KN type algebra only finitely many λs are involved, hence the following is immediate

Theorem 4.6. *The Krichever-Novikov type vector field algebras $\mathcal{L}$, function algebras $\mathcal{A}$, differential operator algebras $\mathcal{D}^1$, Lie superalgebras $\mathcal{S}$, and Jordan superalgebras $\mathcal{J}$ are (strongly) almost-graded algebras and the corresponding modules $\mathcal{F}^{\lambda}$ are almost-graded modules.*

We obtain with $n \in \mathbb{J}_\lambda$

$$
\begin{gathered}
\dim \mathcal{L}_n = \dim \mathcal{A}_n = \dim \mathcal{F}^{\lambda}_n = K, \\
\dim \mathcal{S}_n = 2K, \quad \dim \mathcal{D}^1_n = 3K\,.
\end{gathered}
\tag{4.15}
$$

If $\mathcal{U}$ is any of these algebras, with product denoted by $[\ ,\ ]$ then

$$
[\mathcal{U}_n, \mathcal{U}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_i} \mathcal{U}_h,
\tag{4.16}
$$

with $R_i = R_1$ for $\mathcal{U} = \mathcal{A}$ and $R_i = R_2$ otherwise.

The lowest degree term component in (4.13) for certain special cases reads as

$$
\begin{aligned}
A_{n,p} \cdot A_{m,r} &= A_{n+m,r}\,\delta^p_r \ + \ \text{h.d.t.} \\
A_{n,p} \cdot f^{\lambda}_{m,r} &= f^{\lambda}_{n+m,r}\,\delta^p_r \ + \ \text{h.d.t.} \\
[e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\,\delta^p_r \ + \ \text{h.d.t.} \\
e_{n,p} \,.\, f^{\lambda}_{m,r} &= (m + \lambda n) \ \cdot f^{\lambda}_{n+m,r}\,\delta^p_r \ + \ \text{h.d.t.}
\end{aligned}
\tag{4.17}
$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_i$, Finally, the almost-grading of $\mathcal{A}$ induces an almost-grading of the current algebra $\overline{\mathfrak{g}}$ by setting $\overline{\mathfrak{g}}_n = \mathfrak{g} \otimes \mathcal{A}_n$. We obtain

$$\overline{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \overline{\mathfrak{g}}_n, \quad \dim \overline{\mathfrak{g}}_n = K \cdot \dim \mathfrak{g}. \tag{4.18}$$

4.5. **Triangular decomposition and filtrations.** Let $\mathcal{U}$ be one of the above introduced algebras (including the current algebra). On the basis of the almost-grading we obtain a triangular decomposition of the algebras

$$\mathcal{U} = \mathcal{U}_{[+]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[-]}, \tag{4.19}$$

where

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{4.20}$$

By the almost-gradedness the $[+]$ and $[-]$ subspaces are (infinite dimensional) subalgebras. The $[0]$ spaces in general not. Sometimes we will use *critical strip* for them.

With respect to the almost-grading of $\mathcal{F}^\lambda$ we introduce a filtration

$$\begin{gathered} \mathcal{F}^\lambda_{(n)} := \bigoplus_{m\ge n} \mathcal{F}^\lambda_m, \\ \ldots\ldots \quad \supseteq \quad \mathcal{F}^\lambda_{(n-1)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n+1)} \quad \ldots\ldots \end{gathered} \tag{4.21}$$

Proposition 4.7. [43, Prop. 3.15]

$$\mathcal{F}^\lambda_{(n)} = \{\ f \in \mathcal{F}^\lambda \mid \operatorname{ord}_{P_i}(f) \ge n - \lambda,\ \forall i = 1, \ldots, K\ \}. \tag{4.22}$$

In case that O has more than one point there are certain choices, e.g. numbering of the points in O, different rules, etc. involved in defining the almost-grading. Hence, if the choices are made differently the subspaces $\mathcal{F}^\lambda_n$ might depend on them, and consequently also the almost-grading. But by this proposition the induced filtration is indeed canonically defined via the splitting of A into I and O.

Moreover, different choices will give equivalent almost-grading. We stress the fact, that under a KN type algebra we will mostly understand one of the introduced algebras together with an almost-grading (respectively equivalence class of almost-grading, respectively filtration) introduced by the splitting $A = I \cup O$.

5. Central Extensions

Central extension of our algebras appear naturally in the context of quantization and regularization of actions. Of course they are also of independent mathematical interest.

5.1. **Central extensions and cocycles.** Recall that equivalence classes of central extensions of a Lie algebra W are classified by the elements of the Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ with values in the trivial module $\mathbb{C}$. The central extension $\widehat{\mathcal{W}}$ decomposes as vector space $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ then its Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \psi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W\,. \tag{5.1}$$

Here $\psi \in [\psi]$ where $[\psi] \in \mathrm{H}^2(W, \mathbb{C})$.

Recall that the Lie algebra 2-cocycle condition reads as

$$0 = d_2\psi(x,y,z) := \psi([x,y],z) + \psi([y,z],x) + \psi([z,x],y). \tag{5.2}$$

This condition is equivalent to the fact, that $\widehat{\mathcal{W}}$ fulfills the Jacobi identity.

5.2. **Geometric cocycles.** For the Witt algebra (i.e. the vector field algebra in the classical situation) we have $\dim \mathrm{H}^2(\mathcal{W},\mathbb{C}) = 1$. hence there are only two essentially different central extensions, the splitting one given by the direct sum $\mathbb{C} \oplus \mathcal{W}$ of Lie algebras, and the up to equivalence and rescaling unique non-trivial one, the Virasoro algebra $\mathcal{V}$. For $\mathcal{V}$ we already gave the algebraic structure above (3.19). Recall that the defining cocycle reads as

$$\psi(e_n,e_m) = \frac{1}{12}(n^3-n)\delta_n^{-m}. \tag{5.3}$$

Obviously it does not make any sense in the higher genus and/or multi-point case. We need to find a geometric description. For this we have first to introduce connections.

5.2.1. *Projective and affine connections.* Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$.

Definition 5.1. (a) A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{5.4}$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

(b) A system of local (holomorphic, meromorphic) functions $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *affine connection* if it transforms as

$$T_\beta(z_\beta)\cdot(f'_{\beta,\alpha}) = T_\alpha(z_\alpha) + \frac{f''_{\beta,\alpha}}{f'_{\beta,\alpha}}. \tag{5.5}$$

Every Riemann surface admits a holomorphic projective connection [16],[14]. Given a point P then there exists always a meromorphic affine connection holomorphic outside of P and having maximally a pole of order one there [34].

From their very definition it follows that the difference of two affine (projective) connections will be a (quadratic) differential. Hence, after fixing one affine (projective) connection all others are obtained by adding (quadratic) differentials.

5.2.2. *The function algebra $\mathcal{A}$.* We consider it as abelian Lie algebra. In the following let C always be an arbitrary smooth not necessarily connected curve not meeting A. We obtain the following cocycle

$$\psi^1_C(g,h) := \frac{1}{2\pi \mathrm{i}}\int_C g dh, \quad g,h\in\mathcal{A}. \tag{5.6}$$

5.2.3. *The current algebra $\overline{\mathfrak{g}}$.* For $\overline{\mathfrak{g}} = \mathfrak{g}\otimes\mathcal{A}$ we fix a symmetric, invariant, bilinear form β on $\mathfrak{g}$ (not necessarily non-degenerate). Recall, that invariance means that we have $\beta([x,y],z) = \beta(x,[y,z])$ for all $x,y,z\in\mathfrak{g}$. Then a cocycle is given by

$$\psi^2_{C,\beta}(x\otimes g, y\otimes h) := \beta(x,y)\cdot\frac{1}{2\pi \mathrm{i}}\int_C g dh, \quad x,y\in\mathfrak{g},\ g,h\in\mathcal{A}. \tag{5.7}$$

5.2.4. *The vector field algebra $\mathcal{L}$.* Here it is a little bit more delicate. First we have to choose a (holomorphic) projective connection R. We define

$$\psi^3_{C,R}(e,f) := \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \; . \tag{5.8}$$

Only by the term coming with the projective connection it will be a well-defined differential, i.e. independent of the coordinate chosen. It is shown in [34] (and [43]) that it is a cocycle. Another choice of a projective connection will result in a cohomologous cocycle. Hence, the equivalence class of the central extension will be the same.

5.2.5. *The differential operator algebra $\mathcal{D}^1$.* For the differential operator algebra the cocycles of type (5.6) for $\mathcal{A}$ can be extended by zero on the subspace $\mathcal{L}$. The cocycles for $\mathcal{L}$ can be pulled back. In addition there is a third type of cocycles mixing $\mathcal{A}$ and $\mathcal{L}$:

$$\psi^4_{C,T}(e,g) := \frac{1}{24\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A}, \tag{5.9}$$

with an affine connection T, with at most a pole of order one at a fixed point in O. Again, a different choice of the connection will not change the cohomology class. For more details on the cocycles see [37], [43].

5.2.6. *The Lie superalgebra $\mathcal{S}$.* The Lie superalgebra $\mathcal{S}$ has the vector field algebra $\mathcal{L}$ as Lie subalgebra. In particular, every cocycle of $\mathcal{S}$ will define a cocycle of $\mathcal{L}$. It is shown in [41], [43] that for C as above and R any (holomorphic) projective connection the bilinear extension of

$$\begin{aligned} \Phi_{C,R}(e,f) &:= \psi^3_{C,R}(e,f) \\ \Phi_{C,R}(\varphi,\psi) &:= -\frac{1}{24\pi \mathrm{i}} \int_C (\varphi'' \cdot \psi + \varphi \cdot \psi'' - R \cdot \varphi \cdot \psi)\, dz \\ \Phi_{C,R}(e,\varphi) &:= 0 \end{aligned} \tag{5.10}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. A different projective connection will yield a cohomologous cocycle.

A similar formula was given by Bryant in [7]. By adding the projective connection in the second part of (5.10) he corrected some formula appearing in [6]. He only considered the two-point case and only the integration over a separating cycle. See also [23] for the multi-point case, where still only the integration over a separating cycle is considered.

In contrast to the differential operator algebra case the two parts cannot be prescribed independently. Only with the same integration path (more precisely, homology class) and the given factors in front of the integral it will work. The reason for this is that there are cocycle conditions relating vector fields and $(-1/2)$-forms.

5.3. **Uniqueness and classification of central extensions.** The above introduced cocycles depend on the choice of the connections R and T. Different choices will not change the cohomology class. Hence, this ambiguity does not disturb us. What really matters is that they depend on the integration curve C chosen.

In contrast to the classical situation, for the higher genus and/or multi-point situation there are many non-homologous different closed curves inducing non-equivalent central extensions defined by the integration.

But we should take into account that we want to extend the almost-grading from our algebras to the centrally extended ones. This means we take $\deg \hat{x} := \deg x$ and assign a degree $\deg(t)$ to the central element t, and still we want to obtain almost-gradedness.

This is possible if and only if our defining cocycle ψ is "local" in the following sense (the name was introduced in the two point case by Krichever and Novikov in [24]). There exists $M_1, M_2 \in \mathbb{Z}$ such that

$$\forall n, m: \quad \psi(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies M_1 \leq n + m \leq M_2. \tag{5.11}$$

Here $\mathcal{U}$ stands for any of our algebras (including the supercase). It is very important that "local" is defined in terms of the almost-grading, and the almost-grading itself depends on the splitting $A = I \cup O$. Hence what is "local" depends on the splitting too.

We will call a cocycle *bounded* (from above) if there exists $M \in \mathbb{Z}$ such that

$$\forall n, m: \quad \psi(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies n + m \leq M. \tag{5.12}$$

Similarly bounded from below can be defined. Locality means bounded from above and from below. Given a cocycle class we call it bounded (respectively local) if and only if it contains a representing cocycle which is bounded (respectively local). Not all cocycles in a bounded (local) class have to be bounded (local).

If we choose as integration path a separating cocycle C_S, or one of the C_i then the above introduced geometric cocycles are local, respectively bounded. Recall that in this case integration can be done by calculating residues at the in-points or at the out-points. All these cocycles are cohomologically nontrivial. The theorems in the following concern the opposite direction. They were treated in my works [37], [38], [41]. See also [43] for a complete and common treatment.

The following result for the vector field algebra $\mathcal{L}$ gives the principal structure of the classification results.

Theorem 5.2. [37], [43, Thm. 6.41] *Let $\mathcal{L}$ be the Krichever–Novikov vector field algebra with a given almost-grading induced by the splitting $A = I \cup O$.*
(a) The space of bounded cohomology classes is K-dimensional ($K = \#I$). A basis is given by setting the integration path in (5.8) to C_i, $i = 1, \ldots, K$ the little (deformed) circles around the points $P_i \in I$.
(b) The space of local cohomology classes is one-dimensional. A generator is given by integrating (5.8) over a separating cocycle C_S, i.e.

$$\psi^3_{C_S,R}(e, f) = \frac{1}{24\pi \mathrm{i}} \int_{C_S} \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \ . \tag{5.13}$$

(c) Up to equivalence and rescaling there is only one non-trivial one-dimensional central extension $\widehat{\mathcal{L}}$ of the vector field algebra $\mathcal{L}$ which allows an extension of the almost-grading.

Remark 5.3. In the classical situation, Part (c) shows also that the Virasoro algebra is the unique non-trivial central extension of the Witt algebra (up to equivalence and rescaling). This result extends to the more general situation under the condition that one fixes the almost-grading, hence the splitting $A = I \cup O$. Here I like to repeat the fact that for $\mathcal{L}$, depending on the set A and its possible splittings into two disjoint subsets, there are different almost-gradings. Hence, the "unique" central extension finally obtained will also depend on the splitting. Only in the two point case there is only one splitting possible. In the case that the genus $g \geq 1$ there are even integration paths possible in the definition of (5.8) which are not homologous to a separating cycle of any splitting. Hence, there are other central extensions possible not corresponding to any almost-grading.

The above theorem is a model for all other classification results. We will always obtain a statement about the bounded (from above) cocycles and then for the local cocycles. The cocycles will be explicitely given as geometric cocycles introduced above with integration over the C_i and C_S respectively.

If we consider the function algebra $\mathcal{A}$ as an abelian Lie algebra then every skew-symmetric bilinear form will be a non-trivial cocycle. Hence, there is no hope of uniqueness. But if we add the condition of $\mathcal{L}$-invariance, which is given as

$$\psi(e.g, h) + \psi(g, e.h) = 0, \quad \forall e \in \mathcal{L},\ g, h \in \mathcal{A} \tag{5.14}$$

things will change.

Let us denote the subspace of local cohomology classes by H^2_{loc}, and the subspace of local and $\mathcal{L}$-invariant cohomology classes by $\mathrm{H}^2_{\mathcal{L},loc}$. Note that the condition is only required for at least one representative in the cohomology class. We collect a part of the results for the cocycle classes of the other algebras in the following theorem.

Theorem 5.4. [43, Cor. 6.48]

(a) $\dim \mathrm{H}^2_{\mathcal{L},loc}(\mathcal{A}, \mathbb{C}) = 1$,

(b) $\dim \mathrm{H}^2_{loc}(\mathcal{L}, \mathbb{C}) = 1$,

(c) $\dim \mathrm{H}^2_{loc}(\mathcal{D}^1, \mathbb{C}) = 3$,

(d) $\dim \mathrm{H}^2_{loc}(\overline{\mathfrak{g}}, \mathbb{C}) = 1$ *for* $\mathfrak{g}$ *a simple finite-dimensional Lie algebra,*

(e) $\dim \mathrm{H}^2_{loc}(\mathcal{S}, \mathbb{C}) = 1$.

A basis of the cohomology spaces are given by taking the cohomology classes of the cocycles (5.6), (5.8), (5.9), (5.7), (5.10) obtained by integration over a separating cycle C_S.

Consequently, we obtain also for these algebras the corresponding result about uniqueness of almost-graded central extensions. For the differential operator algebra we get three independent cocycles. This generalizes results of [1] valid for the classical case.

For the results on the bounded cocycle classes we have to multiply the dimensions above by $K = \#I$. In the supercase the central element which we consider here is of even parity. For the supercase with odd central element the bounded cohomology vanishes [41].

For $\mathfrak{g}$ a reductive Lie algebra and if the cocycle is $\mathcal{L}$-invariant if restricted to the abelian part, a complete classification of local cocycle classes for both $\overline{\mathfrak{g}}$ and $\mathcal{D}^1_{\mathfrak{g}}$ can be found in [38], [43, Chapter 9].

I like to mention that in all the applications I know of, the cocycles coming from representations, regularizations, etc. are local. Hence, the uniqueness or classification results presented above can be used.

Remark 5.5. I classified in [45] for the multi-point genus zero situation all 2nd cohomology for the above algebras. In particular, we showed that all classes are geometric. This is done by showing that all classes are bounded classes with respect to the "standard splitting" of $N-1$ points in I and one point in O. Hence, from [37],

[38] they can be explicitly given as geometric cocycles. In such a way the universal central extensions can be obtained. This chain of arguments do not work in higher genus as there are cocycle classes which will never be bounded with respect to any splitting.

6. Sugawara Representation

In the classical set-up the Sugawara construction relates to a representation of the classical affine Lie algebra $\widehat{\mathfrak{g}}$ a representation of the Virasoro algebra, see e.g. [20], [21]. In some sense it assigns to a gauge symmetry a conformal symmetry.

In joint work with O. Sheinman the author succeeded in extending it to arbitrary genus and the multi-point setting [46]. For an updated treatment, incorporating also the uniqueness results of central extensions, see [43, Chapter 10].

We will need the Sugawara operators to define the Knizhnik-Zamolodchikov (KZ) connection. Hence, we will give a very rough sketch of it here.

We start with an admissible representation V of a centrally extended current algebra $\widehat{\mathfrak{g}}$ (i.e. the affine Lie algebra of KN type). Admissible means, that the central element operates as constant $\times$ identity, and that every element v in the representation space will be annihilated by the elements in $\widehat{\mathfrak{g}}$ of sufficiently high degree (the degree might depend on the element v).

For simplicity let $\mathfrak{g}$ be either abelian or simple and β the non-degenerate symmetric invariant bilinear form used to construct $\widehat{\mathfrak{g}}$ (now we need that it is non-degenerate). Note that in the case of a simple Lie algebra every symmetric, invariant bilinear form β is a multiple of the Cartan-Killing form.

Let $\{u_i\},\ \{u^j\},\ i = 1, \ldots, \dim \mathfrak{g}$ be a system of dual basis elements for $\mathfrak{g}$ with respect to β, i.e. $\beta(u_i, u^j) = \delta_i^j$. Note that the Casimir element of $\mathfrak{g}$ can be given by $\sum_i u_i u^i$. For $x \in \mathfrak{g}$ we consider the family of operators $x(n,p)$ given by the operation of $x \otimes A_{n,p}$ on V. Recall that the $\{A_{n,p}\}$, $n \in \mathbb{Z}$, $p = 1, \ldots, K$ is the collection of basis elements introduced above of the algebra $\mathcal{A}$. We group the operators together in a formal sum

$$\widehat{x}(Q) := \sum_{n\in\mathbb{Z}} \sum_{p=1}^{K} x(n,p)\omega^{n,p}(Q), \quad Q \in \Sigma. \tag{6.1}$$

Such a formal sum is called a field if applied to a vector $v \in V$, i.e.,

$$\widehat{x}(Q)\,.\,v := \sum_{n\in\mathbb{Z}} \sum_{p=1}^{K} (x(n,p)\,.\,v)\omega^{n,p}(Q), \quad Q \in \Sigma, \tag{6.2}$$

it gives again a formal sum (now of elements from V) which is bounded from above. By the condition of admissibility $\widehat{x}(Q)$ is a field. It is of conformal weight one, as the one-differentials $\omega^{n,p}$ show up.

The current operator fields are defined as [4]

$$J_i(Q) := \widehat{u_i}(Q) = \sum_{n,p} u_i(n,p)\omega^{n,p}(Q). \tag{6.3}$$

[4]For simplicity we drop mentioning the range of summation here and in the following when it is clear.

The Sugawara operator field $T(Q)$ is defined by

$$T(Q) := \frac{1}{2}\sum_i :J_i(Q)J^i(Q): \ . \tag{6.4}$$

Here :...: denotes some normal ordering, which is needed to make the product of two fields again a field. The *standard normal ordering* is defined as

$$:x(n,p)y(m,r): \quad := \begin{cases} x(n,p)y(m,r), & (n,p) \le (m,r) \\ y(m,r)x(n,p), & (n,p) > (m,r) \end{cases} \tag{6.5}$$

where the indices (n,p) are lexicographically ordered. By this prescription the annihilation operator, i.e. the operators of positive degree, are brought as much as possible to the right so that they act first.

As the current operators are fields of conformal weights one the Sugawara operator field is a field of weight two. Hence we write it as

$$T(Q) = \sum_{k\in\mathbb{Z}}\sum_{p=1}^{K} L_{k,p}\cdot\Omega^{k,p}(Q) \tag{6.6}$$

with certain operators $L_{k,p}$. The $L_{k,p}$ are called *modes of the Sugawara field* T or just Sugawara operators. In fact we can express them via

$$\begin{aligned} L_{k,p} &= \frac{1}{2\pi \mathrm{i}}\int_{C_S} T(Q)e_{k,p}(Q) \\ &= \frac{1}{2}\sum_{n,m}\sum_{r,t}\sum_i :u_i(n,r)u^i(m,t): \, l_{k,p}^{(n,r)(m,t)}, \\ \text{with} \quad & l_{k,p}^{(n,r)(m,t)} = \frac{1}{2\pi\mathrm{i}}\int_{C_S} w^{n,r}(Q)w^{m,t}(Q)e_{k,p}(Q) \ . \end{aligned} \tag{6.7}$$

Let 2κ be the eigenvalue of the Casimir operator in the adjoint representation. For $\mathfrak{g}$ abelian $\kappa = 0$. If $\mathfrak{g}$ simple and β normalized such that the longest roots have square length 2 then κ is the dual Coxeter number. Recall that the central element t acts on the representation space V as $c\cdot id$ with a scalar c. This scalar is called the *level* of the representation. The key result is (where $x(g)$ denotes the operator corresponding to the element $x\otimes g$)

Proposition 6.1. [43, Prop. 10.8] *Let $\mathfrak{g}$ be either an abelian or a simple Lie algebra. Then*

$$[L_{k,p}, x(g)] = -(c+\kappa)\cdot x(e_{k,p}\,.\,g)\,. \tag{6.8}$$

$$[L_{k,p}, \widehat{x}(Q)] = (c+\kappa)\cdot(e_{k,p}\,.\,\widehat{x}(Q))\,. \tag{6.9}$$

Recall that $e_{k,p}$ are the KN basis elements for the vector field algebra $\mathcal{L}$.

In the next step the commutators of the operators $L_{k,p}$ can be calculated. The case $c+\kappa = 0$ is called the *critical level*. In this case these operators generate a subalgebra of the center of $\mathfrak{gl}(V)$. If $c+\kappa\neq 0$ (i.e. at a non-critical level) the $L_{k,p}$ can be replaced by rescaled elements $L^*_{k,p} = \frac{-1}{c+\kappa}L_{k,p}$ and we we denote by $T[..]$ the linear representation of $\mathcal{L}$ induced by

$$T[e_{k,p}] = L^*_{k,p}. \tag{6.10}$$

From (6.8) we obtain

$$[T[e], u(g)] = u(e\,.\,g) \tag{6.11}$$

Let V be an admissible representation of $\widehat{\mathfrak{g}}$ of non-critical level, then the Sugawara operators define a projective representation of $\mathcal{L}$ with a local cocycle. This cocycle is up to rescaling our geometric cocycle $\psi^3_{C_S,R}$ with a projective connection [5] R. In detail,

$$T[[e,f]] = [T[e],T[f]] + \frac{c\dim\mathfrak{g}}{c+\kappa}\,\psi^3_{C_S,R}(e,f)id. \tag{6.12}$$

Consequently,

Theorem 6.2. *In the non-critical level, by setting*

$$T[\hat{e}] := T[e], \quad T[t] := \frac{c\dim\mathfrak{g}}{c+\kappa}\,id\ . \tag{6.13}$$

we obtain a honest Lie representation of the centrally extended vector field algebra $\widehat{\mathcal{L}}$ *given by the local cocycle* $\psi_{C_S,R}$.

For the general reductive case, see [43, Section 10.2.1].

7. Wess-Zumino-Novikov-Witten models and Knizhnik–Zamolodchikov Connection

Wess-Zumino-Novikov-Witten models (WZNW) are important examples of models of two-dimensional conformal field theories and their quantized versions. They can roughly be described as follows. The gauge algebra of the theory is the affine algebra associated to a finite-dimensional gauge algebra (i.e. a simple finite-dimensional Lie algebra). The geometric data consists of a compact Riemann surface (with complex structure) of genus g and a finite number of marked points on this surface. Starting from representations of the gauge algebra the space of conformal blocks can be defined. It depends on the geometric data. Varying the geometric data should yield a bundle over the moduli space of the geometric data.

In [22] Knizhnik and Zamolodchikov considered the case of genus 0 (i.e. the Riemann sphere). There, changing the geometric data consists in moving the marked points on the sphere. The space of conformal blocks could completely be found inside the part of the representation associated to the finite-dimensional gauge algebra. On this space an important set of equations, the Knizhnik–Zamolodchikov (KZ) equations, were introduced. In a geometric setting, solutions are the flat sections of the bundle of conformal blocks over the moduli space with respect to the Knizhnik–Zamolodchikov connection.

For higher genus it is not possible to realize the space of conformal blocks inside the representation space associated to the finite-dimensional algebra. There exists different attacks to the generalization. Some of them add additional structure on these representation spaces (e.g. twists, representations of the fundamental group,..). A very incomplete list of names are Bernard, [4], [5], Felder, Wieczerkowski, Enriquez, [9], [10], [8], Hitchin, [17], and Ivanov [18].

An important approach very much in the spirit of the original Knizhnik–Zamolodchikov approach was given by Tsuchiya, Ueno and Yamada [51]. The main point in their approach is that at the marked points, after choosing local coordinates, local constructions are done. In this setting the well-developed theory of representations of the traditional affine Lie algebras (Kac-Moody algebras of affine type) can be

[5] The projective connection takes care of the "up to coboundary" and of different normal orderings.

used. It appears a mixture between local and global objects and considerable effort is necessary to extend the local constructions to global ones.

Oleg Sheinman and the author presented in [47] and [48] a different approach to the WZNW models which uses global objects. These objects are the Krichever–Novikov algebras and their representations. A crucial point is that a certain subspace of the Krichever–Novikov algebra of vector fields is identified with tangent directions on the moduli space of the geometric data. Conformal blocks can be defined. Finally, with the help of the global Sugawara construction given in Section 6 it is possible to define the higher genus multi-point Knizhnik–Zamolodchikov connection (see (11.6)). This connection is well-defined on the vector bundle of conformal blocks at least for those representations of the higher genus multi-point affine algebras which fulfill certain conditions. These conditions are fulfilled e.g. for Verma modules induced by representations of the finite-dimensional Lie algebra $\mathfrak{g}$ at the marked points [43, Sect. 9.9], or for fermionic Fock space representations [43, Sect. 9.10]. It turns out that the connection is projectively flat.

This global operator approach to WZNW models will be presented in the following. Complete proofs appeared in [47], [48]. More details (and an improved presentation) can be found in [43, Chapter 11] and [50].

I like to point out that up to now the global construction works only over an open dense subset of the moduli space. This is in contrast to the approach of Tsuchiya, Ueno and Yamada [51] which gives a theory valid for the compactified moduli space and hence includes stable singular curves. In this way they were able to proof the Verlinde formula.

8. Moduli space of curves with marked points

We will switch to the point of view that we consider compact Riemann surfaces Σ as smooth projective curves C over $\mathbb{C}$. For simplicity we call them just curves but will continue to use also the symbol Σ for them. Without further explanations we will use the language of algebraic geometry here. Using the fact that for compact Riemann surfaces all global holomorphic objects will be algebraic we will allow ourselves, if needed, to switch between the two languages.

Our set A where poles are allowed will be separated into

$$I := \{P_1, P_2, \ldots, P_N\}, \quad O := \{P_\infty\}. \tag{8.1}$$

In particular, our usual K will be N and our usual N will be $N+1$. This change of notation reflects the fact the point P_∞ will play a special role as reference point.

Ignoring for the moment the necessary reference point P_∞ our principal moduli space object is the moduli space $\mathcal{M}_{g,N}$ of genus g curves with N marked points. A point $b \in \mathcal{M}_{g,N}$ is given as

$$b = [\Sigma, P_1, \ldots, P_N] \tag{8.2}$$

where $[\ldots]$ denotes equivalence with respect to algebraic isomorphisms $\phi : \Sigma \to \Sigma'$ with $\phi(P_i) = P_i'$. We will work over a generic open subset W of $\mathcal{M}_{g,N}$ and hence there will be an universal family of curves $\mathcal{U} \to W$.

We will need an additional reference point, hence we should consider $\mathcal{M}_{g,N+1}$. Furthermore we will need (at least) first order jets of coordinates at the points in I and for technical intermediate reasons also some jets of coordinates at the point P_∞. Recall that a k-jet of coordinates at P_i is an equivalence class of coordinates where two coordinates z_i and z_i' are identified if they coincide up to order k in their power series expansion. This says $z_i'(z_i) = z_i + O(z_i^{k+1})$. Note that the zero order term of a coordinate is always fixed by the point itself and consequently a 0-jet says that we ignore the coordinates at the point P_i.

We start rather general and denote by $\mathcal{M}^{(k,p)}_{g,N+1}$ the moduli space of smooth projective curves of genus g (over $\mathbb{C}$) with $N+1$ ordered distinct marked points and fixed k-jets of local coordinates at the first N points and a fixed p-jet of a local coordinate at the last point. The elements of $\mathcal{M}^{(k,p)}_{g,N+1}$ are given as

$$\tilde{b}^{(k,p)} = [\Sigma, P_1, \ldots, P_N, P_\infty, z_1^{(k)} \ldots, z_N^{(k)}, z_\infty^{(p)}] \;, \tag{8.3}$$

where Σ is a smooth projective curve of genus g, P_i, $i = 1, \ldots, N, \infty$ are distinct points on Σ, z_i is a coordinate at P_i with $z_i(P_i) = 0$, and $z_i^{(l)}$ is a l-jet of z_i ($l \in \mathbb{N}_0$). Here [..] denotes an equivalence class of such tuples in the following sense. Two tuples representing $\tilde{b}^{(k,p)}$ and $\tilde{b}^{(k,p)\prime}$ are equivalent if there exists an algebraic isomorphism $\phi : \Sigma \to \Sigma'$ which respects the points and the corresponding jets of coordinates. For the following two special cases we introduce the notation

$$\mathcal{M}_{g,N+1} := \mathcal{M}^{(0,0)}_{g,N+1} \quad \text{and} \quad \mathcal{M}^{(1)}_{g,N+1} := \mathcal{M}^{(1,0)}_{g,N+1} \,. \tag{8.4}$$

Also we will use $\tilde{b}^{(1)} := \tilde{b}^{(1,0)}$. In the first case there will be no coordinates at all, in the second case only the first jets of coordinates at the points in I appear. These elements will be given by

$$\begin{aligned} \tilde{b} &= [\Sigma, P_1, \ldots, P_N, P_\infty] \quad \in \quad \mathcal{M}_{g,N+1} \\ \tilde{b}^{(1)} &= [\Sigma, P_1, \ldots, P_N, P_\infty, z_1^{(1)} \ldots, z_N^{(1)}] \quad \in \quad \mathcal{M}^{(1)}_{g,N+1}. \end{aligned} \tag{8.5}$$

By forgetting either coordinates or higher order jets we obtain natural projections

$$\nu^{k,p} : \mathcal{M}^{(k,p)}_{g,N+1} \to \mathcal{M}_{g,N+1} \tag{8.6}$$

or more generally $\mathcal{M}^{(k,p)}_{g,N+1} \to \mathcal{M}^{(k',p')}_{g,N+1}$ for any $k' \le k$ and $p' \le p$.

Here we deal with the situation in the neighborhood of a moduli point corresponding to a generic curve Σ with a generic marking $(P_1, P_2, \ldots, P_N, P_\infty)$. Let $\widetilde{W} \subseteq \mathcal{M}_{g,N+1}$ be an open subset around such a generic point $\tilde{b} = [\Sigma, P_1, P_2, \ldots, P_N, P_\infty]$. A generic curve of $g \ge 2$ admits no nontrivial infinitesimal automorphism, and we may assume that there exists a universal family of curves with marked points over $\widetilde{W}$. In particular, this says that there is a proper, flat family of smooth curves over $\widetilde{W}$

$$\pi : \mathcal{U} \to \widetilde{W} \;, \tag{8.7}$$

such that for the points $\tilde{b} = [\Sigma, P_1, P_2, \ldots, P_N, P_\infty] \in \widetilde{W}$ we have $\pi^{-1}(\tilde{b}) = \Sigma$ and that the sections defined as

$$\sigma_i : \widetilde{W} \to \mathcal{U}, \quad \sigma_i(\tilde{b}) = P_i, \quad i = 1, \ldots, N, \infty \tag{8.8}$$

will not meet and are algebraic.

The subset $\widetilde{W}$ can be pullbacked via $\nu^{k,p}$ to $\mathcal{M}^{(k,p)}_{g,N+1}$ and we obtain $\widetilde{W}^{(k,p)}$ as open subset. Now the sections (8.8) can additionally be complemented by choosing infinitesimal neighbourhoods of the order under consideration. As cases of special importance we obtain the subsets

$$\widetilde{W}^{(1)} = \widetilde{W}^{(1,0)} \subseteq \mathcal{M}^{(1)}_{g,N+1}, \quad \widetilde{W}^{(1,1)} \subseteq \mathcal{M}^{(1,1)}_{g,N+1}. \tag{8.9}$$

There is another relation to be taken into account. If we "forget" the last point P_∞, the reference point, we obtain maps

$$\mathcal{M}_{g,N+1} \to \mathcal{M}_{g,N}, \quad \mathcal{M}^{(0,p)}_{g,N+1} \to \mathcal{M}_{g,N}, \quad \mathcal{M}^{(k,p)}_{g,N+1} \to \mathcal{M}^{(k)}_{g,N}. \tag{8.10}$$

Let us fix an algebraic section $\widehat{\sigma}_\infty$ of the universal family of curves (without marking). In particular, for every curve there is a point chosen in a manner depending algebraically on the moduli. The subset

$$W' := \{\tilde{b} = [\Sigma, P_1, P_2, \dots, P_N, P_\infty] \mid P_\infty = \widehat{\sigma}_\infty([\Sigma])\} \quad \subseteq \quad \widetilde{W} \tag{8.11}$$

can be identified with the open subset W of $\mathcal{M}_{g,N}$ via

$$\tilde{b} = [(\Sigma, P_1, P_2, \dots, P_N, \widehat{\sigma}_\infty([\Sigma]))] \ \to \ b = [(\Sigma, P_1, P_2, \dots, P_N)] \ . \tag{8.12}$$

By genericity, the map is one-to-one.

By choosing not only a section $\widehat{\sigma}_\infty$ but also a p-th order infinitesimal neighborhood of this section we even get an identification of the open subset W of $\mathcal{M}_{g,N}$ with an analytic subset $W'^{,(p)}$ of $\widetilde{W}^{(0,p)}$ of $\mathcal{M}^{(0,p)}_{g,N+1}$. It is defined in a similar way as W'.

All constructions done below for the $N+1$ point situation will yield results for the N point situation. But it is important to keep in mind that the results will in general depend on the chosen section $\widehat{\sigma}_\infty$ yielding the reference points P_∞.

All these considerations can be extended to the case where we allow infinite jets of local coordinates at P_∞. In this way we obtain the moduli space $\mathcal{M}^{(k,\infty)}_{g,N+1}$.

Next we consider the tangent space at the moduli spaces. Recall that our moduli point is generic, hence the moduli space there is smooth. The Kodaira–Spencer map for a versal family of complex analytic manifolds $Y \to B$ over the base B at the base point $b \in B$,

$$\mathrm{T}_b(B) \to \mathrm{H}^1(Y_b, T_{Y_b}) \tag{8.13}$$

is an isomorphism. Here $\mathrm{T}_b(B)$ denotes the tangent space of B at the point b, Y_b is the fiber over b and T_{Y_b} the (holomorphic) tangent sheaf of Y_b. The space $\mathrm{H}^1(Y_b, T_{Y_b})$ is also sometime called *Kuranishi tangent space*.

We are in the local generic situation where we have a universal family. Hence we can employ (8.13). Let Σ be the curve fixed by b, $\tilde{b}$, respectively $\tilde{b}^{(k,p)}$. We obtain

$$\mathrm{T}_{[\Sigma]}(\mathcal{M}_{g,0}) \cong \mathrm{H}^1(\Sigma, T_\Sigma) \ . \tag{8.14}$$

Denote by S the divisor $S = \sum_{i=1}^N P_i$ on Σ. Then the Kodaira Spencer map gives

$$\mathrm{T}_{\tilde{b}}\mathcal{M}_{g,N+1} \cong \mathrm{H}^1(\Sigma, T_\Sigma(-S - P_\infty)), \tag{8.15}$$

$$\mathrm{T}_{\tilde{b}^{(1,p)}}\mathcal{M}^{(1,p)}_{g,N+1} \cong \mathrm{H}^1(\Sigma, T_\Sigma(-2S - (p+1)P_\infty)), \tag{8.16}$$

$$\mathrm{T}_{\tilde{b}^{(k,p)}}\mathcal{M}^{(k,p)}_{g,N+1} \cong \mathrm{H}^1(\Sigma, T_\Sigma(-(k+1)S - (p+1)P_\infty)) \tag{8.17}$$

With the help of Riemann-Roch and Serre duality the dimension of the moduli spaces can be calculated. The following cases will be of importance for us.

$$\dim_{[\Sigma]}(\mathcal{M}_{g,0}) = \begin{cases} 3g-3, & g \geq 2 \\ 1, & g = 1 \\ 0, & g = 0 \, , \end{cases} \tag{8.18}$$

and for $N \neq 0$

$$\dim_b(\mathcal{M}_{g,N}) = \begin{cases} 3g-3+N, & g \geq 1 \\ \max(0, N-3), & g = 0 \end{cases}, \tag{8.19}$$

and correspondingly for $\dim_{\tilde{b}}(\mathcal{M}_{g,N+1})$. Finally

$$\dim_{\tilde{b}^{(1,p)}}(\mathcal{M}^{(1,p)}_{g,N+1}) = \begin{cases} 3g-2+2N+p, & g \geq 1 \\ \max(0, -2+2N+p), & g = 0 \end{cases}. \tag{8.20}$$

Remark 8.1. For genus 0 and 1 the situations are in certain sense special. In this case it is better to work with the configuration space. For illustration let us consider $g = 0$. There is only one isomorphy type. By an automorphism of $\mathbb{P}^1$ it is always possible to move three distinct points to the triple $(0, 1, \infty)$. If this is done there are no further automorphisms. Hence the moduli space $\mathcal{M}_{0,N}$ has a non-zero dimension exactly for $N \geq 4$. Its dimension is $\min(0, N-3)$ in accordance with the formula (8.19). In this case one usually works with the configuration space of N points which reads as

$$\widehat{W} := \{(P_1, P_2, \ldots, P_N) \mid P_i \in \mathbb{C},\ P_i \neq P_j,\ \text{for } i \neq j\}$$

and studies the remaining invariance at the end to pass to the moduli space. It is quite useful to map the reference point P_∞ always to ∞.

9. Tangent spaces of the moduli spaces and the Krichever-Novikov vector field algebra

In this section we will relate the tangent spaces at a moduli point of the above introduced moduli spaces with certain parts of the Krichever–Novikov vector field algebra associated to the corresponding curve. This is done by showing that the above identified cohomology spaces can be identified with elements of the critical strips of the Krichever–Novikov vector field algebra. Hence the elements of the latter can be identified with tangent vectors to the moduli spaces.

To do this we first have to recall the triangular decomposition of Section 4.5 of the vector field algebra $\mathcal{L}$, (and will do it for later use also already for $\mathcal{A}$ and hence for $\overline{\mathfrak{g}}$).

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_+ \oplus \mathcal{L}_{(0)} \oplus \mathcal{L}_-, \\ \mathcal{A} &= \mathcal{A}_+ \oplus \mathcal{A}_{(0)} \oplus \mathcal{A}_-. \end{aligned} \tag{9.1}$$

These decompositions are defined with the help of the Krichever–Novikov basis elements.

Recall that due to the almost-grading the subspaces $\mathcal{A}_\pm$, and $\mathcal{L}_\pm$ are subalgebras but the subspaces $\mathcal{A}_{(0)}$, and $\mathcal{L}_{(0)}$ in general are not. We used the term *critical strip* for them.

Note that $\mathcal{A}_+$, respectively $\mathcal{L}_+$ can be described as the algebra of functions (vector fields) having a zero of at least order one (two) at the points $P_i, i = 1, \ldots, N$. These algebras can be enlarged by adding all elements which are regular at all P_i's. This can be achieved by moving the set of basis elements $\{A_{0,i}, i = 1, \ldots, N\}$, (respectively $\{e_{0,i}, e_{-1,i}, i = 1, \ldots, N\}$ from the critical strip to these algebras. We denote the enlarged algebras by $\mathcal{A}^*_+$, respectively by $\mathcal{L}^*_+$.

On the other hand $\mathcal{A}_-$ and $\mathcal{L}_-$ could also be enlarged such that they contain all elements which are regular at P_∞. We obtain $\mathcal{A}_-^*$ and $\mathcal{L}_-^*$ respectively.

In the same way for every $p \in \mathbb{N}_0$ let $\mathcal{L}_-^{(p)}$ be the subalgebra of vector fields vanishing of order $\ge p+1$ at the point P_∞, and $\mathcal{A}_-^{(p)}$ the subalgebra of functions vanishing of order $\ge p$ at the point P_∞. We obtain decompositions

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_+ \oplus \mathcal{L}_{(0)}^{(p)} \oplus \mathcal{L}_-^{(p)}, \quad \text{for } p \ge 0, \\ \mathcal{A} &= \mathcal{A}_+ \oplus \mathcal{A}_{(0)}^{(p)} \oplus \mathcal{A}_-^{(p)}, \quad \text{for } p \ge 1, \end{aligned} \tag{9.2}$$

with "critical strips" $\mathcal{L}_{(0)}^{(p)}$ and $\mathcal{A}_{(0)}^{(p)}$, which are only subspaces.

Of particular interest to us is $\mathcal{L}_{(0)}^{(1)}$ which we call *reduced critical strip*. For $g \ge 2$ its dimension is

$$\dim \mathcal{L}_{(0)}^{(1)} = N + N + (3g-3) + 1 + 1 = 2N + 3g - 1\ . \tag{9.3}$$

The first two terms correspond to $\mathcal{L}_0$ and $\mathcal{L}_{-1}$. The intermediate term comes from the vector fields in the basis which have poles at least at one the $P_i, i = 1, \dots, N$ and a pole at P_∞. The "$1+1$" corresponds to the vector fields in the basis with exact order zero respectively one at P_∞. Also a special role will be played by the *reduced regular subalgebras*

$$\mathcal{A}^r = \mathcal{A}_-^{(1)} \subset \mathcal{A}_-^*, \qquad \mathcal{L}^r = \mathcal{L}_-^{(0)} \subset \mathcal{L}_-^*\ , \tag{9.4}$$

containing the function respectively the vector fields vanishing at P_∞.

Recall that the almost-grading extends to the *higher genus current algebra* $\overline{\mathfrak{g}}$ by setting $\deg(x \otimes A_{n,p}) := n$, and we obtain obtain a triangular decomposition as above

$$\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_+ \oplus \overline{\mathfrak{g}}_{(0)} \oplus \overline{\mathfrak{g}}_-, \quad \text{with} \quad \overline{\mathfrak{g}}_\beta = \mathfrak{g} \otimes \mathcal{A}_\beta, \quad \beta \in \{-, (0), +\}\ , \tag{9.5}$$

In particular, $\overline{\mathfrak{g}}_\pm$ are subalgebras. The corresponding is true for the enlarged subalgebras. Among them, $\overline{\mathfrak{g}}^r := \overline{\mathfrak{g}}_-^{(1)} = \mathfrak{g} \otimes \mathcal{A}_-^{(1)}$ is of special importance. It is called the *reduced regular subalgebra* of $\overline{\mathfrak{g}}$.

Remark 9.1. All these subalgebras can be considered as subalgebras of the corresponding almost-graded central extensions $\widehat{\mathcal{A}}$, $\widehat{\mathcal{L}}$, and $\widehat{\mathfrak{g}}$ respectively. This is obvious as the defining cocycles which are integrated over a separating cycle C_S can be calculated by calculating residues either at the points $P_1, \dots, P_N$ or at P_∞. But the elements of the subalgebras are holomorphic at one these sets. Also note that the finite-dimensional Lie algebra $\mathfrak{g}$ via $x \mapsto x \otimes 1$ can naturally be considered as subalgebra of $\overline{\mathfrak{g}}$. As $1 = \sum_p A_{0,p}$ it lies in the subspace $\overline{\mathfrak{g}}_0$.

Let Σ be the Riemann surface we are dealing with and let U_∞ be a coordinate disc around P_∞, such that $P_1, \dots, P_N \notin U_\infty$. Let $U_1 = \Sigma \setminus \{P_\infty\}$. Because U_1 and U_∞ are affine (respectively Stein) [15, p.297] we get $\mathrm{H}^1(U_j, F) = 0$, $j = 1, \infty$ for every coherent sheaf F. Hence, the sheaf cohomology can be given as Cech cohomology with respect to the covering $\{U_1, U_\infty\}$. Set $U_\infty^* = U_1 \cap U_\infty = U_\infty \setminus \{P_\infty\}$. The Cech two-cocycles are given by $s_{1,\infty} \in F(U_\infty^*)$ ($s_{0,0} = s_{\infty,\infty} = 0$), hence by arbitrary sections over the punctured coordinate disc U_∞^*. We recall that it does not make any difference whether we calculate the Cech cohomology in the algebraic-geometric category or in the complex-analytic category.

Coming back to the holomorphic (algebraic) tangent bundle T_Σ. For every element f of the Krichever–Novikov vector field algebra its restriction to U_∞^* is holomorphic and indeed algebraic as the poles (outside of U_∞^*) are of finite order. Hence, it defines an element of $\mathrm{H}^1(\Sigma, T_\Sigma)$. Note that it defines also an element of $\mathrm{H}^1(\Sigma, T_\Sigma(D))$, where D is any divisor supported outside of U_∞^*. We introduce the map

$$\theta_D : \mathcal{L} \;\to\; \mathrm{H}^1(\Sigma, T_\Sigma(D)), \qquad f \mapsto \theta_D(f) := [f_{|U_\infty^*}]\;. \tag{9.6}$$

If the divisor D is clear from the context we will suppress it in the notation. For us only the divisors

$$D_{k,p} := (k+1)S + (p+1)P_\infty, \qquad k,p \in \mathbb{Z},\; k,p \ge -1\,. \tag{9.7}$$

are of importance and we set $\theta_{k,p} = \theta_{D_{k,p}}$.

Theorem 9.2. [43, Thm. 11.6] *Let $g \ge 2$ and $k,p > -1$ then there is a surjective linear map from the Krichever-Novikov vector field algebra $\mathcal{L}$ to the cohomology space*

$$\theta = \theta_{k,p} : \quad \mathcal{L} \;\to\; \mathrm{H}^1(\Sigma, T_\Sigma(-(k+1)S - (p+1)P_\infty) \tag{9.8}$$

such that θ restricted to the following subspace gives an isomorphism

$$\begin{aligned} \mathcal{L}_{k-1} \oplus \cdots \oplus \mathcal{L}_{(0)}^{(p)} &\cong \mathrm{H}^1(\Sigma, T_\Sigma(-(k+1)S - (p+1)P_\infty) \\ &\cong \mathrm{T}_{\tilde{b}^{(k,p)}}\mathcal{M}_{g,N+1}^{(1,p)}. \end{aligned} \tag{9.9}$$

Moreover,

$$\ker \theta_{k,p} = \bigoplus_{n \ge k} \mathcal{L}_n \oplus \mathcal{L}_-^{(p)}. \tag{9.10}$$

For $g = 1$ the same is true if at least k or p is ≥ 0. For $g = 0$ it is true except for some small values N, k, or p.

This can be specialized to the following important cases:

$$\begin{aligned} \mathcal{L}_0 \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^{(p)} &\cong \mathrm{H}^1(\Sigma, T_\Sigma(-2S - (p+1)P_\infty) \cong \mathrm{T}_{\tilde{b}^{(1,p)}}\mathcal{M}_{g,N+1}^{(1,p)}, \\ \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^{(p)} &\cong \mathrm{H}^1(\Sigma, T_\Sigma(-S - (p+1)P_\infty)) \cong \mathrm{T}_{\tilde{b}^{(0,p)}}\mathcal{M}_{g,N+1}, \\ \mathcal{L}_{(0)}^{(p)} &\cong \mathrm{H}^1(\Sigma, T_\Sigma(-(p+1)P_\infty) \cong \mathrm{T}_{[\Sigma, P_\infty^{(p)}]}\mathcal{M}_{g,1}^{(p)}. \end{aligned} \tag{9.11}$$

For the infinite jets we obtain

$$\mathrm{T}_{\tilde{b}^{(1,\infty)}}\mathcal{M}_{g,N+1}^{(1,\infty)} = \lim_{p\to\infty} \mathrm{H}^1(\Sigma, T_\Sigma(-2S - pP_\infty)) \cong \mathcal{L}_{(0)} \oplus \mathcal{L}_-. \tag{9.12}$$

Remark 9.3. The spaces $\ker\theta_{k,p}$ are invariantly defined. They neither depend on the ordering of the points, nor on any special recipe for fixing the Krichever–Novikov basis elements. Because the vector fields in $\ker\theta \subset \mathcal{L}$ are not corresponding to deformations in the moduli we sometimes call this vector fields *vertical vector fields*. They should not be confused with the sections of the relative tangent sheaf of the universal family. The following should also be kept in mind. The definition of the critical strip, hence of the complementary subspace to $\ker\theta$, is only fixed by the order prescription for the basis. A different prescription (which involves changing the required orders) will yield a different identification of tangent vectors on the moduli space with vector fields in the fiber.

The inverse of the map $\theta_{k,p}$ we will denote by

$$\overline{\rho}: \mathrm{T}_{\tilde{b}^{(k,p)}}\mathcal{M}^{(1,p)}_{g,N+1} \to \mathcal{L}, \qquad X \mapsto \overline{\rho}(X) \tag{9.13}$$

where strictly speaking the image should lie in the subspace $\mathcal{L}_{k-1}\oplus\cdots\oplus\mathcal{L}^{(p)}_{(0)}$. In fact we could also describe $\overline{\rho}$ in a more natural manner as a composition of the canonical map to $\mathcal{L}/\ker\theta_{k,p}$ with a non-canonical identification with the critical strip.

$$\overline{\rho}: \mathrm{T}_{\tilde{b}^{(k,p)}}\mathcal{M}^{(k,p)}_{g,N+1} \ \to\ \mathcal{L}/\ker\theta_{k,p} \cong \mathcal{L}_{k-1}\oplus\cdots\oplus\mathcal{L}^{(p)}_{(0)}, \tag{9.14}$$

In this more natural description $\overline{\rho}(X)$ is an element of the critical strip modulo vertical vector fields which does not depend on choices. For the equation (9.14) we could instead of the critical strip take any complement of $\ker\theta_{k,p}$. The map $\overline{\rho}$ is the Kodaira–Spencer map.

Remark 9.4. In genus 0 and 1 there are global holomorphic vector fields. Hence the decomposition of the critical strip and its identification in (9.9) are not valid anymore. The space $\mathcal{L}_0\oplus\mathcal{L}_{-1}$ and $\mathcal{L}^*_{(0)}\oplus\mathcal{L}^{(0)}_{-}$ have a non-trivial intersection. This is in complete conformity with the corrected dimensions of the moduli space.

10. Sheaf versions of the Krichever-Novikov type algebras

Let $\widetilde{W}$ be a generic open subset of $\mathcal{M}_{g,N+1}$ which we consider here. Let

$$\tilde{b} = [\Sigma, P_1, P_2, \ldots, P_N, P_\infty] \tag{10.1}$$

be a point of $\widetilde{W}$. For each $\tilde{b}$ we construct the Krichever-Novikov objects

$$\mathcal{A}_{\tilde{b}},\ \mathcal{L}_{\tilde{b}},\ \widehat{\mathcal{L}}_{\tilde{b}},\ \overline{\mathfrak{g}}_{\tilde{b}},\ \widehat{\mathfrak{g}}_{\tilde{b}},\ \mathcal{F}^\lambda_{\tilde{b}},\ \text{etc.} \tag{10.2}$$

with respect to the splitting $I = \{P_1, P_2, \ldots, P_N\}$ and $O = \{P_\infty\}$.

We sheafify these objects over $\widetilde{W}$. This is done with the help of the universal family. For details we have to refer to [43, Chapter 11.3].

This construction can be extended to the affine algebra situation. Given a finite-dimensional Lie algebra $\mathfrak{g}$, the *sheaf of the associated current algebras* $\overline{\mathfrak{g}}_{\widetilde{W}}$ and the *sheaf of the associated affine algebra* $\widehat{\mathfrak{g}}_{\widetilde{W}}$ are defined as

$$\overline{\mathfrak{g}}_{\widetilde{W}} := \mathcal{A}_{\widetilde{W}}\otimes\mathfrak{g}, \qquad \widehat{\mathfrak{g}}_{\widetilde{W}} := \overline{\mathfrak{g}}_{\widetilde{W}}\oplus\mathcal{O}_{\widetilde{W}}\cdot t\ , \tag{10.3}$$

where the Lie structure is given by the naturally extended form of Section 5 (respectively without its central term for $\overline{\mathfrak{g}}_{\widetilde{W}}$).

Clearly, these are $\mathcal{O}_{\widetilde{W}}$-sheaves. Furthermore, let $\tilde{b}\in\widetilde{W}$ and let $\mathcal{O}_{\widetilde{W},\tilde{b}}$ be the local ring at $\tilde{b}$ and $M_{\tilde{b}}$ its maximal ideal. Set $\mathbb{C}_{\tilde{b}}\cong\mathcal{O}_{\widetilde{W},\tilde{b}}/M_{\tilde{b}}$, then we obtain the following canonical isomorphisms of localizing

$$\mathbb{C}_{\tilde{b}}\otimes\mathcal{A}_{\widetilde{W}}\cong\mathcal{A}_{\tilde{b}}, \qquad \mathbb{C}_{\tilde{b}}\otimes\widehat{\mathcal{A}}_{\widetilde{W}}\cong\widehat{\mathcal{A}}_{\tilde{b}}, \qquad \mathbb{C}_{\tilde{b}}\otimes\overline{\mathfrak{g}}_{\widetilde{W}}\cong\overline{\mathfrak{g}}_{\tilde{b}}, \qquad \mathbb{C}_{\tilde{b}}\otimes\widehat{\mathfrak{g}}_{\widetilde{W}}\cong\widehat{\mathfrak{g}}_{\tilde{b}}\ .$$

Definition 10.1. A sheaf $\mathfrak{V}$ of $\mathcal{O}_{\widetilde{W}}$-modules is called a *sheaf of representations* for the affine algebra $\widehat{\mathfrak{g}}_{\widetilde{W}}$ if the $\mathfrak{V}(U)$ are modules over $\widehat{\mathfrak{g}}_{\widetilde{W}}(U)$ for all open subsets of $\widetilde{W}$.

For a sheaf of representations $\mathfrak{V}$ we obtain that $\mathfrak{V}_{\tilde{b}}$ is a module over $\widehat{\mathfrak{g}}_{\tilde{b}}$ for every point in $\widetilde{W}$. We already introduced the moduli space

$$\widetilde{W}^{(1)}\subset\mathcal{M}^{(1)}_{g,N+1} = \mathcal{M}^{(1,0)}_{g,N+1} \tag{10.4}$$

containing first order jets of coordinates at P_i. Let $\eta : \mathcal{M}^{(1)}_{g,N+1} \to \mathcal{M}_{g,N+1}$ be the surjective analytic map obtained by forgetting the coordinates. It is a surjective analytic map.

Proposition 10.2. [43, Prop.11.11] *The sheaves over* $\widetilde{W}^{(1)}$

$$\mathcal{A}_{\widetilde{W}^{(1)}},\ \mathcal{L}_{\widetilde{W}^{(1)}},\ \widehat{\mathcal{L}}_{\widetilde{W}^{(1)}},\ \overline{\mathfrak{g}}_{\widetilde{W}^{(1)}},\ \widehat{\mathfrak{g}}_{\widetilde{W}^{(1)}},\ \mathcal{F}^{\lambda}_{\widetilde{W}^{(1)}} \tag{10.5}$$

are free sheaves of $\mathcal{O}_{\widetilde{W}^{(1)}}$*-modules of infinite rank.*

The reason for this is that if the first order jets of the coordinates at the points in I are fixed, then the basis elements of the corresponding algebras are uniquely fixed, not only up to a scalar. By their explicite expressions valid for generic points as described in [32] it is clear that they constitute a basis over $\widetilde{W}^{(1)}$.

Pulling back a sheaf of representation $\mathfrak{V}$ over W we obtain a sheaf of representation $\mathfrak{V}^{(1)} = \eta^*\mathfrak{V}$ of $\widehat{\mathfrak{g}}_{W^{(1)}}$. More generally, we can define sheaves of representations over $W^{(1)}$ directly. In particular for these sheaves of representations, operators depending on the Krichever–Novikov basis are well-defined.

With the help of the Krichever–Novikov basis elements also an almost-grading was introduced for the algebraic objects. Hence, clearly the sheaves (10.5) carry an almost-grading too and the homogeneous subspaces globalize immediately to the sheaf version. The definition of the homogeneous subspaces does not depend on a rescaling of the basis elements. Hence, already the sheaves (10.3) over $\widetilde{W}$ carry an almost-graded structure. This allows to define a sheaf of admissible representations to be a sheaf of representations either over $\widetilde{W}$ or over $\widetilde{W}^{(1)}$, where all fiber-wise representations are admissible. In addition we will usually require (if nothing else is said) that the central element t operates as $c \cdot id$ with c a function on W. This function is called the level function. Usually one assumes c to be a constant, which is just called the level of the representation.

Remark 10.3. Sheaf version of the Sugawara construction.
Recall from Section 6 the Sugawara construction which is well-defined for admissible representations of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated to a finite dimensional reductive Lie algebra $\mathfrak{g}$. The construction again can be sheafified. Hence let $\mathfrak{V}$ be an admissible representation sheaf of the sheaf $\widehat{\mathfrak{g}}_{\widetilde{W}^{(1)}}$. For simplicity let $\mathfrak{g}$ be either abelian or simple. Let 2κ be the eigenvalue of the Casimir in the adjoint representation of the finite Lie algebra and let c be the level of the representation under consideration. Assume that the level function c obeys $c + \kappa \neq 0$ for the moduli points we are considering. For every sheaf of admissible representations over $\widetilde{W}^{(1)}$ or $\widetilde{W}$ the Sugawara operators

$$T[e] := \frac{-1}{c+\kappa} \cdot \frac{1}{2\pi \mathrm{i}} \int_{C_S} T(Q)e(Q) \tag{10.6}$$

are well-defined. But note that the individual $T[e_{n,p}] = L^*_{n,p}$ will depend on the first order jet of the coordinates.

Recall the definition of the *reduced regular subalgebras* $\mathcal{A}^r$, $\mathcal{L}^r$ and $\overline{\mathfrak{g}}^r = \mathfrak{g} \otimes \mathcal{A}^r$. as consisting of those elements vanishing at P_∞. Denote by $\overline{\mathfrak{g}}^r_{\widetilde{W}}$, $\mathcal{L}^r_{\widetilde{W}}$ etc. the corresponding sheaves.

Definition 10.4. Let $\mathfrak{V}_{\widetilde{W}}$ be a sheaf of (fiber-wise) representations of $\widehat{\mathfrak{g}}_{\widetilde{W}}$. The sheaf of *conformal blocks* (associated to the representation $\mathfrak{V}_{\widetilde{W}}$) is defined as the sheaf of coinvariants

$$C_{\widetilde{W}} := C_{\widetilde{W}}(\mathfrak{V}) := \mathfrak{V}_{\widetilde{W}} / \overline{\mathfrak{g}}^{r}_{\widetilde{W}} \mathfrak{V}_{\widetilde{W}}. \tag{10.7}$$

Here $\overline{\mathfrak{g}}^{r}_{\widetilde{W}} \mathfrak{V}_{\widetilde{W}}$ should mean that we take the sheaf of point-wise vector spaces generated by these elements. Of course, we can define also conformal blocks for representation sheaves over $\widetilde{W}^{(1)}$ and even more generally.

For a general representation sheaf the sheaf of conformal blocks will only be a sheaf. For the examples given by the Verma modules (at least for simple Lie algebras $\mathfrak{g}$) and for certain fermionic Fock modules the sheaf of conformal blocks will be locally free of finite rank over $\widetilde{W}^{(1)}$. For more information see [48], [50], [43].

11. The Knizhnik–Zamolodchikov connection

For the following let $\mathfrak{V}$ be a locally free representation sheaf of $\widehat{\mathfrak{g}}$ over $\widetilde{W}$ or more generally over $\widetilde{W}^{(1,1)} = (\nu^{(1,1)})^{-1}(\widetilde{W})$. To simplify the presentation we restrict ourselves on $\widetilde{W}$. We assume that for $\mathfrak{V}$ the fiber-wise representations are admissible. Furthermore we assume that the sheaf of conformal blocks is locally free of finite rang. This is e.g. the case for the Verma module sheaf and the fermionic module sheaf.

11.1. **Variation of the complex structure.** We will now sketch how our objects behave under variation of the moduli point (moduli parameter) $\tau \in \mathcal{M}$. Let $\mathcal{A}_\tau$ be the stalk of the sheaf $\mathcal{A}_{\widetilde{W}}$ at the moduli point τ. This says that elements are functions on Σ_τ. If we differentiate e.g. an $g \in \mathcal{A}_\tau$ with respect to moduli parameters it will not be a Krichever–Novikov function on the same Σ_τ anymore. But it can be given as infinite series with respect to the expansion at the point P_∞. To take care of this, we have to consider completions of the algebras at P_∞ by allowing (infinite) Laurent series of Krichever–Novikov basis elements. These series are only defined locally in a neighbourhood of P_∞ and might have algebraic poles there. The appearing sums are of the type

$$f = \sum_{n=-\infty}^{n<M} \sum_{p=1}^{N} \alpha_{n,p} f^{\lambda}_{n,p} \,. \tag{11.1}$$

In this way we obtain the algebra of local functions $\widetilde{\mathcal{A}}$, local vector fields $\widetilde{\mathcal{L}}$, and local currents $\widetilde{\mathfrak{g}}$.

Our cocycles for the above algebras can be obviously extended as cocycles to the algebra of local elements, as they are calculated via residues at P_∞. Furthermore, they define sheaves $\widetilde{\mathcal{A}}_{\widetilde{W}}$, $\widetilde{\mathcal{L}}_{\widetilde{W}}$, and $\widetilde{\mathfrak{g}}_{\widetilde{W}}$ over $\widetilde{W}$.

To avoid cumbersome notations we will sometimes drop the mention of the space if we talk about a sheaf. For example $\mathcal{A}$ could mean one special copy of the algebra $\mathcal{A}$ or the sheaf over $\widetilde{W}$, etc.

In the following we will need the action of $u(g)$ and $T[e]$ on a representation $\mathfrak{V}$ not only for honest Krichever–Novikov elements g and e, but for local elements. To make this well-defined we have to complement the representation space $\mathfrak{V}$ in negative degree direction (or equivalently in positive degree direction with respect to the P_∞ degree). This completion is denoted by $\overline{\mathfrak{V}}$. For those representation which we are considering here such a completion is possible. By the almost-gradedness the above operators will be well-defined and the important result (6.11) stays valid for local objects. Indeed, only finitely many terms of the expansions (of the operator and of the corresponding element in $\overline{\mathfrak{V}}$) contribute in the result to the component of a given degree. See [48] and [50] for more details. We will define the Knizhnik–Zamolodchikov connection further-down anyway only for conformal blocks

$$C(\overline{\mathfrak{V}}) = \overline{\mathfrak{V}}/\widetilde{\mathfrak{g}}^r\overline{\mathfrak{V}} = \mathfrak{V}/\overline{\mathfrak{g}}^r\mathfrak{V} = C(\mathfrak{V}), \tag{11.2}$$

and low negative degrees will be truncated, see e.g. [43, Lemma 11.19]. Next we deal with the variation of the complex structure. Above we introduced U_∞^* and made via Cech cohomology the identification of the vector field with tangent vectors on moduli. Choose a generic point in the moduli space with moduli parameters τ_0 in $\mathcal{M}$.

In particular, Σ_{τ_0} has a fixed conformal structure representing the algebraic curve corresponding to the moduli parameters τ_0. For τ lying in a small enough neighbourhood of τ_0, the conformal structure on Σ_τ can be obtained by deforming the conformal structure Σ_{τ_0}. Roughly speaking we cut a coordinate patch at P_∞, deform the gluing function and re-glue it back again. This is exactly the way the Kodaira–Spencer cocycle is constructed. If X is an infinitesimal direction on the moduli space $\mathcal{M}_{g,N+1}^{(1,1)}$ then via the Kodaira–Spencer cocycle map ρ we can assign to it a local vector field $\rho(X)$ (lying in $\widetilde{\mathcal{L}}_\tau$) on $\widetilde{W}$. By adding suitable coboundary terms (which corresponds to taking a different local coordinate at P_∞) we could even obtain that $\rho(X) \in \mathcal{L}$. Furthermore, with (9.14) we get

$$\overline{\rho}(X) \in \mathcal{L}/\ker\theta_{k,p} \cong \overline{\mathcal{L}}, \tag{11.3}$$

where $\overline{\mathcal{L}}$ is a fixed chosen complement. For example we can take the corresponding critical strip.

11.2. **Defining the connection.** Let $\mathfrak{V}$ be a locally free representation sheaf of the sheaf $\overline{\mathfrak{g}}_{\widetilde{W}}$. In the following we will ignore the fact, that we temporary have to use the completion $\overline{\mathfrak{V}}$. By passing to the conformal blocks it will disappear again.

Consider a sheaf of operators on the local sections of the sheaf $\mathfrak{V}$. Assume B to be a local section of it. Then its derivative is defined as

$$\partial_X B\,.\,v = [\partial_X, B]\,.\,v = \partial_X(B\,.\,v) - B(\partial_X v)\,, \tag{11.4}$$

where v denotes a local section of the sheaf $\mathfrak{V}$. We have to deal with the following cases

(1) $B = u(g)$ where $u \in \mathfrak{g}$ and $g \in \mathcal{A}$
(2) $B = T[e] =: T(e)$, the Sugawara operator associated to $e \in \mathcal{L}$[6].

Recall that we extended the operation to the local objects. To avoid cumbersome notation we already used $\mathcal{A}$, $\mathcal{L}$, etc. but meant of course the sheaves, and the elements are supposed to be sections of the sheaves. We will use this simplified notation also in the following.

We assume now that for the derivative of the operators $u(g)$ on $\mathfrak{V}$ we have

$$\partial_X u(g) = u(\partial_X g). \tag{11.5}$$

As it is shown in [48] that this is fulfilled both for the fermionic representation and for the Verma module representations.

[6]In this section we choose to denote the Sugawara operator by $T(e)$ to avoid confusion with the Lie bracket.

Given an X we choose the local vector field $\rho(X)$ as described above and define the first order differential operator on local sections of $\mathfrak{V}$

$$\nabla_X = \partial_X + T(\rho(X)), \tag{11.6}$$

The operator ∇_X on $\mathfrak{V}$ will depend on the pullback of $\rho(X)$, i.e. on the coordinate at P_∞.

The following statements are the key results.

Proposition 11.1. [48, Prop.4.4.] *The operator ∇_X is well-defined on conformal blocks $C(\mathfrak{V})$ and does not depend on a choice of the pull-back $\rho(X)$ of X.*

Now all relations in the following should be understood with respect to conformal blocks.

Proposition 11.2. [48, Prop.4.6] *Let $\mathfrak{V}$ be either an irreducible representation of $\overline{\mathfrak{g}}$ or a fermionic representation then for every $X \in \mathrm{T}\widetilde{W}$ we have*

$$\partial_X T(e) = T(\partial_X e) + \lambda \cdot id,$$

where $\lambda = \lambda(X,e) \in \mathbb{C}$.

Proposition 11.3. [51, Lemma 1.3.8]

$$\rho([X,Y]) = [\rho(X), \rho(Y)] + \partial_X \rho(Y) - \partial_Y \rho(X). \tag{11.7}$$

Theorem 11.4. [48, Thm. 4.8.], [50, Thm. 3.14] *The operator ∇_X is a projectively flat connection on the vector bundle of conformal blocks, i.e.*

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + \lambda(X,Y) \cdot id, \quad \lambda(X,Y) \in \mathbb{C}. \tag{11.8}$$

11.3. **Knizhnik–Zamolodchikov equations.** Let $\mathfrak{V}$ be a sheaf of admissible representations of the affine algebra $\widehat{\mathfrak{g}}$ over $\widetilde{W}$ or $\widetilde{W}^{(1,1)}$ as introduced above. We assume the level c to be constant and obeying the condition $(c+\kappa) \neq 0$. Moreover, let all the additional assumptions be fulfilled which are needed in the previous section to define the Knizhnik–Zamolodchikov connection on conformal blocks.

Definition 11.5. [47] The Knizhnik–Zamolodchikov equation are the equations

$$\nabla_X \Phi = (\partial_X + T(\rho(X)))\Phi = 0, \quad X \in X \in \mathrm{H}^0(\widetilde{W}, \mathrm{T}\mathcal{M}_{g,N+1}), \tag{11.9}$$

respectively $\mathrm{H}^0(\widetilde{W}^{(1,1)}, \mathrm{T}\mathcal{M}^{(1,1)}_{g,N+1})$ where Φ is a section of the sheaf of conformal blocks $C(\mathfrak{V})$.

Hence, the Knizhnik–Zamolodchikov equations are the equations which have as solutions the horizontal sections of the connection ∇_X.

After fixing $\widehat{\sigma}_\infty$ and a first order infinitesimal neighbourhood of it we can again formulate this over the moduli space $\mathcal{M}_{g,N}$. This says that we do not consider the point P_∞ as a moving point, hence there is no tangent direction corresponding to this.

Denote the tangent vectors by X_k, $k = 1, \ldots, 3g-3+N$ and set $e_k = \rho(X_k)$ for the corresponding element of the critical strip.

We set $\partial_k := \partial_{X_k}$. and define for sections Φ of $\mathfrak{V}$ and for every k the operator

$$\nabla_k \Phi := (\partial_k + T[e_k])\, \Phi \ . \tag{11.10}$$

Then the Knizhnik–Zamolodchikov equations read as

$$\nabla_k\, \Phi = 0\ , \qquad k = 1, \ldots, 3g-3+N\ . \tag{11.11}$$

Using (6.7) and

$$l_k^{(n,p)(m,s)} := \frac{1}{2\pi \mathrm{i}} \int_{C_\tau} \omega^{n,p} \omega^{m,s} e_k \tag{11.12}$$

we rewrite this as

$$\left(\partial_k - \frac{1}{c+\kappa} \sum_{\substack{n,m \\ p,s}} l_k^{(n,p)(m,s)} \sum_a :u_a(n,p) u^a(m,s): \right) \Phi = 0 \ , \qquad k = 1, \ldots, 3g-3+N \ . \tag{11.13}$$

Here the summation over a is a summation over a system of dual basis elements in $\mathfrak{g}$. The coefficients $l_k^{(n,p)(m,s)}$ encode the geometric information about the complex structure and the positions of the points.

Remark 11.6. If we pass over to the Kodaira–Spencer class $\overline{\rho}(X)$ we obtain the Knizhnik–Zamolodchikov equations in terms of a fixed critical strip. By Theorem 9.2 the elements of the standard critical strip correspond to tangent vectors along the moduli space $\mathcal{M}_{g,N}$. For example the N equations related to $e_{-1,p}$, $p = 1, \ldots, N$. correspond to moving the points, and the other ones corresponding to the $3g-3$ elements $e_k \in \mathcal{L}^*_{(0)}$ (for $g \geq 2$) are responsible for changing the complex structure of the curve. Without further assumptions on the representations under consideration, respectively additional conditions on the solutions of the Knizhnik–Zamolodchikov equations the equations (with $\overline{\rho}(X)$) will depend on the critical strip chosen. One such assumption guaranteeing independence is that we require from the solutions Φ that $\widehat{\mathfrak{g}}_+ \Phi = 0$. But this might be too restrictive for certain applications.

Remark 11.7. In genus zero the only relevant variations are moving the points Let z_i, $i = 1, \ldots, N$ be the N moving points and fix the reference point z_∞ to be ∞. It is shown in [48] and in the book [50] that in the Verma module case the equation

$$\left(\frac{\partial}{\partial z_i} - \frac{2}{c+\kappa} \sum_{j \neq i} \sum_a \frac{t_i^a t_j^a}{z_i - z_j} \right) \Phi = 0 \ , \qquad i = 1, \ldots, N \tag{11.14}$$

will be obtained. This is exactly the usual form of the rational Knizhnik–Zamolodchikov equation found in [22].

Here u_a is a self-dual basis of $\mathfrak{g}$ and the action of $u_a(0,i) = u_a(A_{0,i})$ can be described as

$$u_a(0,i)\, \Phi = t_i^a\, \Phi,$$

where i indicates the Point P_i. We get

$$u_a(0,j) u_a(0,i)\, \Phi = t_j^a t_i^a\, \Phi = t_i^a t_j^a\, \Phi \,, \quad \text{for } j \neq i \ .$$

In genus zero, the conformal blocks can be realized in degree zero. Finally, we obtain the expression (11.14). The degree zero property is not true anymore for higher genus, see e.g. already the genus $g = 1$ case in the above mentioned references.

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