Lie superalgebras of Krichever–Novikov type

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Abstract. Classically, starting from the Witt and Virasoro algebra important examples of Lie superalgebras were constructed. In this write-up of a talk presented at the Białowieża meetings we report on results on Lie superalgebras of Krichever-Novikov type. These algebras are multi-point and higher genus equivalents of the classical algebras. The grading in the classical case is replaced by an almost-grading. It is induced by a splitting of the set of points, were poles are allowed, into two disjoint subsets. With respect to a fixed splitting, or equivalently with respect to a fixed almost-grading, it is shown that there is up to rescaling and equivalence a unique non-trivial central extension of the Lie superalgebra of Krichever–Novikov type. It is given explicitly.

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1. Introduction

In the context of conformal field theory (CFT) the Witt algebra and its universal central extension, the Virasoro algebra, play an important role. These algebras encode conformal symmetry. To incorporate superconformal symmetry one is forced to extend the algebras to Lie superalgebras. Examples of them are the Neveu–Schwarz and the Ramond type superalgebras.

These algebras we call the classical algebras. They correspond to the genus zero situation. Krichever–Novikov algebras are higher genus and multi-point analogs of them. For higher genus, but still only for two points where poles are allowed, some of the algebras were generalised in 1986 by Krichever and Novikov [9], [10], [11]. In 1990 the author [15], [16], [17], [18] extended

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the approach further to the general multi-point case. These extensions were not straight-forward generalizations. The crucial point was to introduce a replacement of the graded algebra structure present in the “classical” case. Krichever and Novikov found that an almost-grading, see Definition 6, will be enough to allow constructions in representation theory, like triangular decompositions, highest weight modules, Verma modules and so on. In [17], [18] it was realized that a splitting of the set A of points where poles are allowed into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading. For every such splitting the corresponding almost-grading was given. Essentially different splittings (not just corresponding to interchanging of I and O) will yield essentially different almost-gradings. For the general theory (including the classical case) see the recent monograph [22].

In the context of conformal field theory and string theory the Neveu-Schwarz and the Ramond type superalgebras appear as superextensions of the classical algebras. Some physicists also studied superanalogs of the algebra of Krichever-Novikov type, but still only with two points where poles are allowed, e.g. [1], [2], [3], [4], [23]. The multi-point case was developed by the author some time ago and recently published in [21]. See also [22].

Starting from Krichever–Novikov type superalgebras interesting explicite infinite dimensional examples of Jordan superalgebras and antialgebras can be constructed. In this respect, see the work of Leidwanger and Morier-Genoud [12], [13], and Kreusch [8].

In this write-up we will recall the construction of the Krichever–Novikov (KN) type algebras for the multi-point situation and for arbitrary genus. The classical situation will be a special case. In particular, the construction of the Lie superalgebra is recalled. Its almost-graded structure, induced by a fixed splitting $A = I \cup O$, is given.

The main result presented here is the fact that up to rescaling the central element and equivalence of extension, there is only one non-trivial almost-graded central extension of the Lie superalgebra of KN type with even central element. We stress the fact, that this does not mean that there is essentially only one central extension. For an essentially different splitting we get an essentially different central extension. For higher genus there are even central extensions not related to any splitting. In the classical situation in this way uniqueness of the non-trivial central extension is again obtained. Recall that “classical” means genus zero and two points where poles are allowed.

We will give a geometric description for the defining cocycle, see (43). For the two-point case the form of the cocycle was given by Bryant in [4], correcting some ommission in [1].

In the case of odd central elements we obtained that the corresponding central extension of the Lie superalgebra will split. For the proofs we have to refer to the original article [21], respectively to [22].

2. The Classical Lie Superalgebra

For the convenience of the reader we start with recalling the definition of a Lie superalgebra. Let $\mathcal{S}$ be a vector space which is decomposed into even and odd elements $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, i.e. $\mathcal{S}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bilinear map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that for elements x, y of pure parity

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]. \tag{1}$$

This says that

$$[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subseteq \mathcal{S}_{\bar{0}}, \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{1}}, \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{0}}, \tag{2}$$

and $[x, y]$ is symmetric for x and y odd, otherwise anti-symmetric. Furthermore, $\mathcal{S}$ is a Lie superalgebra if in addition the super-Jacobi identity (x, y, z of pure parity)

$$(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0 \tag{3}$$

is valid. As long as the type of the arguments is different from *(even, odd, odd)* all signs can be put to $+1$ and we obtain the form of the usual Jacobi identity. In the remaining case we get

$$[x, [y, z]] + [y, [z, x]] - [z, [x, y]] = 0. \tag{4}$$

By the very definitions $\mathcal{S}_0$ is a Lie algebra.

In purely algebraic terms the Virasoro algebra $\mathcal{V}$ is given by generators $\{e_n (n \in \mathbb{Z}), t\}$ with relations

$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t, \qquad [t, e_n] = 0\,. \tag{5}$$

Without the central term t we obtain the Witt algebra $\mathcal{W}$.

For the classical (Neveu–Schwarz) Lie superalgebra we add an additional set of generators $\{\varphi_m \mid m \in \mathbb{Z} + \frac{1}{2}\}$ and complete the relations to

$$\begin{aligned}
[e_n, e_m] &= (m-n)e_{m+n} + \frac{1}{12}(n^3 - n)\,\delta_n^{-m}\,t, \\
[e_n, \varphi_m] &= (m - \frac{n}{2})\,\varphi_{m+n}, \\
[\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\,(n^2 - \frac{1}{4})\,\delta_n^{-m}\,t, \\
[t, e_n] &= [t, \varphi_m] = 0\,.
\end{aligned} \tag{6}$$

These algebras can be realized in a geometric manner by considering vector fields and forms of weight $-1/2$ (see below) on the Riemann sphere S^2 (i.e. the Riemann surface of genus zero). If we take the vector fields $e_n = z^{n+1}\frac{d}{dz}$, the forms of weight $-1/2$ given by $\varphi_m = z^{m+1/2}(dz)^{-1/2}$, let the vector field e_n act by taking the Lie derivative, and set $[\varphi_m, \varphi_n] = \varphi_m \cdot \varphi_n$ then we obtain the relations above without central terms. The element t is an additional element and the factors in front of it seem to be rather ad-hoc for the moment[1]. One verifies by direct calculations that by our prescription we obtain indeed a Lie superalgebra.

3. Higher genus generalization

Starting from the geometric realization above we can extend this to arbitrary genus. For the whole contribution let Σ be a compact Riemann surface (without boundary). We do not put any restriction on the genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \dots, P_K), \quad \text{and} \quad O = (Q_1, \dots, Q_M) \tag{7}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets.

Our objects, algebras, structures, ... will be meromorphic objects defined on Σ which are holomorphic outside the points in A. To introduce them let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ, respectively. the locally free canonically sheaf. The local sections of the bundle are the local holomorphic differentials. If $P \in \Sigma$ is a point and z a local holomorphic coordinate at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function f defined in a neighbourhood of P. A global holomorphic section can be described locally for a covering by coordinate charts $(U_i, z_i)_{i\in J}$ by a system of local holomorphic functions $(f_i)_{i\in J}$, which are related by the transformation rule induced by the coordinate change map $z_j = z_j(z_i)$ and the condition $f_i dz_i = f_j dz_j$. This says

$$f_j = f_i \cdot \left(\frac{dz_j}{dz_i}\right)^{-1}. \tag{8}$$

With respect to a coordinate covering a meromorphic section of $\mathcal{K}$ is given as a collection of local meromorphic functions $(h_i)_{i\in J}$ for which the transformation law (8) is true.

In the following λ is either an integer or a half-integer. If λ is an integer then
(1) $\mathcal{K}^\lambda = \mathcal{K}^{\otimes\lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.
Here as usual $\mathcal{K}^*$ denotes the dual line bundle to the canonical line bundle. The dual line bundle is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$. If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle,

[1] Below we will give for the factors a geometric expression.

sometimes called a *theta-characteristics*. This means we fix a line bundle L for which $L^{\otimes 2} = \mathcal{K}$.

After such a choice of L is done we set $\mathcal{K}^\lambda = \mathcal{K}_L^\lambda = L^{\otimes 2\lambda}$. In most cases we will drop mentioning L, but we have to keep the choice in mind. Also the structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark. A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. For $g = 0$ we have $\mathcal{K} = \mathcal{O}(-2)$ and $L = \mathcal{O}(-1)$, the tautological bundle which is the unique square root. Already for $g = 1$ we have 4 non-isomorphic ones. As in this case $\mathcal{K} = \mathcal{O}$ one solution is $L_0 = \mathcal{O}$. But we have also the other bundles L_i, $i = 1, 2, 3$.

As above we can talk about holomorphic and meromorphic sections of K^λ. In local coordinates z_i we can write such sections as $f_i dz_i^\lambda$, with f_i being a local holomorphic, respectively meromorphic function.

We set

$$\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid \\ \text{such that } f \text{ is holomorphic over } \Sigma \setminus A\}. \tag{9}$$

Here the set of A is fixed and we drop it in the notation. Obviously, $\mathcal{F}^\lambda$ is an infinite dimensional $\mathbb{C}$-vector space. Recall that in the case of half-integer λ everything depends on the theta characteristic L. We call the elements of the space $\mathcal{F}^\lambda$ *meromorphic forms of weight* λ (with respect to the theta characteristic L). Altogether we set

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{10}$$

4. Algebraic Structure

4.1. Associative Multiplication

The natural map of the locally free sheaves of rank one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{11}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{12}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\, dz^{\lambda+\nu}. \tag{13}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

The following is obvious

Proposition 1. *The vector space $\mathcal{F}$ with operation $\cdot$ is an associative and commutative graded (over $\frac{1}{2}\mathbb{Z}$) algebra. Moreover, $\mathcal{F}^0$ is a subalgebra.*

We also use $\mathcal{A} := \mathcal{F}^0$. Of course, it is the algebra of meromorphic functions on Σ which are holomorphic outside of A. The spaces $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

4.2. Lie algebra structure

Next we define a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t], \tag{14}$$

which is defined in local representatives of the sections by

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda)s\frac{dt}{dz} + \nu\, t\frac{ds}{dz}\right) dz^{\lambda+\nu+1}, \tag{15}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 2. *(a) The bilinear map $[.,.]$ defines a Lie algebra structure on $\mathcal{F}$.*
(b) The space $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra.

Proof. This is done by local calculations. For details see [19], [22]. □

Proposition 3. *The subspace $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra with respect to the operation $[.,.]$, and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.*

As forms of weight -1 are vector fields, $\mathcal{L}$ could also be defined directly as the Lie algebra of those meromorphic vector fields on the Riemann surface Σ which are holomorphic outside of A. The product (15) gives the usual Lie bracket of vector fields and the Lie derivative for their actions on forms:

$$[e,f]_{|}(z) = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ , \tag{16}$$

$$\nabla_e(f)_{|}(z) = L_e(f)_{|} = e\,.\,f_{|} = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{17}$$

4.3. Superalgebra of half forms

Next we consider the associative product

$$\cdot\ \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L}, \tag{18}$$

and introduce the vector space and the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f] + \varphi\cdot\psi, e\,.\,\varphi - f\,.\,\psi). \tag{19}$$

Usually we will denote the elements of $\mathcal{L}$ by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

Definition (19) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a "super-bracket" (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi,e] := e\,.\,\varphi =_{|} (e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2}, \tag{20}$$

and

$$[\varphi,\psi] = \varphi\cdot\psi. \tag{21}$$

We call the elements of $\mathcal{L}$ elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ elements of odd parity. For such elements x we denote by $\bar{x} \in \{\bar{0}, \bar{1}\}$ their parity.

The sum (19) can be described as $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 4. [21, Prop. 2.5] *The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Definition 5. The algebra $\mathcal{S}$ is the Lie superalgebra of Krichever–Novikov type.

Remark. The introduced Lie superalgebra corresponds classically to the Neveu-Schwarz superalgebra. In string theory physicists considered also the Ramond superalgebra as string algebra (in the two-point case). The elements of the Ramond superalgebra do not correspond to sections of the dual theta characteristics. They are only defined on a 2-sheeted branched covering of Σ, see e.g. [1], [3]. Hence, the elements are only multi-valued sections. As here we only consider honest sections of half-integer powers of the canonical bundle, we do not deal with the Ramond algebra.

The choice of the theta characteristics corresponds to choosing a spin structure on Σ. Furthermore, this bundle is related to graded Riemann surfaces. See Bryant [4] for more details on this aspect.

5. Almost-Graded Structure

Recall the classical situation. This is the Riemann surface $\mathbb{P}^1(\mathbb{C}) = S^2$, i.e. the Riemann surface of genus zero, and the points where poles are allowed are $\{0, \infty\}$. In this case the algebras introduced in the last chapter are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [9] there is a weaker concept, called an almost-grading which to a large extend is a valuable replacement of a honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting certain basis elements in the spaces $\mathcal{F}^\lambda$ as homogeneous.

Definition 6. Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,
(ii) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}. \tag{22}$$

Elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called homogeneous subspace of degree n.

In a similar manner almost-graded modules over almost-graded algebras are defined. Of course, we can extend in an obvious way the definition to superalgebras, respectively even to more general algebraic structures. This definition makes complete sense also for more general index sets $\mathbb{J}$. In fact we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ for our superalgebra. The even elements (with respect to the super-grading) will have integer degree, the odd elements half-integer degree.

As already mentioned above the almost-grading for $\mathcal{F}^\lambda$ is introduced by exhibiting certain elements $f_{m,p}^\lambda \in \mathcal{F}^\lambda$, $p = 1, \dots, K$ which constitute a basis of the subspace $\mathcal{F}_m^\lambda$ of homogeneous elements of degree m. Here $m \in \mathbb{J}_\lambda$ with $\mathbb{J}_\lambda = \mathbb{Z}$ (for λ integer) or $\mathbb{J}_\lambda = \mathbb{Z}+1/2$ (for λ half-integer). The basis elements $f_{m,p}^\lambda$ of degree m are required to have order

$$\operatorname{ord}_{P_i}(f_{m,p}^\lambda) = (n + 1 - \lambda) - \delta_i^p$$

at the point $P_i \in I$, $i = 1, \dots, K$. The prescription at the points in O is made in such a way that the element $f_{m,p}^\lambda$ is essentially uniquely given. For more details on the prescription see [22, Chapter 4] or the original article [16]. In the classical case we have $\deg(e_n) = n$ and $\deg(\varphi_m) = m$. Warning: The spaces $\mathcal{F}_m^\lambda$ depend on the splitting of A.

For the property of being almost-graded the following result is crucial. It is obtained by calculating residues and estimating orders.

Proposition 7. [22, Thm. 3.8] *There exist constants R_1 and R_2 (depending on the number and splitting of the points in A and of the genus g) independent of $n, m \in \mathbb{J}$ and λ and ν such that for the basis elements*

$$\begin{aligned} f_{n,p}^\lambda \cdot f_{m,r}^\nu = \quad & f_{n+m,r}^{\lambda+\nu} \delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu}, \quad a_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}, \end{aligned} \tag{23}$$

$$\begin{aligned} [f_{n,p}^\lambda, f_{m,r}^\nu] = \quad & (-\lambda m + \nu n)\, f_{n+m,r}^{\lambda+\nu+1} \delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu+1}, \quad b_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}. \end{aligned}$$

The constants R_i can be explicitly calculated (if needed).

As a direct consequence we obtain

Theorem 8. *The algebras $\mathcal{L}$ and $\mathcal{S}$ are almost-graded Lie, respectively Lie superalgebras. The almost-grading depends on the splitting of the set A into I and O. More precisely,*

$$\mathcal{F}^\lambda = \bigoplus_{m \in \mathbb{J}_\lambda} \mathcal{F}_m^\lambda, \qquad \textit{with} \quad \dim \mathcal{F}_m^\lambda = K. \tag{24}$$

and there exist R_2, R_3 (independent of n and m) such that

$$[\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h\ , \qquad [\mathcal{S}_n, \mathcal{S}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_3} \mathcal{S}_h.$$

Also from (23) we directly conclude

Proposition 9. *For all $m, n \in \mathbb{J}_\lambda$ and $r, p = 1, \ldots, K$ we have*

$$\begin{aligned} [e_{n,p}, e_{m,r}] &= (m-n)\cdot e_{n+m,r}\,\delta_r^p + h.d.t. \\ e_{n,p}\cdot\varphi_{m,r} &= (m-\frac{n}{2})\cdot\varphi_{n+m,r}\,\delta_r^p + h.d.t. \\ \varphi_{n,p}\cdot\varphi_{m,r} &= e_{n+m,r}\,\delta_r^p + h.d.t. \end{aligned} \tag{25}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_i$.

See (6) for an example in the classical case (by ignoring the central extension appearing there for the moment).

Remark. Leidwanger and Morier-Genoux introduced in [12] also a Jordan superalgebra based on the Krichever-Novikov objects given by

$$\mathcal{J} := \mathcal{F}^0 \oplus \mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}. \tag{26}$$

Recall that $\mathcal{F}^0$ is the associative algebra of meromorphic functions. The (Jordan) product is defined via the algebra structure for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f\circ g &:= f\cdot g \quad &\in \mathcal{F}^0, \\ f\circ\varphi &:= f\cdot\varphi \quad &\in \mathcal{F}^{-1/2}, \\ \varphi\circ\psi &:= [\varphi,\psi] \quad &\in \mathcal{F}^0. \end{aligned} \tag{27}$$

By rescaling the second definition with the factor 1/2 one obtains a Lie antialgebra. See [12] for more details and additional results on representations. Using the results presented here one obtains an almost-grading (depending on a splitting $A = I \cup O$) of the Jordan superalgebra

$$\mathcal{J} = \bigoplus_{m\in 1/2\mathbb{Z}} \mathcal{J}_m. \tag{28}$$

Hence, it makes sense to call it a Jordan superalgebra of KN type. Calculated for the introduced basis elements we get (using Proposition 7)

$$\begin{aligned} A_{n,p}\circ A_{m,r} &= A_{n+m,r}\,\delta_r^p + \text{h.d.t.} \\ A_{n,p}\circ\varphi_{m,r} &= \varphi_{n+m,r}\,\delta_r^p + \text{h.d.t.} \\ \varphi_{n,p}\circ\varphi_{m,r} &= \frac{1}{2}(m-n)\mathcal{A}_{n+m,r}\,\delta_r^p + \text{h.d.t.} \end{aligned} \tag{29}$$

6. Central Extensions

A central extension of a Lie algebra W is defined on the vector space direct sum $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ its Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W. \tag{30}$$

Then $\widehat{W}$ will be a Lie algebra, e.g. fulfill the Jacobi identity, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y). \tag{31}$$

There is the notion of equivalence of central extensions. It turns out that two central extensions are equivalent if and only if the difference of their defining 2-cocycles Φ and Φ' is a coboundary, i.e. there exists a $\phi : W \to \mathbb{C}$ such that

$$\Phi(x, y) - \Phi'(x, y) = d_1\phi(x, y) = \phi([x, y]). \tag{32}$$

In this way the second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ of W with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions. The class $[0]$ corresponds to the trivial (i.e. split) central extension. Hence, to construct central extensions of our Lie algebras we have to find such Lie algebra 2-cocycles.

For the superalgebra case central extension are obtained with the help of a bilinear map

$$\Phi : \mathcal{S} \times \mathcal{S} \to \mathbb{C} \tag{33}$$

via an expression completely analogous to (30). Additional conditions for Φ follow from the fact that the resulting extension should be again a superalgebra. This implies that for homogeneous elements $x, y, z \in \mathcal{S}$ ($\mathcal{S}$ might be an arbitrary Lie superalgebra) we need

$$\Phi(x, y) = -(-1)^{\bar{x}\bar{y}}\Phi(x, y). \tag{34}$$

If x and y are odd then the bilinear map Φ will be symmetric, otherwise it will be antisymmetric. The super-cocycle condition reads in complete analogy with the super-Jacobi relation as

$$(-1)^{\bar{x}\bar{z}}\Phi(x, [y, z]) + (-1)^{\bar{y}\bar{x}}\Phi(y, [z, x]) + (-1)^{\bar{z}\bar{y}}\Phi(z, [x, y]) \;=\; 0. \tag{35}$$

As we will need it anyway, I will write it out for the different type of arguments. For *(even,even,even)*, *(even,even, odd)*, and *(odd,odd,odd)* it will be of the "usual form" of the cocycle condition

$$\Phi(x, [y, z]) + \Phi(y, [z, x]) + \Phi(z, [x, y]) \;=\; 0. \tag{36}$$

For *(even,odd,odd)* we obtain

$$\Phi(x, [y, z]) + \Phi(y, [z, x]) - \Phi(z, [x, y]) \;=\; 0. \tag{37}$$

Now we have to decide which parity our central element should have. In our context, as we want to extend the central extension of the vector field algebra to the superalgebra, the natural choice is that the central element should be even. This implies that our bilinear form Φ has to be an even form. Consequently,

$$\Phi(x,y) = \Phi(y,x) = 0, \quad \text{for} \ \ \bar{x} = 0, \bar{y} = 1. \tag{38}$$

In this case only (37) for the *(even,odd,odd)* and (36) for the *(even,even,even)* case will give relations which are not trivially zero.

Given a linear form $\phi : \mathcal{S} \to \mathbb{C}$ we assign to it

$$\delta_1\phi(x,y) = \phi([x,y]). \tag{39}$$

As in the classical case $\delta_1\phi$ will be a super-cocycle. A super-cocycle Φ will be a coboundary if and only if there exists a linear form $\phi : \mathcal{S} \to \mathbb{C}$ such that $\Phi = \delta_1\phi$. As ϕ is a linear form it can be written as $\phi = \phi_{\bar{0}} \oplus \phi_{\bar{1}}$ where $\phi_{\bar{0}} : \mathcal{S}_{\bar{0}} \to \mathbb{C}$ and $\phi_{\bar{1}} : \mathcal{S}_{\bar{1}} \to \mathbb{C}$. Again we have the two cases of the parity of the central element. Let Φ be a coboundary $\delta_1\phi$. If the central element is even then Φ will also be a coboundary with respect to a ϕ with $\phi_{\bar{1}} = 0$. In other words this ϕ is even. In the odd case we can take $\phi_{\bar{0}} = 0$ and ϕ is odd.

After fixing a parity of the central element we consider the quotient spaces

$$\mathrm{H}^2_{\bar{0}}(\mathcal{S},\mathbb{C}) := \{\text{even cocycles}\}/\{\text{even coboundaries}\}, \tag{40}$$

$$\mathrm{H}^2_{\bar{1}}(\mathcal{S},\mathbb{C}) := \{\text{odd cocycles}\}/\{\text{odd coboundaries}\}. \tag{41}$$

These cohomology spaces classify central extensions of $\mathcal{S}$ with even (respectively odd) central elements up to equivalence. Equivalence is defined as in the non-super setting.

To define a super-cocycle we have to introduce the following objects.

Definition 10. Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A system of local holomorphic functions $R = (R_\alpha(z_\alpha))$ is called a holomorphic *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{42}$$

the Schwartzian derivative. Here $'$ means differentiation with respect to the coordinate z_α.

It is a classical result [6], [5] that every Riemann surface admits a holomorphic projective connection R. From the definition it follows that the difference of two projective connections is a quadratic differential. In fact starting from one projective connection we will obtain all of them by adding quadratic differentials to it.

If we have a cocycle Φ for the algebra $\mathcal{S}$ we obtain by restriction a cocycle for the algebra $\mathcal{L}$. For arguments with mixed parity we know that $\Phi(e,\psi) = 0$. A naive try to put just anything for $\Phi(\varphi,\psi)$ will not work as (37) relates the restriction of the cocycle on $\mathcal{L}$ with its values on $\mathcal{F}^{-1/2}$.

Proposition 11. [21, Prop. 5.1] *Let C be any closed (differentiable) curve on Σ not meeting the points in A, and let R be any (holomorphic) projective connection, then the bilinear extension of*

$$\begin{aligned}
\Phi_{C,R}(e,f) &:= \frac{1}{24\pi\mathrm{i}}\int_C \left(\frac{1}{2}(e'''f - ef''') - R\cdot(e'f-ef')\right)dz \\
\Phi_{C,R}(\varphi,\psi) &:= -\frac{1}{24\pi\mathrm{i}}\int_C (\varphi''\cdot\psi + \varphi\cdot\psi'' - R\cdot\varphi\cdot\psi)\,dz \\
\Phi_{C,R}(e,\varphi) &:= 0
\end{aligned} \tag{43}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. The cocycle class does not depend on the chosen connection R.

A similar formula was given by Bryant in [4]. By adding the projective connection in the second part of (43) he corrected some formula appearing in [1]. He only considered the two-point case and only the integration over a separating cycle. See also [8] for the multi-point case, where still only the integration over a separating cycle is considered.

The following remarks are in order. For the proof of the claims see [21].

1. Adding the projective connection is necessary to make the integrand a well-defined differential.
2. Different R will yield cohomologous cocycles.
3. With respect to the curve C only its homology class in $\mathrm{H}_1(\Sigma\setminus A,\mathbb{Z})$ is relevant for the cocycle.

But a different cycle class will change the cocycle in an essential manner. We cannot expect uniqueness if $g>0$ or $N>2$. We should not forget, that we want to extend our almost-grading of $\mathcal{S}$ to the centrally extended Lie superalgebra by assigning a degree to the central element. This only works if our cocycle is “local” [9] in the following sense: There exists $M_1, M_2\in\mathbb{Z}$ such that

$$\forall n,m: \quad \psi(W_n,W_m)\neq 0 \implies M_1\le n+m\le M_2.$$

What is local is defined in terms of the almost-grading and hence depends on the splitting $A = I\cup O$.

If the integration path C in (43) is a *separating cycle* C_S, i.e. a cycle which separates the points in I from the points in O, then the cocycle $\Phi_{C_S,R}$ is local. A special choice for C_S is the collection of circles around the points in I. This shows that in this case the cocycle can be calculated via residues.

How about the opposite direction? Given a local cocycle, can it be described as such a geometric cocycle? The answer is yes and this is the main result which we present.

Theorem 12. [21, Thm. 5.5] *Given a local (even) cocycle for the Lie superalgebra $\mathcal{S}$ of Krichever–Novikov type then up to coboundary it is a multiple of $\Phi_{C_S,R}$. Hence, up to rescaling and equivalence there is a unique non-trivial almost-graded central extension of $\mathcal{S}$.*

Recall that the almost-grading is fixed by the splitting. We will only give some general remarks on the proof.

1. We start with a local cocycle for $\mathcal{S}$ and restrict it to $\mathcal{L}$, the vector field subalgebra.
2. This gives a local cocycle for the vector field algebra $\mathcal{L}$.
3. For such local cocycles my earlier classification results [20] show that it is unique (in the above sense) and can by given by an expression of the type $\Phi_{C_S,R}$ for $\mathcal{L}$.
4. This expression we extend to $\mathcal{S}$ by using the full expression of $\Phi_{C_S,R}$.
5. The difference between the initial cocycle and the extended one vanishes if restricted to $\mathcal{L}$.
6. Next we show that each local cocycle of $\mathcal{S}$ which vanishes on $\mathcal{L}$ vanishes in total. This is done by some induction process using the almost-gradedness and the locality (more precisely, the boundedness from above is enough).

Theorem 13. [21, Thm. 5.6] *All local cocycles of odd type are coboundaries. Hence, there does not exist non-trivial almost-graded odd central extensions of $\mathcal{S}$.*

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