On four Koszul-Tate resolutions

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Abstract

We suggest a $\mathcal{D}$-geometric definition of a Koszul-Tate (KT) resolution for a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism (thought of as the projection onto an on-shell function algebra). Here $\mathtt{DG}\mathcal{D}\mathtt{A}$ denotes the category of differential non-negatively graded algebras over linear differential operators $\mathcal{D}$ acting on functions of a smooth base scheme. Such a $\mathcal{D}$-geometric KT resolution does always exist: no locality, regularity, or reducibility assumptions are needed. In the case of a smooth affine base, a $\mathcal{D}$-geometric KT resolution can be obtained from the functorial cofibrant replacement functor on $\mathtt{DG}\mathcal{D}\mathtt{A}$ that has been explicitly constructed in [BPP15b]. Also the latter resolution exists without any of the mentioned restrictive hypotheses. It turns out that the classical KT resolution constructed in coordinates [Bar10], for any regular on-shell irreducible gauge theory (as the Tate extension of the Koszul resolution of a regular surface), as well as the compatibility complex KT resolution built in coordinates [Ver02], under regularity and off-shell reducibility conditions (existence of a finite formally exact compatibility complex), are KT resolutions in the $\mathcal{D}$-geometric sense. The relationships between the classical and the cofibrant replacement KT resolutions, as well as between the classical and the compatibility complex KT resolutions, are studied. In the appendix, we construct from scratch some of the knowledge needed to study PDE-s and corresponding resolutions in the $\mathcal{D}$-algebraic and the physical settings, as well as in the jet bundle formalism. For the model categorical approach, we refer to [BPP15a] and [BPP15b].

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Contents

1 Notation and conventions

When $\mathcal{F}_X$ is a sheaf over a topological space X and $U \subset X$ is an open subset, we write $\mathcal{F}(U)$ for $\mathcal{F}_X(U) = \Gamma(U, \mathcal{F}_X)$.

For any unital ring R, we denote by $D_\bullet^k$ the k-disc chain complex

$$D_\bullet^k : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overset{(k)}{R} \overset{\mathrm{id}}{\longrightarrow} \overset{(k-1)}{R} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \overset{(0)}{0} \ , \tag{1}$$

and by $S_\bullet^k$ the k-sphere chain complex

$$S_\bullet^k : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overset{(k)}{R} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \overset{(0)}{0} \ . \tag{2}$$

Moreover, we set

$$I = \{i_k : S_\bullet^{k-1} \to D_\bullet^k,\ k \geq 0\}$$

and

$$J = \{\zeta_k : 0 \to D_\bullet^k,\ k \geq 1\} \ ,$$

where i_k, ζ_k are the canonical chain maps.

2 Preliminaries

This paper is the third of a series of works on the BV-formalism. In [BPP15a] and [BPP15b] we proved the following

Theorem 1. *The category* $\mathtt{DG}\mathcal{D}\mathtt{A}$ *of differential non-negatively graded commutative unital algebras over the ring* $\mathcal{D} = \mathcal{D}_X(X)$ *of total sections of the sheaf* $\mathcal{D}_X$ *of differential operators of a smooth affine variety* X*, is a finitely (and thus a cofibrantly) generated model category (in the sense of [GS06] and in the sense of [Hov07]), with* $\mathcal{S}(I) = \{\mathcal{S}(\iota_k) : \iota_k \in I\}$ *as its generating set of cofibrations and* $\mathcal{S}(J) = \{\mathcal{S}(\zeta_k) : \zeta_k \in J\}$ *as its generating set of trivial cofibrations, where* $\mathcal{S}$ *denotes the graded symmetric tensor algebra functor. The weak equivalences are the* $\mathtt{DG}\mathcal{D}\mathtt{A}$*-morphisms that induce an isomorphism in homology, the fibrations are the* $\mathtt{DG}\mathcal{D}\mathtt{A}$*-morphisms that are surjective in all positive degrees* $p > 0$*, and the cofibrations are exactly the retracts of the relative Sullivan* $\mathcal{D}$*-algebras.*

Further, we describe in these articles explicit functorial cofibration-fibration factorizations, as well as explicit functorial fibrant and cofibrant replacement functors. We then use the latter to build a model categorical Koszul-Tate resolution for $\mathcal{D}$-algebraic on-shell function algebras.

3 $\mathcal{D}$-geometric KT resolution

Let X be a smooth scheme and let $\mathcal{O}_X$ (resp., $\mathcal{D}_X$) be the sheaf of rings of functions (resp., differential operators) of X. Denote by $\mathtt{qcCAlg}(\mathcal{O}_X)$ (resp., $\mathtt{qcCAlg}(\mathcal{D}_X)$) the category of commutative unital $\mathcal{O}_X$-algebras (resp., commutative unital $\mathcal{D}_X$-algebras) that are quasi-coherent as $\mathcal{O}_X$-modules. In the following, we refer to the objects of this category as $\mathcal{O}_X$-algebras (resp., $\mathcal{D}_X$-algebras). The forgetful functor has a left adjoint [BD04]

$$\mathcal{J}^\infty : \mathtt{qcCAlg}(\mathcal{O}_X) \to \mathtt{qcCAlg}(\mathcal{D}_X) \ ,$$

called the **jet functor**.

Proposition 1. *Let $\pi : E \to X$ be a vector bundle of finite rank over X and denote by $\mathcal{O}_E$ the structure sheaf of the scheme E. Then $\mathcal{O}_X^E := \pi_* \mathcal{O}_E \in \mathtt{qcCAlg}(\mathcal{O}_X)$ and thus $\mathcal{J}^\infty(\mathcal{O}_X^E) \in \mathtt{qcCAlg}(\mathcal{D}_X)$.*

The latter can be interpreted as the $\mathcal{D}$-geometric counterpart of the function sheaf $\mathcal{O}_{J^\infty E}$ of the infinite jet bundle of a smooth vector bundle. See Appendix 7.3 for additional information, as well as for the proof of Proposition 1.

The algebraization of a scalar partial differential equation (PDE) acting on sections of a smooth vector bundle E may be viewed as a function $F \in \mathcal{O}(J^\infty E)$ of $J^\infty E$. The function algebra $\mathcal{O}(\Sigma^\infty)$ of the infinite prolongation $\Sigma^\infty \subset J^\infty E$ (also called the 'stationary surface' or the 'shell') of this equation is the quotient of the algebra $\mathcal{O}(J^\infty E)$ by the ideal I of all functions that vanish on Σ^∞. Hence, we think about an ideal $\mathcal{I} \subset \mathcal{J}^\infty(\mathcal{O}_X^E)$ as a scalar polynomial PDE acting on sections of $\pi : E \to X$ and about $\mathcal{J}^\infty(\mathcal{O}_X^E)/\mathcal{I}$ as the sheaf of corresponding on-shell function $\mathcal{D}_X$-algebras. The latter ideal is of course a $\mathcal{D}_X$-ideal, i.e., an $\mathcal{O}_X$-ideal and a $\mathcal{D}_X$-submodule that is quasi-coherent as $\mathcal{O}_X$-module. Our goal is to resolve this $\mathcal{D}_X$-algebra.

In the following, we write $\mathcal{J}$ instead of $\mathcal{J}^\infty(\mathcal{O}_X^E)$. We will explain below that in classical Koszul-Tate resolutions [HT92, Ver02], the natural type of differential operators are the 'total or horizontal differential operators', which can be identified with the sheaf $\mathcal{J}[\mathcal{D}_X] := \mathcal{J} \otimes \mathcal{D}_X$ of rings of differential operators with coefficients in $\mathcal{J}$. Moreover, as mentioned in [BPP15b], a Koszul-Tate resolution of $\mathcal{R} := \mathcal{J}/\mathcal{I}$, or, of the canonical $\mathcal{D}_X$-algebra morphism $f : \mathcal{J} \to \mathcal{R}$, should be a DG $\mathcal{D}_X$-algebra, as well as a $\mathcal{J}$-algebra, or, still better, a DG $\mathcal{J}[\mathcal{D}_X]$-algebra. Hence, in addition to the category $\mathtt{DG_+qcCAlg}(\mathcal{D}_X)$ of differential non-negatively graded quasi-coherent commutative unital $\mathcal{D}_X$-algebras, which we studied in [BPP15a, BPP15b], we will in the sequel also consider the category $\mathtt{DG_+qcCAlg}(\mathcal{J}[\mathcal{D}_X])$, with self-explaining notation. We suggest to the reader, who considers himself as not familiar with this topic, to skim Appendices 7.3.4 and 7.3.5, which contain some details that will be freely used in the sequel.

The computations of [BPP15b] suggest the following $\mathcal{D}$-geometric definition:

Definition 1. *Let X be a smooth scheme, let $\mathcal{A}$ be a $\mathcal{D}_X$-algebra, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a DG $\mathcal{D}_X$-algebra morphism. A* **Koszul-Tate resolution** *(a KTR for short) of ϕ is a DG $\mathcal{A}[\mathcal{D}_X]$-algebra morphism $\psi : \mathcal{C} \to \mathcal{B}$, which is a quasi-isomorphism in the category of DG $\mathcal{A}[\mathcal{D}_X]$-modules, and whose source $\mathcal{C}$ is of Sullivan type. Here, $\mathcal{C}$ is of* **Sullivan type** *means that $\mathcal{C}$ admits an increasing filtration $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots$ by DG $\mathcal{D}_X$-subalgebras, such that there is a DG $\mathcal{D}_X$-algebra morphism $\mathcal{A} \to \mathcal{C}_0$ (we set $\mathcal{C}_{-1} := \mathcal{A}$) and that $\mathcal{C}_k$ ($k \geq 0$) is isomorphic as DG $\mathcal{D}_X$-algebra to $\mathcal{C}_k \simeq \mathcal{C}_{k-1} \otimes \mathcal{S}V_k$, where V_k is a locally projective graded $\mathcal{D}_X$-submodule of $\mathcal{C}_k$ such that $d_{\mathcal{C}_k} V_k \subset \mathcal{C}_{k-1}$.*

Remark 1. *Observe first that a quasi-isomorphism in the category of DG $\mathcal{A}[\mathcal{D}_X]$-modules is a morphism that induces a bijection in homology, i.e., is an $\mathcal{A}$-linear quasi-isomorphism in the category of DG $\mathcal{D}_X$-modules. Further, the differential on $\mathcal{C}_{k-1} \otimes \mathcal{S}V_k$ is $d_{\mathcal{C}_k}$ and it is completely defined by the facts that $d_{\mathcal{C}_k}$ is a degree -1 graded derivation and that $\mathcal{C}_{k-1}$ is a DG $\mathcal{D}_X$-subalgebra of $\mathcal{C}_k$.*

The requirement that $\mathcal{C}$ be equipped with an increasing filtration by DG $\mathcal{D}_X$-subalgebras $\mathcal{C}_k$ $(k \geq 0)$ and that there exist a DG $\mathcal{D}_X$-algebra morphism $j_0 : \mathcal{A} \to \mathcal{C}_0$, is equivalent to the condition that $\mathcal{C}$ be filtered by a sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots$ of DG $\mathcal{A}[\mathcal{D}_X]$-subalgebras. Indeed, since $j_0 : \mathcal{A} \to \mathcal{C}_0$, as well as the canonical inclusions $i_k : \mathcal{C}_{k-1} \to \mathcal{C}_k$ $(k \geq 1)$, are DG $\mathcal{D}_X$-algebra morphisms, we have DG $\mathcal{D}_X$-algebra morphisms $j_k = i_k \circ \ldots \circ i_1 \circ j_0 : \mathcal{A} \to \mathcal{C}_k$ that provide a filtring sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots$ of DG $\mathcal{A}[\mathcal{D}_X]$-subalgebras. Conversely, such a sequence gives a DG $\mathcal{D}_X$-algebra morphism $\mathcal{A} \ni a \mapsto a \triangleleft 1_{\mathcal{C}_0} \in \mathcal{C}_0$. Hence, a resolution of Sullivan type is the same as an $\mathcal{A}$-semi-free resolution [BD04]. It follows [BD04] that the next proposition holds.

Proposition 2. *Let X be a smooth scheme. If $\mathcal{A}$ is a $\mathcal{D}_X$-algebra, any DG $\mathcal{D}_X$-algebra morphism $\mathcal{A} \to \mathcal{B}$ admits a Koszul-Tate resolution. In particular, if $\pi : E \to X$ is a finite rank vector bundle and if $\mathcal{J} := \mathcal{J}^\infty(\mathcal{O}_E)$, any DG $\mathcal{D}_X$-algebra morphism $\mathcal{J} \to \mathcal{B}$ admits a KTR; for instance, if $\mathcal{I}$ is a $\mathcal{D}_X$-ideal, the $\mathcal{D}_X$-algebra morphism $\mathcal{J} \to \mathcal{J}/\mathcal{I}$ has a KTR.*

Remark 2. *Let us stress that the $\mathcal{D}$-geometric KTR is defined in the algebraic geometric setting, over any smooth scheme X, and for any DG $\mathcal{D}_X$-algebra map with arguments in a $\mathcal{D}_X$-algebra – thought of as morphism from infinite jet space functions to on-shell functions of some partial differential equation – . However, in fact, no equation is considered, and the $\mathcal{D}$-geometric KTR does always exist, although, unlike the more classical situations discussed below, no locality, no regularity, and no reducibility assumptions have been made.*

4 Cofibrant replacement KT resolution and $\mathcal{D}$-geometric KT resolution

The 'classical' Koszul-Tate resolutions [HT92] and [Ver02] are 'local' results, see below. If in the context of the preceding section, we assume locality, in the sense that the underlying smooth scheme X is smooth affine, or is even a smooth affine algebraic variety, we can replace sheaves by global sections, see [BPP15a].

Let now $\pi : E \to X$ be a smooth morphism of smooth affine algebraic varieties. The jet algebra $J := \mathcal{J}^\infty(\mathcal{O}_X^E(X))$ is a $\mathcal{D}$-algebra, $\mathcal{D} = \mathcal{D}_X(X)$. If $I \subset J$ is a $\mathcal{D}$-ideal, i.e., a scalar polynomial PDE acting on the sections of π, the quotient J/I is the $\mathcal{D}$-algebra of 'on-shell' functions. In view of [BPP15b], the canonical $\texttt{DG}\mathcal{D}\texttt{A}$-morphism $\phi : J \to J/I$ admits a 'cofibration - trivial fibration' decomposition given by the functorial 'Cof - TrivFib' factorization of the cofibrantly generated model structure of $\texttt{DG}\mathcal{D}\texttt{A}$, see Theorem 1:

$$J \rightarrowtail J \otimes \mathcal{S}V \overset{\sim}{\twoheadrightarrow} J/I\ . \tag{3}$$

Theorem 2. *The cofibrant replacement (3) of J/I in the undercategory $J \downarrow \texttt{DG}\mathcal{D}\texttt{A}$ [BPP15b], or, better, the morphism $J \otimes \mathcal{S}V \to J/I$ is a $\mathcal{D}$-geometric Koszul-Tate resolution of the morphism $\phi : J \to J/I$ in the sense of Definition 1.*

Indeed, the constructions in Section 4 of [BPP15b] directly imply that the minimal relative Sullivan $\mathcal{D}$-algebra $J \to J \otimes \mathcal{S}V$ is clearly of Sullivan type, and the $\texttt{DG}\mathcal{D}\texttt{A}$-morphisms $\iota : J \ni$

$j \mapsto j \otimes 1_{\mathcal{O}} \in J \otimes \mathcal{S}V$ and $\phi : J \ni j \mapsto [j] \in J/I$ allow to endow the two target algebras $J \otimes \mathcal{S}V$ (with multiplication $\diamond$) and J/I (with multiplication $*$) with natural $\mathtt{DG}J[\mathcal{D}]\mathtt{A}$-structures

$$j \triangleleft T = (j \otimes 1_{\mathcal{O}}) \diamond T \quad \text{and} \quad j \triangleleft [j'] = [j] * [j'] \, .$$

It thus suffices to show that the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $q : J \otimes \mathcal{S}V \to J/I$ is J-linear (see also Remark 1). The latter is obvious from the definition [BPP15b] and the properties of q. Indeed,

$$q(j \triangleleft T) = q((j \otimes 1_{\mathcal{O}}) \diamond T) = q(j \otimes 1_{\mathcal{O}}) * q(T) = \phi(j) * q(T) = [j] * q(T) = j \triangleleft q(T) \, .$$

Remark 3. *The context for the cofibrant replacement KTR is again algebraic geometric, but a locality assumption is necessary, in the sense that we must work over a smooth affine algebraic variety X. Moreover, we start from a $\mathcal{D}$-ideal I of the jet $\mathcal{D}$-algebra J associated to a morphism $E \to X$ – thought of as some partial differential equation – . Again no regularity and no reducibility hypotheses are needed. The KTR is the cofibrant replacement of J/I in $J \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$.*

5 Classical and $\mathcal{D}$-geometric KT resolutions

Remark 4. *In the following, we use without reference results and notation of Subsection 7.1 and Subsection 7.2.*

5.1 Regular on-shell irreducible gauge theory

We consider a **regular irreducible gauge theory**, i.e., a field theory, whose dynamical equations are the Euler-Lagrange equations of some Lagrangian $\mathcal{L}$, which admits non-trivial Noether identities (i.e., non-trivial gauge symmetries in characteristic form) and satisfies the regularity and irreducibility assumptions 1-5 of Subsection 7.2.3. It follows that **we work locally**, in a trivialization of a smooth rank r vector bundle $\pi : E \to X$ over a coordinate patch of a smooth manifold of dimension n. The fiber (resp., base) coordinates are denoted by $u = (u^a)$ (resp., $x = (x^i)$), with $a \in \{1, \ldots, r\}$ (resp., $i \in \{1, \ldots, n\}$).

The assumptions 1-5 imply that the considered regular gauge theory is irreducible in the sense that

Proposition 3. *In a regular irreducible gauge theory, there exists an irreducible set of non-trivial Noether operators.*

More precisely, a gauge theory admits, by definition, non-trivial Noether identities $N^a_\alpha D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0$, so that the $D^\alpha_x \delta_{u^a}\mathcal{L}$ are not independent. More precisely, at least one of the functions $N^a_\alpha \in \mathcal{F}(\pi)$ of the infinite jet space $J^\infty(\pi)$ of π, does not vanish on the constraint surface $\Sigma \subset J^\infty(\pi)$, which is defined by the total derivatives $D^\alpha_x \delta_{u^a}\mathcal{L} = 0$ of the algebraized Euler-Lagrange equations $\delta_{u^a}\mathcal{L} = 0$. The hypotheses 1-5 entail that the $D^\alpha_x \delta_{u^a}\mathcal{L}$ can be separated into independent and dependent equations $E_{\mathfrak{a}}$ and E_Δ. Further, the dependent equations $E_\Delta = F^{\mathfrak{a}}_\Delta E_{\mathfrak{a}}$, where $F^{\mathfrak{a}}_\Delta \in \mathcal{F}(\pi)$, are the total derivatives $E_\Delta = D^\beta_x E_\delta$ of a finite number of dependent equations $E_\delta = F^{\mathfrak{b}}_\delta E_{\mathfrak{b}}$ ($\delta \in \{1, \ldots, K\}$), and the Noether identities $E_\Delta - F^{\mathfrak{a}}_\Delta E_{\mathfrak{a}} \equiv 0$

associated to the E_Δ are the total derivatives $D_x^\beta(E_\delta - F_\delta^{\mathfrak{b}} E_{\mathfrak{b}}) \equiv 0$ of the Noether identities $E_\delta - F_\delta^{\mathfrak{b}} E_{\mathfrak{b}} \equiv 0$ associated to the E_δ (this hypothesis is called the irreducibility assumption for the considered gauge theory (IA)). We write the latter *Noether identities*

$$R^a_{\delta\alpha}\, D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0 \quad (\delta \in \{1, \ldots, K\})\ . \tag{4}$$

It is easy to see that they are *non-trivial*, i.e., that, for any δ, there is at least one coefficient $R^a_{\delta\alpha}$ that does not vanish on the constraint surface $\Sigma \subset J^\infty(\pi)$ (note that the tuple of the $D^\alpha_x \delta_{u^a}\mathcal{L}$ is given by the action of an invertible matrix $\mathfrak{I}$ on the tuple made of the $E_{\mathfrak{a}}, E_\Delta$ (we sometimes assume for simplicity that this matrix is identity)).

A compatibility operator (roughly, non-trivial linear total differential relations between the equations) can itself admit a compatibility operator (relations between the relations). Similarly, Noether identities can be related by so-called first-stage Noether identities, which satisfy second-stage Noether identities... It is naturel to refer to the existence of non-trivial higher-stage Noether identities as the reducibility of the considered gauge theory. Since we deal in this text with an *irreducible* gauge theory, no non-trivial first-stage Noether identity should exist, i.e., any linear total differential operator $(S^1_\beta \ldots S^K_\beta) D^\beta_x$ such that $S^\delta_\beta D^\beta_x \circ R^a_{\delta\alpha} D^\alpha_x = 0$ should be trivial, should vanish. Such an operator vanishes if and only if all its coefficients vanish. In the present approach to the Koszul-Tate resolution, 'trivial' (resp., 'non-trivial') means that all the coefficients vanish (resp., at least one coefficient does not vanish) on Σ. Hence, we actually deal with **on-shell irreducibility**. This means that

$$S^\delta_\beta D^\beta_x \circ R^a_{\delta\alpha} D^\alpha_x \approx 0 \quad \text{must imply that} \quad S^\delta_\beta \approx 0 \quad (\forall\, \delta \in \{1, \ldots, K\})\ . \tag{5}$$

It can be shown [Bar10] that this *on-shell irreducibility condition* really holds – in view of the above irreducibility assumption (IA).

In view of (4) and (5), the linear total / horizontal differential operators $R^a_\delta = R^a_{\delta\alpha} D^\alpha_x$ are the announced irreducible set of non-trivial Noether operators.

5.2 Classical KTR as Tate extension of the Koszul resolution of a regular surface

The **Koszul-Tate resolution** of the algebra $C^\infty(\Sigma)$ of functions of the constraint surface is a generalization of the Koszul resolution of a regular surface, see Subsection 7.2.1. The difference between the case of a regular surface $\Sigma \subset \mathbb{R}^n$ and the case of a constraint surface $\Sigma \subset J^\infty(\pi)$ in a regular irreducible gauge theory, is the existence of the irreducible set of non-trivial Noether operators R^a_δ, or, still, of the Noether identities $R^a_{\delta\alpha}\, D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0$ and their extensions

$$D^\beta_x\, R^a_{\delta\alpha}\, D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0\ . \tag{6}$$

It turns out that, to kill the homology in higher degrees, we must introduce additional generators that take into account these extensions. More precisely, we do not only associate degree 1 generators $\phi^{\alpha *}_a$ to the equations $D^\alpha_x \delta_{u^a}\mathcal{L} = 0$ of Σ, but we assign further degree 2 generators

$C^{\beta *}_{\delta}$ to the relations (6). The Koszul-Tate resolution of $C^\infty(\Sigma)$ is then the chain complex, whose chains are the elements of the free Grassmann algebra

$$\mathrm{KT} = \mathcal{F}(\pi) \otimes \mathcal{S}[\phi^{\alpha *}_a, C^{\beta *}_{\delta}] \;, \tag{7}$$

and whose differential is defined, in analogy with the Koszul differential of a regular surface, by

$$\delta_{\mathrm{KT}} = D^\alpha_x \delta_{u^a} \mathcal{L}\, \partial_{\phi^{\alpha *}_a} + D^\beta_x\, R^a_{\delta\alpha}\, D^\alpha_x \phi^*_a\, \partial_{C^{\beta *}_\delta} \;, \tag{8}$$

where we substituted ϕ^*_a to $\delta_{u^a}\mathcal{L}$ and where the total derivatives have to be interpreted in the extended sense that puts the 'antifields' ϕ^* and C^* on an equal footing with the 'fields' ϕ. This means that D_{x^i} must be defined as

$$D_{x^i} = \partial_{x^i} + \phi^a_{i\alpha} \partial_{\phi^a_\alpha} + \phi^{i\alpha *}_a \partial_{\phi^{\alpha *}_a} + C^{i\beta *}_\delta \partial_{C^{\beta *}_\delta} \;.$$

Note that the replacement in δ_{KT} of the $\delta_{u^a}\mathcal{L}$ by the ϕ^*_a is necessary to get a degree -1 operator and that this replacement lends naturalness to the extended interpretation of the total derivatives. The bosonic antifield C^* is referred to as the Tate part of the Koszul-Tate complex $(\mathrm{KT}, \delta_{\mathrm{KT}})$.

The homology of $(\mathrm{KT}, \delta_{\mathrm{KT}})$ is actually concentrated in degree 0, where it coincides with $C^\infty(\Sigma)$. Indeed, the 0-cycles are the functions $\mathcal{F}(\pi)$ and the 0-boundaries are the

$$\delta_{\mathrm{KT}} \left(\sum F^a_\alpha \phi^{\alpha *}_a \right) = \sum F^a_\alpha D^\alpha_x \delta_{u^a} \mathcal{L} \approx 0 \;.$$

In view of the regularity assumption 2 in Subsection 7.2.3, the equations $E_{\mathfrak{a}}$ play the same role as the equations $f^{\mathfrak{a}}$ play in Subsection 7.2.1, so that the ideal $I(\Sigma)$ of those functions of $\mathcal{F}(\pi)$ that vanish on Σ is made of the combinations $\sum F^{\mathfrak{a}} E_{\mathfrak{a}}$. Therefore, not only any 0-boundary belongs to $I(\Sigma)$, but, conversely, any function of $I(\Sigma)$ reads

$$\sum F^{\mathfrak{a}} E_{\mathfrak{a}} = \sum F^{\mathfrak{a}} (\mathfrak{I}^{-1})^a_{\mathfrak{a}\alpha}\, D^\alpha_x \delta_{u^a} \mathcal{L} = \delta_{\mathrm{KT}} \left(\sum F^{\mathfrak{a}}\, (\mathfrak{I}^{-1})^a_{\mathfrak{a}\alpha}\, \phi^{\alpha *}_a \right)$$

and is therefore a 0-boundary. It follows that $H_0(\mathrm{KT}) = \mathcal{F}(\pi)/I(\Sigma) = C^\infty(\Sigma)$. To show that the homology vanishes in higher degrees, one needs the antifield C^*, as well as the irreducibility assumption (IA).

In fact, the above **irreducible set of non-trivial Noether operators R^a_δ is generating**, in the sense that any Noether operator $(N^1_\alpha \dots N^r_\alpha) D^\alpha_x$, i.e., any total differential operator (e.g., from $\mathcal{F}(\pi, \pi)$ to $\mathcal{F}(\pi)$) such that $N^a_\alpha\, D^\alpha_x \delta_{u^a} \mathcal{L} \equiv 0$, uniquely reads

$$N^a_\alpha\, D^\alpha_x = S^\delta_\gamma\, D^\gamma_x \circ R^a_{\delta\beta}\, D^\beta_x + M^{[a,b}_{\alpha,\beta]}\, D^\beta_x \delta_{u^b} \mathcal{L}\, D^\alpha_x \;, \tag{9}$$

where the coefficients belong to $\mathcal{F}(\pi)$ and satisfy $S^\delta_\gamma \not\approx 0$ and $M^{[a,b}_{\alpha,\beta]} = -M^{[b,a}_{\beta,\alpha]}$. Hence, in a regular irreducible gauge theory, any Noether operator $(N^1 \dots N^r)$ coincides on-shell with a composite $(S^\delta \circ R^1_\delta \dots S^\delta \circ R^r_\delta)$ of the irreducible set of Noether operators with some total differential operators. This result is actually a quite straightforward corollary of the fact that $H_1(\mathrm{KT}) = 0$.

5.3 Change of perspective

In the classical Koszul-Tate complex $\mathrm{KT} = \mathcal{F} \otimes \mathcal{S}\mathfrak{V}$, where $\mathcal{F} = \mathcal{F}(\pi)$ and

$$\mathfrak{V} = \bigoplus_{\alpha,a} \mathbb{R} \cdot \phi_a^{\alpha *} \oplus \bigoplus_{\beta,\delta} \mathbb{R} \cdot C_\delta^{\beta *} \ ,$$

the tensor products are over $\mathbb{R}$ and $(\mathrm{KT}, \delta_{\mathrm{KT}})$ is viewed as a chain complex in the category of $\mathcal{F}$-modules.

However, the algebra $\mathcal{F}$ can be endowed with a $\mathcal{D}$-module structure. Since we work in fixed coordinates, any $D \in \mathcal{D}$ uniquely reads $D = \sum_{|\alpha|\leq k} D_\alpha(x)\partial_x^\alpha$, for some integer $k \in \mathbb{N}$ and functions $D_\alpha \in \mathcal{O} := C^\infty(X)$. As observed in Subsection 7.3.2, the action of D on $F \in \mathcal{F}$ is defined by

$$D \cdot F = \mathcal{C}(D)F = \sum_{|\alpha|\leq k} D_\alpha(x) D_x^\alpha F \ , \tag{10}$$

where $\mathcal{C}$ denotes the horizontal lift. It is easily seen that this definition actually provides a $\mathcal{D}$-module structure, since, for any composable linear differential operators $\Delta_1 \in \mathrm{Diff}(\eta_1, \eta_2)$ and $\Delta_2 \in \mathrm{Diff}(\eta_2, \eta_3)$ between vector bundles η_i over X, the horizontal lifts

$$\mathcal{C}(\Delta_1) \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2)) \quad \text{and} \quad \mathcal{C}(\Delta_2) \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_2), \pi_\infty^*(\eta_3))$$

satisfy

$$\mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C}(\Delta_2) \circ \mathcal{C}(\Delta_1) \ .$$

This result holds for any vector bundle $\pi : E \to X$, in particular for the trivial one we fixed at the beginning of Subsection 5.1 – see [KV98].

It is clear that this $\mathcal{D}$-module structure and the $\mathcal{O}$-algebra structure of $\mathcal{F}$ are compatible in the sense that vector fields act as derivations. Hence, $\mathcal{F}$ is a $\mathcal{D}$-algebra. Moreover, the ideal $I(\Sigma)$ is an $\mathcal{O}$-ideal and a $\mathcal{D}$-submodule, hence a $\mathcal{D}$-ideal. As for the submodule structure, note that if $F \in I(\Sigma)$ and $D \in \mathcal{D}$, one has

$$(D \cdot F)|_\Sigma = (\mathcal{C}(D)F)|_\Sigma = (\mathcal{C}(D))|_\Sigma F|_\Sigma = 0 \ ,$$

see Subsection 7.1. Finally, the quotient $C^\infty(\Sigma) = \mathcal{F}/I(\Sigma)$ is a $\mathcal{D}$-algebra for the action $D \cdot [F] = [D \cdot F]$ and the multiplication $[F][G] = [FG]$. It follows that the passage

$$\phi : \mathcal{F} \ni F \mapsto [F] \in C^\infty(\Sigma)$$

to the quotient is a $\mathcal{D}$-algebra map. However, not only differential operators act on $C^\infty(\Sigma)$, also jet functions do act: it suffices to set $F \triangleleft [G] := [F][G] = [FG]$. This $\mathcal{F}$-algebra and the former $\mathcal{D}$-algebra structures on $C^\infty(\Sigma)$ are compatible, so that $C^\infty(\Sigma)$ is an $\mathcal{F}[\mathcal{D}]$-algebra.

Since $\mathcal{F}$ is a $\mathcal{D}$-algebra, hence an $\mathcal{O}$-algebra, it is natural to replace $\mathfrak{V}$ by the free non-negatively graded $\mathcal{O}$-module

$$\mathcal{V} = \bigoplus_{\alpha,a} \mathcal{O} \cdot \phi_a^{\alpha *} \oplus \bigoplus_{\beta,\delta} \mathcal{O} \cdot C_\delta^{\beta *}$$

over the generators $\phi_a^{\alpha *}$ and $C_\delta^{\beta *}$ of degree 1 and 2, respectively. Just as the variables u_α^a or ϕ_α^a are algebraizations of the derivatives $\partial_x^\alpha \phi^a$ of the components of a section ϕ of a vector bundle $E \to X$ (fields), the generators $\phi_a^{\alpha *}$ and $C_\delta^{\beta *}$ symbolize the total derivatives $D_x^\alpha \phi_a^*$ and $D_x^\beta C_\delta^*$ of the components of sections ϕ^* and C^* of some vector bundles $\pi_\infty^* F_1 \to J^\infty E$ and $\pi_\infty^* F_2 \to J^\infty E$ (antifields). Hence, the $\phi_a^{\alpha *}$ and $C_\delta^{\beta *}$ can be thought of as horizontal jet space coordinates of $\pi_\infty^* F_1$ and $\pi_\infty^* F_2$. These coordinates may of course be denoted by other symbols, e.g., by $\partial_x^\alpha \cdot \phi_a^*$ and $\partial_x^\beta \cdot C_\delta^*$, provided we define the $\mathcal{D}$-action as the action $D_x^\alpha \phi_a^*$ and $D_x^\beta C_\delta^*$ by the corresponding horizontal lift. This is not only in accordance with (10), but leads to appropriate interpretations when the ϕ_a^*-s and the C_δ^*-s are the components of true sections, as well as when interpreting the total derivatives in the above-mentioned extended sense that puts antifields on the same level as fields:

$$\partial_x^\alpha \cdot \phi_a^* := D_x^\alpha \phi_a^* = \phi_a^{\alpha *} \quad \text{and} \quad \partial_x^\beta \cdot C_\delta^* = D_x^\beta C_\delta^* = C_\delta^{\beta *} \ . \tag{11}$$

Eventually, the best choice for the underlying module $\mathfrak{V}$ or $\mathcal{V}$ is the free non-negatively graded $\mathcal{D}$-module

$$V = \bigoplus_a \mathcal{D} \cdot \phi_a^* \oplus \bigoplus_\delta \mathcal{D} \cdot C_\delta^*$$

over the components of the antifields ϕ^* and C^*. The $\mathcal{F}$-module of Koszul-Tate chains then reads

$$\mathrm{KT} = \mathcal{F} \otimes_{\mathbb{R}} \mathcal{S}_{\mathbb{R}} \mathfrak{V} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V \ , \tag{12}$$

where the RHS is also a graded $\mathcal{D}$-algebra.

Any element c of this graded $\mathcal{D}$-algebra reads non-uniquely as a finite sum

$$c = \sum F \, (D^a \cdot \phi_a^*) \dots (\Delta^\delta \cdot C_\delta^*) \ ,$$

where $F \in \mathcal{F}$ and $D^a, \Delta^\delta \in \mathcal{D}$, and where we omitted the tensor products. The Koszul-Tate differential δ_{KT}, which is well-defined on KT, acts as a graded derivation and *is thus completely known, if it is known on the $D^a \cdot \phi_a^*$ and the $\Delta^\delta \cdot C_\delta^*$*. For any $D = D_\alpha \partial_x^\alpha$, we have, in view of the definitions given above,

$$\delta_{\mathrm{KT}}(D \cdot \phi_a^*) = D_\alpha \, \delta_{\mathrm{KT}}(\partial_x^\alpha \cdot \phi_a^*) = D_\alpha \, \delta_{\mathrm{KT}}(\phi_a^{\alpha *}) = D_\alpha D_x^\alpha \delta_{u^a} \mathcal{L} = D \cdot (\delta_{u^a} \mathcal{L}) = D \cdot \delta_{\mathrm{KT}}(\phi_a^*) \ . \tag{13}$$

Similarly, we get

$$\delta_{\mathrm{KT}}(D \cdot C_\delta^*) = D_\alpha \, \delta_{\mathrm{KT}}(\partial_x^\alpha \cdot C_\delta^*) = D_\alpha \, \delta_{\mathrm{KT}}(C_\delta^{\alpha *}) = D_\alpha D_x^\alpha (R_{\delta\beta}^a \, D_x^\beta \phi_a^*) = D_\alpha D_x^\alpha (R_{\delta\beta}^a \, \phi_a^{\beta *}) \ .$$

The extended total derivative D_x^α of $R_{\delta\beta}^a \, \phi_a^{\beta *}$ is a sum of terms of the type

$$D_x^{\alpha_1} R_{\delta\beta}^a \, D_x^{\alpha_2} \phi_a^{\beta *} = (\partial_x^{\alpha_1} \cdot R_{\delta\beta}^a) \, (\partial_x^{\alpha_2} \cdot \phi_a^{\beta *}) \ ,$$

so that, in view of the definition of the $\mathcal{D}$-action on the tensor product of $\mathcal{F}$ and $\mathcal{S}_{\mathcal{O}} V$, we find that

$$D_x^\alpha (R_{\delta\beta}^a \, \phi_a^{\beta *}) = \partial_x^\alpha \cdot (R_{\delta\beta}^a \, \phi_a^{\beta *}) \ .$$

Eventually,

$$\delta_{\mathrm{KT}}(D \cdot C_\delta^*) = D \cdot \delta_{\mathrm{KT}}(C_\delta^*) \ . \tag{14}$$

5.4 Classical KTR viewed as $\mathcal{D}$-geometric KTR

In the following, we apply, without further reference, [BPP15b, Lemma 1] that allows to construct non-split relative Sullivan $\mathcal{D}$-algebras (RS$\mathcal{D}$A-s), as well as $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphisms from such a Sullivan algebra to another differential graded $\mathcal{D}$-algebra. For convenience, we recall this lemma in Subsection 7.3.1.

Let $V_1 := \bigoplus_a \mathcal{D} \cdot \phi_a^*$. To endow the graded $\mathcal{D}$-algebra

$$\mathcal{C}_1 := \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_1 \tag{15}$$

with a differential graded $\mathcal{D}$-algebra structure d, we set,

$$d\phi_a^* := \delta_{u^a}\mathcal{L} \in \mathcal{F} \ , \tag{16}$$

extend d to V_1 by $\mathcal{D}$-linearity, and equip $\mathcal{C}_1$ with the differential d given by

$$d(F(D \cdot \phi_a^*)\,(\Delta \cdot \phi_b^*)) := (F\, d(D \cdot \phi_a^*))(\Delta \cdot \phi_b^*) - (F\, d(\Delta \cdot \phi_b^*))(D \cdot \phi_a^*) \ ,$$

where we omitted the tensor products and considered, to increase clarity, an element of degree 2. Then the natural $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $\imath : (\mathcal{F}, 0) \ni F \mapsto F \otimes 1_{\mathcal{O}} \in (\mathcal{C}_1, d)$ is a RS$\mathcal{D}$A. Since δ_{KT} is also a graded derivation that is $\mathcal{D}$-linear in the sense of Equation (13) and coincides with d on the generators ϕ_a^*, the RS$\mathcal{D}$A is actually a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\imath : (\mathcal{F}, 0) \ni F \mapsto F \otimes 1_{\mathcal{O}} \in (\mathcal{C}_1, \delta_{\mathrm{KT}}) \ . \tag{17}$$

Consider now the $\mathcal{D}$-algebra $C^\infty(\Sigma) = \mathcal{F}/I(\Sigma)$ and the $\mathcal{D}\mathtt{A}$-morphism $\phi : \mathcal{F} \to C^\infty(\Sigma)$. To define a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_1 : \mathcal{C}_1 \to C^\infty(\Sigma) \ , \tag{18}$$

it suffices to set

$$q_1(\phi_a^*) = 0 \in (C^\infty(\Sigma))_1 \cap 0^{-1}(\phi(d\phi_a^*)) \ , \tag{19}$$

to extend q_1 by $\mathcal{D}$-linearity to V_1, and to define q_1 in degree 0 by $q_1(F) = [F]$ and in degree ≥ 1 by $q_1 = 0$. As for Condition (48), note that $\phi(d\phi_a^*) = [\delta_{u^a}\mathcal{L}] = 0$, in view of the definition of Σ.

An anew application of Lemma 1 in [BPP15b], where the role played above by $(\mathcal{F}, 0)$ (resp., V_1) is now assumed by $(\mathcal{C}_1, \delta_{\mathrm{KT}})$ (resp., $V_2 := \bigoplus_\delta \mathcal{D} \cdot C_\delta^*$), endows the graded $\mathcal{D}$-algebra

$$\mathcal{C}_2 := \mathcal{C}_1 \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_2 \tag{20}$$

with a differential graded $\mathcal{D}$-algebra structure d that, similar to d above, is fully defined by

$$\mathrm{d}\, C_\delta^* = R_{\delta\alpha}^a (\partial_x^\alpha \cdot \phi_a^*) \in (\mathcal{C}_1)_1 \cap \delta_{\mathrm{KT}}^{-1}\{0\} \ . \tag{21}$$

Indeed, we have

$$\delta_{\mathrm{KT}}(R_{\delta\alpha}^a (\partial_x^\alpha \cdot \phi_a^*)) = R_{\delta\alpha}^a\, D_x^\alpha \delta_{u^a}\mathcal{L} \equiv 0 \ .$$

To compare the differential d with the differential δ_{KT}, note that d is extended to V_2 by $\mathcal{D}$-linearity and that its value on $c = F(D \cdot \phi_a^*)\,(\Delta \cdot C_\delta^*)\,(\nabla \cdot C_\varepsilon^*)$, for instance, is

$$\begin{aligned}\mathrm{d}\,c = \quad & \delta_{\mathrm{KT}}(F(D \cdot \phi_a^*))\,(\Delta \cdot C_\delta^*)\,(\nabla \cdot C_\varepsilon^*) \\ & -(F(D \cdot \phi_a^*)\ \mathrm{d}(\Delta \cdot C_\delta^*))\ (\nabla \cdot C_\varepsilon^*) \\ & -(F(D \cdot \phi_a^*)\ \mathrm{d}(\nabla \cdot C_\varepsilon^*))\ \ (\Delta \cdot C_\delta^*)\ .\end{aligned}$$

As δ_{KT} is a graded derivation that is $\mathcal{D}$-linear in the sense of Equation (14) and coincides with d on the generators C_δ^*, we get $\mathrm{d} = \delta_{\mathrm{KT}}$ on $\mathcal{C}_2$. Hence, the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\jmath : (\mathcal{C}_1, \delta_{\mathrm{KT}}) \ni c \mapsto c \otimes 1_{\mathcal{O}} \in (\mathcal{C}_2, \delta_{\mathrm{KT}}) \tag{22}$$

is a relative Sullivan $\mathcal{D}$-algebra.

Start now from the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism q_1, and define a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_2 : \mathcal{C}_2 \to C^\infty(\Sigma) \tag{23}$$

by setting

$$q_2(C_\delta^*) = 0 \in (C^\infty(\Sigma))_2 \cap 0^{-1}(q_1(\delta_{\mathrm{KT}}\,C_\delta^*))\ ,$$

extending q_2 by $\mathcal{D}$-linearity to V_2 and by defining q_2 in degree 0 by $q_2(F) = [F]$ and in degree ≥ 1 by $q_2 = 0$.

Since $V = V_1 \oplus V_2$ as graded $\mathcal{D}$-module, the graded $\mathcal{D}$-algebras $\mathcal{S}_{\mathcal{O}} V = \mathcal{S}_{\mathcal{O}}(V_1 \oplus V_2)$ and $\mathcal{S}_{\mathcal{O}} V_1 \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_2$ are isomorphic. Hence, the same holds for the graded $\mathcal{D}$-algebras

$$\mathrm{KT} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V \quad \text{and} \quad \mathcal{C}_2 = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_1 \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_2\ .$$

It follows that $\jmath \circ \imath : (\mathcal{F}, 0) \to (\mathrm{KT}, \delta_{\mathrm{KT}})$ is a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism and thus allows to endow $(\mathrm{KT}, \delta_{\mathrm{KT}})$ with a $\mathrm{DG}\mathcal{F}[\mathcal{D}]\mathrm{A}$-structure – see Example 1.

Theorem 3. *The classical Koszul-Tate resolution* $(\mathrm{KT}, \delta_{\mathrm{KT}})$ *is a $\mathcal{D}$-geometric Koszul-Tate resolution of the $\mathcal{D}$-algebra map $\phi : \mathcal{F} \to C^\infty(\Sigma)$, in the sense of Definition 1 (in the smooth setting).*

Proof. Most of the proof is given in the preparation that precedes the theorem. For instance, it is clear from what has been said that $\mathrm{KT} \simeq \mathcal{C}_2$ admits an increasing filtration $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_2 \subset \ldots$ by DG $\mathcal{D}$-subalgebras, such that there is a DG $\mathcal{D}$-algebra morphism $\mathcal{F} \to \mathcal{C}_1$ (we set $\mathcal{C}_0 := \mathcal{F}$) and that $\mathcal{C}_k$ ($k \geq 1$) is isomorphic as DG $\mathcal{D}$-algebra to $\mathcal{C}_k \simeq \mathcal{C}_{k-1} \otimes_O \mathcal{S}_O V_k$, where V_k is a free graded $\mathcal{D}$-submodule of $\mathcal{C}_k$ such that $\delta_{\mathrm{KT}} V_k \subset \mathcal{C}_{k-1}$: KT is of Sullivan type. We already mentioned that $\mathrm{KT} \simeq \mathcal{C}_2$ and $C^\infty(\Sigma)$ are $\mathrm{DG}\mathcal{F}[\mathcal{D}]$-algebras. It now suffices to show that the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $q := q_2 : \mathrm{KT} \to C^\infty(\Sigma)$ is $\mathcal{F}$-linear and induces an $\mathcal{F}$- and $\mathcal{D}$-linear bijection $q_\sharp$ of degree 0 between the graded module $H_\bullet(\mathrm{KT})$ and the module $C^\infty(\Sigma)$ concentrated in degree 0. First, q is $\mathcal{F}$-linear, as, if $F, G \in \mathcal{F}$, we obtain

$$F \triangleleft q(G) = F \triangleleft [G] = [FG] = q(FG)\ .$$

Hence, the induced map $q_\sharp$ has the required properties, except, maybe, bijectivity. In degree ≥ 1, the homology $H_\bullet(\mathrm{KT})$ vanishes, just as $C^\infty(\Sigma)$. In degree 0, the homology is given by $C^\infty(\Sigma) = \mathcal{F}/I(\Sigma)$, where $\mathcal{F}$ (resp., $I(\Sigma)$) are the 0-cycles (resp., 0-boundaries), and $q_\sharp[F] = q(F) = [F]$ is the identity. □

5.5 Classical KTR versus cofibrant replacement KTR

Recall that the classical KT resolution $(\mathrm{KT}, \delta_{\mathrm{KT}})$ is the $\mathrm{DG}\mathcal{F}[\mathcal{D}]\mathrm{A}$

$$\mathrm{KT} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V \ ,$$

where V is the free graded $\mathcal{D}$-module with homogeneous basis

$$\bigcup \{\phi_a^*, C_\delta^*\}$$

(the degrees of the generators are 1, 2), endowed with the degree -1, $\mathcal{F}$- and $\mathcal{D}$-linear graded derivation defined by

$$\delta_{\mathrm{KT}}(\phi_a^*) = \delta_{u^a}\mathcal{L} \quad \text{and} \quad \delta_{\mathrm{KT}}(C_\delta^*) = R_{\delta\alpha}^a \left(\partial_x^\alpha \cdot \phi_a^*\right) .$$

The results of [BPP15b], applied (formally) to the $\mathtt{DG}\mathcal{D}\mathtt{A}$-map $\phi : (\mathcal{F}, 0) \to (C^\infty(\Sigma), 0)$, show that the cofibrant replacement KT resolution $(\mathcal{KT}, \delta_{\mathcal{KT}})$ is the $\mathrm{DG}\mathcal{F}[\mathcal{D}]\mathrm{A}$

$$\mathcal{KT} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} \mathcal{V} \ ,$$

where $\mathcal{V}$ is the free graded $\mathcal{D}$-module with homogeneous basis

$$\bigcup \{\mathbb{I}_f, \mathbb{I}^1_{\sigma_n,0}, \mathbb{I}^2_{\sigma_n,0}, \ldots, \mathbb{I}^k_{\sigma_n,0}, \ldots\} \ ,$$

for all $f \in C^\infty(\Sigma)$ and 'numerous' σ_n $(n \geq 0)$ that are described in [BPP15b, Theorem 5] and in the proof that precedes this result (the degrees of the generators are 0, $n+1$, $n+1, \ldots, n+1, \ldots$). Here $\delta_{\mathcal{KT}}$ is the degree -1, $\mathcal{F}$- and $\mathcal{D}$-linear graded derivation defined by

$$\delta_{\mathcal{KT}}(\mathbb{I}_f) = 0 \quad \text{and} \quad \delta_{\mathcal{KT}}(\mathbb{I}^k_{\sigma_n,0}) = \sigma_n \ .$$

When using the just mentioned description in [BPP15b, Theorem 5], one sees rather easily that the map i, defined by

$$i(\phi_a^*) = \mathbb{I}^1_{(\delta_{u^a}\mathcal{L},\, 0)} \in \mathcal{V}_1 \quad \text{and} \quad i(C_\delta^*) = \mathbb{I}^2_{\left(R^a_{\delta\alpha}\left(\partial_x^\alpha \cdot \mathbb{I}^1_{(\delta_{u^a}\mathcal{L},\, 0)}\right),\, 0\right)} \in \mathcal{V}_2 \ ,$$

is a $\mathtt{DG}\mathcal{F}[\mathcal{D}]\mathtt{A}$-morphism

$$i : (\mathrm{KT}, \delta_{\mathrm{KT}}) \to (\mathcal{KT}, \delta_{\mathcal{KT}}) \ .$$

It was clear a priori that the very general functorial cofibrant replacement KT resolution $(\mathcal{KT}, \delta_{\mathcal{KT}})$ would be 'much bigger' than the classical KT resolution $(\mathrm{KT}, \delta_{\mathrm{KT}})$ that is subject to regularity and irreducibility conditions and far from being functorial.

6 Compatibility complex and $\mathcal{D}$-geometric KT resolutions

6.1 Triviality, regularity and off-shell reducibility assumptions

In this and the following subsections, we describe some ideas of [Ver02] adopting a slightly different standpoint and using, as above, results and notation of Subsection 7.1.

The preceding section reminded us of the smooth geometric frame of the classical KT-resolution, as well as of the choice of fixed coordinates. Further, we started from field theoretic Euler-Lagrange equations, with Noether identities relating them, and we made precise regularity and on-shell irreducibility assumptions.

In the present case, the setting will be as well *smooth geometry* and, just as in the classical approach, we will work *locally*, although some aspects are developed in a coordinate-free manner. Our springboard will be any not *necessarily linear PDE*, for which we formulate *regularity* and *off-shell reducibility conditions.*

More precisely, let $\pi : E \to X$ and $\rho_1 : F_1 \to X$ be **smooth** vector bundles of ranks r and r_1, respectively, over a smooth manifold of dimension n. Take a not necessarily linear formally integrable **PDE** $\Sigma^0 \subset J^k(\pi)$ of order k, which is implemented by a not necessarily linear differential operator $D \in \mathrm{DO}_k(\pi, \rho_1)$: $\Sigma^0 = \ker \psi_D$, where $\psi_D \in \mathtt{FB}(J^k(\pi), F_1)$ is the representative fiber bundle morphism of D. Recall (from Subsection 7.1) that

$$\mathrm{DO}_k(\pi,\rho_1) \simeq \mathtt{FB}(J^k(\pi),F_1) \simeq \mathcal{F}_k(\pi,\rho_1) := \Gamma(\pi_k^*(\rho_1)) \subset \Gamma(\pi_\infty^*(\rho_1)) =: \Gamma(R_1) =: \mathcal{R}_1$$

(in the sequel, we often denote a vector bundle over X by a Greek minuscule, its pullback over $J^\infty(\pi)$ by the corresponding Latin capital, and the module of sections of the latter by the same calligraphic letter). As usual, we denote by $\Sigma \subset J^\infty(\pi)$ the infinite prolongation of $\Sigma^0 \subset J^k(\pi)$: $\Sigma = \ker \psi_D^\infty$, where $\psi_D^\infty \in \mathtt{FB}(J^\infty(\pi), J^\infty(\rho_1))$ is the infinite prolongation of ψ_D.

We now recall the locality and regularity hypotheses used in [Ver02]. In fact, the author assumes that Σ is contained in a small open subset $U \subset J^\infty(\pi)$, in which there exist coordinates (x^i, u^a_α). Also in the bundle ρ_1 fiber coordinates – indexed by $\lambda \in \{1, \ldots, r_1\}$ – are fixed. In addition to these **triviality** conditions, he formulates a **regularity** requirement for Σ. Just as for the classical KT-resolution, it is assumed that some equations of Σ can be chosen as first or last coordinates of a new system (of course, the equations of Σ read in the considered trivializations $D_x^\alpha \psi_D^\lambda = 0$, for all $\alpha \in \mathbb{N}^n$ and $\lambda \in \{1, \ldots, r_1\}$.) More precisely, the *neighborhood* U of Σ is assumed to be a trivial bundle over Σ, in the sense that there is an isomorphism $\Phi : U \to \Sigma \times V$, where V is a star-shaped *neighborhood* of 0 in $\mathbb{R}^\infty$, such that the coordinates $v = (v^1, v^2, \ldots)$ in V are precisely certain equations of Σ (not necessarily all of them): for any a, there is an $\alpha_a \in \mathbb{N}^n$ and a $\lambda_a \in \{1, \ldots, r_1\}$, such that $v^a = D_x^{\alpha_a} \psi_D^{\lambda_a}$. This means that the fiber coordinates $v(\kappa)$ of a point $\kappa \in \Sigma$, which are obtained by projecting $\Phi(\kappa)$ on the second factor V, vanish. In addition, as in any trivialization, the projection of $\Phi(\kappa)$, $\kappa \in \Sigma$, on the first factor Σ, is simply κ.

Although in the following we systematically consider the open subset $U \subset J^\infty(\pi)$ instead of the whole jet space, we do not always insist on this restriction (and even write for simplicity sometimes $J^\infty(\pi)$ instead of U).

The latter regularity condition has the same *fundamental consequence* as in Subsections 7.2.1 and 5.2: a function $F \in \mathcal{F}$ vanishes on Σ if and only if it is a finite sum of the type

$$F = \sum F_{\alpha_a,\lambda_a} D_x^{\alpha_a} \psi_D^{\lambda_a} \ ,$$

with $F_{\alpha_a,\lambda_a} \in \mathcal{F}$. In other words, a function $F \in \mathcal{F}$ belongs to the ideal $I(\Sigma)$ if and only if it reads $F = \Psi(\psi_D)$, for some $\Psi \in \mathcal{C}\mathrm{Diff}(\mathcal{R}_1, \mathcal{F})$.

In Subsection 5.1, we assumed on-shell irreducibility, i.e., we assumed that there are no on-shell first stage Noether identities. More precisely, there does exist a generating irreducible set of Noether operators $R^a_{\delta\alpha} D^\alpha_x$, or, still, a horizontal linear differential operator $\Delta_1 \in \mathcal{C}\mathrm{Diff}(\pi^*_\infty(\rho_1), \pi^*_\infty(\rho_2))$. In particular, we have $R^a_{\delta\alpha} D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0$, for all $\delta \in \{1, \ldots, K\}$, or, equivalently, $\Delta_1(\delta_{u^\bullet}\mathcal{L}) \equiv 0$. Note that the LHS of the algebraized Euler-Lagrange equations $\delta_{u^\bullet}\mathcal{L} = 0$ is the representative morphism ψ_D of a not necessarily linear differential operator $D \in \mathrm{DO}(\pi, \rho_1)$. The universal linearization of the latter is a horizontal linear differential operator $\ell_D \in \mathcal{C}\mathrm{Diff}(\pi^*_\infty(\pi), \pi^*_\infty(\rho_1))$. When linearizing the identity $\Delta_1(\psi_D) \equiv 0$, we get $\Delta_1 \circ \ell_D = 0$. Since Δ_1 is generating, it does not vanish and, for any operator $\nabla \in \mathcal{C}\mathrm{Diff}(\pi^*_\infty(\rho_1), \pi^*_\infty(\rho'_2))$, such that $\nabla(\psi_D) \equiv 0$, there is an operator $\Box \in \mathcal{C}\mathrm{Diff}(\pi^*_\infty(\rho_2), \pi^*_\infty(\rho'_2))$, such that $\nabla \approx \Box \circ \Delta_1$, see Equation (9). Hence, roughly speaking, the restriction $\Delta_1|_\Sigma$ is an on-shell compatibility operator for $\ell_D|_\Sigma$, and the mentioned on-shell irreducibility means that there is no on-shell compatibility operator for $\Delta_1|_\Sigma$, see Equation (5).

We now come back to the context of [Ver02]. The restricted linearization $\ell_D|_\Sigma$ of the considered operator D admits a compatibility operator $\Delta_\Sigma \in \mathcal{C}\mathrm{Diff}(\mathcal{R}_1|_\Sigma, \mathcal{R}_2|_\Sigma)$. One of the first results in [Ver02] states that Δ_Σ can be extended to an operator $\Delta_1 \in \mathcal{C}\mathrm{Diff}(\mathcal{R}_1, \mathcal{R}_2)$, such that $\Delta_1(\psi_D) = 0$. Just as any other horizontal linear differential operator, the extension Δ_1 admits a formally exact compatibility complex. However, the latter is a priori neither finite, nor are its $\mathcal{F}$-modules $\mathcal{R}_i$ modules of sections of vector bundles. One of the main assumptions of [Ver02] is that there exists a finite formally exact compatibility complex

$$\mathcal{R}_1 \xrightarrow{\Delta_1} \mathcal{R}_2 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{k-2}} \mathcal{R}_{k-1} \longrightarrow 0 \ , \tag{24}$$

whose $\mathcal{F}$-modules $\mathcal{R}_i$ are all modules $\mathcal{R}_i = \Gamma(R_i) = \Gamma(\pi^*_\infty(\rho_i))$, where the $\rho_i : F_i \to X$ are rank r_i smooth vector bundles, and whose arrows are horizontal operators $\Delta_i \in \mathcal{C}\mathrm{Diff}(\mathcal{R}_i, \mathcal{R}_{i+1})$. This hypothesis is of course an **off-shell reducibility condition**.

6.2 KTR induced by a compatibility complex

Formal exactness implies in particular that, when applying the horizontal infinite jet functor $\bar{\mathcal{J}}^\infty$ to the complex (24), we obtain an exact sequence of $\mathcal{F}$-modules:

$$\bar{\mathcal{J}}^\infty(\mathcal{R}_1) \xrightarrow{\bar{\psi}^\infty_{\Delta_1}} \bar{\mathcal{J}}^\infty(\mathcal{R}_2) \xrightarrow{\bar{\psi}^\infty_{\Delta_2}} \ldots \xrightarrow{\bar{\psi}^\infty_{\Delta_{k-2}}} \bar{\mathcal{J}}^\infty(\mathcal{R}_{k-1}) \longrightarrow 0 \ . \tag{25}$$

Next we use the left exact contravariant Hom functor $\mathrm{Hom}_{\mathcal{F}}(-,\mathcal{F})$, what leads to the exact sequence

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_1),\mathcal{F}) \overset{-\circ\bar{\psi}^\infty_{\Delta_1}}{\longleftarrow} \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_2),\mathcal{F}) \overset{-\circ\bar{\psi}^\infty_{\Delta_2}}{\longleftarrow} \ldots \\ \overset{-\circ\bar{\psi}^\infty_{\Delta_{k-2}}}{\longleftarrow} \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_{k-1}),\mathcal{F}) \longleftarrow 0 \end{aligned} \tag{26}$$

of $\mathcal{F}$-modules. The identification of representative morphisms with the corresponding differential operators finally gives the exact sequence

$$\mathcal{C}\mathrm{Diff}(\mathcal{R}_1,\mathcal{F}) \overset{-\circ\Delta_1}{\longleftarrow} \mathcal{C}\mathrm{Diff}(\mathcal{R}_2,\mathcal{F}) \overset{-\circ\Delta_2}{\longleftarrow} \ldots \overset{-\circ\Delta_{k-2}}{\longleftarrow} \mathcal{C}\mathrm{Diff}(\mathcal{R}_{k-1},\mathcal{F}) \longleftarrow 0\ . \tag{27}$$

The completion

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(\mathcal{R}_{k-1},\mathcal{F}) \overset{-\circ\Delta_{k-2}}{\longrightarrow} \mathcal{C}\mathrm{Diff}(\mathcal{R}_{k-2},\mathcal{F}) \overset{-\circ\Delta_{k-3}}{\longrightarrow} \ldots \overset{-\circ\Delta_1}{\longrightarrow} \mathcal{C}\mathrm{Diff}(\mathcal{R}_1,\mathcal{F}) \overset{-(\psi_D)}{\longrightarrow} \mathcal{F} \longrightarrow 0 \tag{28}$$

of the latter sequence by $-(\psi_D)$ is a complex of $\mathcal{F}$-modules for the natural grading given by the subscripts of the $\mathcal{R}_i$. This complex, which is exact in all spots, except, maybe, in degrees 0 and 1, is actually made of $\mathcal{F}[\mathcal{D}]$-modules. Indeed, in view of Equation (101), we have

$$\mathcal{F}[\mathcal{D}] := \mathcal{F}\otimes\mathcal{D} \simeq \mathcal{C}\mathcal{D}(J^\infty(\pi)) := \mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})\ ,$$

so that the $\mathcal{F}[\mathcal{D}]$-action is given by left composition (except for $\mathcal{F}$). Hence, the arrows of this complex are $\mathcal{F}[\mathcal{D}]$-linear maps and the complex itself is a differential graded $\mathcal{F}[\mathcal{D}]$-module

$$(\mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet,\mathcal{F}),\delta_{\mathsf{KT}}) \in \mathtt{DG}\mathcal{F}[\mathcal{D}]\mathtt{M}\ ,$$

where δ_{KT} is the direct sum of the maps in (28). The graded symmetric tensor algebra functor $\mathcal{S}_{\mathcal{F}}$ sends this underlying module to the free differential graded $\mathcal{F}[\mathcal{D}]$-algebra

$$(\mathsf{KT},\delta_{\mathsf{KT}}) := (\mathcal{S}_{\mathcal{F}}\,\mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet,\mathcal{F}),\delta_{\mathsf{KT}}) \in \mathtt{DG}\mathcal{F}[\mathcal{D}]\mathtt{A}\ , \tag{29}$$

whose differential is a degree -1 graded derivation of the graded symmetric tensor product. The latter complex is the **Koszul-Tate complex**, in the sense of [Ver02], associated to the considered partial differential equation.

The homology space $H_0(\mathsf{KT})$ is easily computed and the above sequences suggest that the higher homology spaces might vanish. Indeed, the module of 0-cycles is $\mathcal{F}$ and the module of 1-chains is $\mathcal{C}\mathrm{Diff}(\mathcal{R}_1,\mathcal{F})$. Due to the above-mentioned fundamental consequence of the regularity condition, the ideal $I(\Sigma)$ coincides with the image of $-(\psi_D)$, i.e., with the module of 0-boundaries. Hence, we get $H_0(\mathsf{KT}) = C^\infty(\Sigma)$.

To prove that the homology spaces $H_p(\mathsf{KT})$, $p\geq 1$, do vanish, it suffices to show that the KT complex (29) coincides – as claimed – with the KT complex defined in [Ver02] and to use the corresponding result therein. The algebra of KT chains is defined in [Ver02] as the graded polynomial function algebra $\mathcal{P}\mathrm{ol}(\bar{J}^\infty(R_\bullet))$. As usual, the polynomial functions $\mathcal{P}\mathrm{ol}(\bar{J}^\infty(R_\bullet))$ are the smooth functions $\mathcal{F}(\bar{J}^\infty(R_\bullet))$ that are polynomial along the fibers of the considered bundle – here $\bar{J}^\infty(R_\bullet) \to J^\infty(\pi)$. Just as the polynomial functions of a vector bundle $G \to X$ are defined by

$$\mathcal{P}\mathrm{ol}(G) := \Gamma(\mathcal{S}G^*) \simeq \mathcal{S}_\mathcal{O}\Gamma(G^*) = \mathcal{S}_\mathcal{O}\,\mathrm{Hom}_\mathcal{O}(\Gamma(G), \mathcal{O})\ ,$$

the polynomial functions considered here are defined by

$$\mathcal{P}\mathrm{ol}(\bar{J}^\infty(R_\bullet)) := \mathcal{S}_\mathcal{F}\,\mathrm{Hom}_\mathcal{F}(\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet), \mathcal{F}) \simeq \mathcal{S}_\mathcal{F}\,\mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet, \mathcal{F})\ .$$

Hence, the KT chains of [Ver02] and those defined above do coincide. Moreover, the KT differential is defined in [Ver02] as an odd evolutionary vector field δ of $\bar{J}^\infty(R_\bullet)$. Such a graded derivation, when restricted as here to $\mathcal{P}\mathrm{ol}(\bar{J}^\infty(R_\bullet))$, is completely defined by its values on the polynomial functions that are linear along the fibers, i.e., on $\mathrm{Hom}_\mathcal{F}(\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet), \mathcal{F}) \simeq \mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet, \mathcal{F})$ – and by its values on $\mathcal{F}$. But on $\nabla_i \in \mathcal{C}\mathrm{Diff}(\mathcal{R}_i, \mathcal{F})$ (resp., $F \in \mathcal{F}$), this evolutionary field is given by $\delta(\nabla_i) = \nabla_i \circ \Delta_{i-1}$, if $i \geq 2$, and by $\delta(\nabla_1) = \nabla_1(\psi_D)$ (resp., $\delta(F) = 0$) [Ver02, Proposition 5.]. Hence, the odd derivations δ and δ_{KT} coincide, the KT complexes $(\mathcal{P}\mathrm{ol}(\bar{J}^\infty(R_\bullet)), \delta)$ and $(\mathsf{KT}, \delta_{\mathsf{KT}})$ coincide, and so do their homologies.

6.3 KTR induced by a compatibility complex versus classical KTR

We compare the coordinate KT complex $(\mathrm{KT}, \delta_{\mathrm{KT}})$ for Euler-Lagrange equations in a regular and on-shell irreducible gauge theory (Section 5) with the coordinate KT complex $(\mathsf{KT}, \delta_{\mathsf{KT}})$ for a not necessarily linear PDE subject to regularity and off-shell reducibility conditions (Section 6).

First, we focus on the KT chains. Since

$$\mathcal{C} : \mathcal{F} \otimes_\mathcal{O} \mathrm{Diff}(\Gamma(\rho_\bullet), \mathcal{O}) \to \mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet, \mathcal{F})$$

is an $\mathcal{F}$-module isomorphism (Equation (100)), we get

$$\mathsf{KT} \simeq \mathcal{S}_\mathcal{F}\,(\mathcal{F} \otimes_\mathcal{O} \mathrm{Diff}(\Gamma(\rho_\bullet), \mathcal{O})) \simeq \mathcal{F} \otimes_\mathcal{O} \mathcal{S}_\mathcal{O}\,\mathrm{Diff}(\Gamma(\rho_\bullet), \mathcal{O})\ .$$

Since we actually work in fixed coordinates, a linear differential operator D from sections of a graded vector bundle $\rho_\bullet : \bigoplus_{i=1}^{k-1} F_i \to X$ to functions of X reads

$$\mathsf{D} = \sum_\alpha (D^1_\alpha(x) \ldots D^{\sum_j r_j}_\alpha(x))\partial^\alpha_x\ , \tag{30}$$

i.e., is nothing but an $(r_1 + \ldots + r_{k-1})$-tuple of operators in $\mathcal{D}$, or, still, an element of the non-negatively graded free $\mathcal{D}$-module

$$V := \bigoplus_{j=1}^{k-1} \bigoplus_{\lambda=1}^{r_j} \mathcal{D} \cdot v^\lambda(j)$$

over formal generators $v^\lambda(j)$ of degree j. Hence, we finally obtain the $\mathcal{F}$-module isomorphism

$$\mathsf{KT} \simeq \mathcal{F} \otimes_\mathcal{O} \mathcal{S}_\mathcal{O} V\ . \tag{31}$$

Whereas the complex KT contains antifields ϕ^* and C^* – with components ϕ^*_a and C^*_δ that correspond to the considered equations and the irreducible relations between them – , the complex KT must contain antifields $v(1), v(2), v(3)$, ... whose components correspond to the equations ψ_D, relations $\Delta_1(-)$ between them, relations $\Delta_2(-)$ between relations, ... Hence, the KT-chains (31) are defined along the same lines than the KT-chains (12).

Also other aspects of the two approaches are analogous. Just as the antifields ϕ^* and C^* have been interpreted as sections of vector bundles $\pi^*_\infty F_i \to J^\infty(\pi)$, $i \in \{1,2\}$, the $v(j)$ will be viewed as sections of the vector bundles $\pi^*_\infty F_j \to J^\infty(\pi)$, $j \in \{1,\ldots,k-1\}$, i.e., of the bundles $R_j \to J^\infty(\pi)$. In other words, the formal parameters $v^\lambda(j)$ are seen as tuples $v^\lambda(j)(x^i, u^a_\alpha)$, where $\lambda \in \{1,\ldots,r_j\}$ and where (x^i, u^a_α) are the base variables. Further, the fundamental definitions (11) will be maintained in the present context:

$$\partial^\beta_x \cdot v^\lambda(j) = D^\beta_x v^\lambda(j) = v^\lambda_\beta(j)\ . \tag{32}$$

Just as derivatives of sections of a vector bundle over X can be interpreted as sections of the corresponding infinite jet bundle, the preceding total derivatives D^β_x of sections $v^\lambda(j)(x^i, u^a_\alpha)$ of the bundle $R_j \to J^\infty(\pi)$, or, even, $R_\bullet \to J^\infty(\pi)$, can be viewed as sections $v^\lambda_\beta(j)(x^i, u^a_\alpha)$ of the horizontal infinite jet bundle $\bar{J}^\infty(R_\bullet)$, with fiber coordinates $v^\lambda_\beta(j)$ and base coordinates (x^i, u^a_α). Hence, the second equality in (32) provides the appropriate result in case the formal parameters $v^\lambda(j)$ are true sections.

Eventually, we previously introduced the lifts of differential operators ∂_{x^ℓ} acting on X-functions $f(x^i) \in \mathcal{O}(X)$ to horizontal differential operators

$$D_{x^\ell} = \partial_{x^\ell} + u^a_{\ell\alpha}\partial_{u^a_\alpha} \tag{33}$$

acting on $J^\infty(\pi)$-functions

$$F(x^i, u^a_\alpha) \in \mathcal{F}(\pi)\ .$$

Similarly, we lift horizontal differential operators D_{x^ℓ} to extended horizontal differential operators

$$\bar{D}_{x^\ell} = \partial_{x^\ell} + u^a_{\ell\alpha}\partial_{u^a_\alpha} + v^\lambda_{\ell\beta}(j)\partial_{v^\lambda_\beta(j)} \tag{34}$$

that act on $\bar{J}^\infty(R_\bullet)$-functions

$$\mathfrak{F}(x^i, u^a_\alpha, v^\lambda_\beta(j)) \in \mathcal{F}(\bar{J}^\infty(R_\bullet))\ .$$

Therefore, the second equality (32) is in accordance with the extended interpretation $\bar{D}^\beta_x$ of D^β_x.

We still have to compare the KT differentials δ_{KT} and δ_{KT}. As mentioned above, the differential δ_{KT} is completely defined by its values on

$$\mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet), \mathcal{F}) \simeq \mathcal{P}\mathrm{ol}^1(\bar{J}^\infty(R_\bullet))$$

and its values on $\mathcal{F}$. Here superscript 1 means of course functions that are linear in the fiber coordinates $v^\lambda_\beta(j)$. In the considered fixed coordinates, such a differential operator ∇, its representative morphism $\bar{\psi}_\nabla$ and the corresponding linear jet bundle function read (with obvious notation)

$$\nabla v = \sum_\beta (\ldots \nabla^\lambda_\beta(j)(x^i, u^a_\alpha)\ldots) D^\beta_x \begin{pmatrix} \vdots \\ v^\lambda(j)(x^i, u^a_\alpha) \\ \vdots \end{pmatrix} \simeq$$

$$\bar{\psi}_\nabla\, \bar{v} = \sum_\beta (\ldots \nabla^\lambda_\beta(j)(x^i, u^a_\alpha)\ldots) \begin{pmatrix} \vdots \\ v^\lambda_\beta(j)(x^i, u^a_\alpha) \\ \vdots \end{pmatrix} \simeq$$

$$\mathfrak{F}_\nabla(x^i, u^a_\alpha, v^\lambda_\beta(j)) = \sum_\beta (\ldots \nabla^\lambda_\beta(j)(x^i, u^a_\alpha)\ldots) \begin{pmatrix} \vdots \\ v^\lambda_\beta(j) \\ \vdots \end{pmatrix} . \tag{35}$$

Since δ_{KT} vanishes on $\mathcal{F}$, it is completely defined by its values on the $v^\lambda_\beta(j)$, exactly as δ_{KT} is fully defined by its values on the $\phi^{\alpha*}_a$ and the $C^{\beta*}_\delta$. Note still, before proceeding, that the identifications for horizontal linear differential operators $\mathcal{C}\mathrm{Diff}(\mathcal{R}_j, \mathcal{R}_{j+1})$ valued in a not necessarily rank 1 bundle, are exactly the same, except that the row of coefficients $\nabla^\lambda_\beta(j)$ is replaced by a matrix of coefficients $\nabla^{\mu\lambda}_\beta(j+1, j)$.

In view of these definitions and identifications, we have

$$\delta_{\mathsf{KT}}(v^\lambda_\beta(1)) = \delta_{\mathsf{KT}}(D^\beta_x v^\lambda(1)) = D^\beta_x(\psi^\lambda_D) \tag{36}$$

– which is entirely similar to the definition

$$\delta_{\mathrm{KT}}(\phi^{\alpha*}_a) = D^\alpha_x(\delta_{u^a}\mathcal{L}) .$$

For $j \in \{2, \ldots, k-1\}$, we find

$$\delta_{\mathsf{KT}}(v^\lambda_\beta(j)) = \delta_{\mathsf{KT}}(\, D^\beta_x v^\lambda(j)) = D^\beta_x \left((\Delta_{j-1}\, v(j-1))^\lambda(j) \right) =$$

$$D^\beta_x \left((\Delta^{\lambda\mu}_\gamma(j, j-1))(x^i, u^a_\alpha)\;\; D^\gamma_x\, v^\mu(j-1) \right) = D^\beta_x \left((\Delta^{\lambda\mu}_\gamma(j, j-1))(x^i, u^a_\alpha)\;\; v^\mu_\gamma(j-1) \right) ,$$

in view of the above remark on matrix coefficients. When interpreting the $v^\mu_\gamma(j-1)$ as purely algebraic fiber coordinates of the horizontal jet bundle, rather than as sections of the latter, we must write

$$\delta_{\mathsf{KT}}(v^\lambda_\beta(j)) = \bar{D}^\beta_x \left((\Delta^{\lambda\mu}_\gamma(j, j-1))(x^i, u^a_\alpha)\;\; v^\mu_\gamma(j-1) \right) = \bar{D}^\beta_x \left(\mathfrak{F}^\lambda_{\Delta_{j-1}} \right) . \tag{37}$$

For $j = 2$, we thus find

$$\delta_{\mathsf{KT}}(v^\lambda_\beta(2)) = \bar{D}^\beta_x \left(\, \Delta^{\lambda\mu}_\gamma(2,1)\;\; \bar{D}^\gamma_x\, v^\mu(1) \right) , \tag{38}$$

where we omitted the variables (x^i, u^a_α) – which is fully analogous to the definition

$$\delta_{\mathrm{KT}}(C^{\beta *}_\delta) = \bar{D}^\beta_x \left(R^\mu_{\delta\gamma} \bar{D}^\gamma_x \phi^*_\mu \right) \, .$$

We conclude with the observation that the KT differential

$$\delta_{\mathsf{KT}} = \sum_{\beta\lambda} \bar{D}^\beta_x \left(\psi^\lambda_D \right) \, \partial_{v^\lambda_\beta(1)} + \sum_{j=2}^{k-1} \sum_{\beta\lambda} \bar{D}^\beta_x \left(\mathfrak{F}^\lambda_{\Delta_{j-1}} \right) \partial_{v^\lambda_\beta(j)} \tag{39}$$

is the evolutionary vector field, or symmetry of the Cartan distribution, that is obtained as the prolongation $\delta_{\mathcal{X}}$ to the horizontal jet bundle $\bar{J}^\infty(R_\bullet) \to J^\infty(\pi)$ of the vertical vector field

$$\mathcal{X} = \sum_\lambda \psi^\lambda_D \, \partial_{v^\lambda(1)} + \sum_{j=2}^{k-1} \sum_\lambda \mathfrak{F}^\lambda_{\Delta_{j-1}} \, \partial_{v^\lambda(j)}$$

of the bundle $R_\bullet \to J^\infty(\pi)$ with coefficients in

$$\mathcal{P}\mathrm{ol}^1(\bar{J}^\infty(R_\bullet)) \subset \mathcal{F}(\bar{J}^\infty(R_\bullet)) \, ,$$

see Equation (117).

6.4 KTR induced by a compatibility complex viewed as $\mathcal{D}$-geometric KTR

Just as in Section 5, the canonical map $\phi : \mathcal{F} \ni F \mapsto [F] \in C^\infty(\Sigma)$ is a $\mathcal{D}$-algebra and even an $\mathcal{F}[\mathcal{D}]$-algebra map.

In the proof that the above Koszul-Tate resolution $(\mathsf{KT}, \delta_{\mathsf{KT}})$ is a $\mathcal{D}$-geometric Koszul-Tate resolution of ϕ, in the sense of Definition 1, one of the difficulties will be to switch between the different viewpoints we used so far:

$$\mathcal{P}\mathrm{ol}^1(\bar{J}^\infty(R_\bullet)) :\simeq \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet), \mathcal{F}) \simeq \mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet, \mathcal{F}) \simeq \mathcal{F} \otimes_{\mathcal{O}} \mathrm{Diff}(\Gamma(\rho_\bullet), \mathcal{O}) \simeq \mathcal{F} \otimes_{\mathcal{O}} V \, , \tag{40}$$

with

$$V = \bigoplus_{j=1}^{k-1} \bigoplus_{\lambda=1}^{r_j} \mathcal{D} \cdot v^\lambda(j) \, .$$

If we set $(x, \mathfrak{u}) = (x^i, u^a_\alpha)$, these isomorphisms of $\mathcal{F}$-modules read, in the considered coordinate context,

$$\nabla^\lambda_\beta(j)(x,\mathfrak{u}) \, v^\lambda_\beta(j) \simeq \nabla^\lambda_\beta(j)(x,\mathfrak{u}) \, v^\lambda_\beta(j)(x,\mathfrak{u}) \simeq \nabla^\lambda_\beta(j)(x,\mathfrak{u}) \, D^\beta_x \left(v^\lambda(j)(x,\mathfrak{u}) \right) \simeq$$

$$\nabla^\lambda_\beta(j)(x,\mathfrak{u}) \, \partial^\beta_x \left(v^\lambda(j)(x) \right) \simeq \nabla^\lambda_\beta(j)(x,\mathfrak{u}) \, \partial^\beta_x \cdot v^\lambda(j) \, , \tag{41}$$

where the $v^\lambda_\beta(j)$ are the polynomial variables, the components of the argument in $\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet)$, in $\mathcal{R}_\bullet$, and in $\Gamma(\rho_\bullet)$, as well as the formal parameters of V, respectively (except in the last case, they are just arguments). In fact, all these $\mathcal{F}$-modules are $\mathcal{F}[\mathcal{D}]$-modules, i.e., are endowed with a compatible $\mathcal{D}$-action (in the sense that vector fields $\theta \in \mathcal{D}$ act as derivations on the $\mathcal{F}$-action). Note first that the $\mathcal{F}$-module isomorphism $\mathcal{F}[\mathcal{D}] \simeq \mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$ acts between two associative unital $\mathbb{R}$-algebras (the statement is obvious for the RHS and comes from Equations (153) and (154) for the LHS), and respects units and multiplications. Hence, $\mathcal{F}[\mathcal{D}]$-modules are the same than $\mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$-modules. We already mentioned that this provides a canonical $\mathcal{F}[\mathcal{D}]$-module structure on $\mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet,\mathcal{F})$, given by left composition. Since $\mathcal{F} \otimes_{\mathcal{O}} \mathrm{Diff}(\Gamma(\rho_\bullet),\mathcal{O})$ and $\mathcal{F} \otimes_{\mathcal{O}} V$ are tensor products of $\mathcal{D}$-modules, they are $\mathcal{D}$-modules, and, as the $\mathcal{F}$- and $\mathcal{D}$-actions are compatible, they are $\mathcal{F}[\mathcal{D}]$-modules. Finally, using the $\mathcal{F}$-module isomorphism

$$\psi : \mathcal{C}\mathrm{Diff}(\mathcal{R}_\bullet,\mathcal{F}) \ni \Delta \mapsto \psi_\Delta \in \mathrm{Hom}_{\mathcal{F}}(\bar{\mathcal{J}}^\infty(\mathcal{R}_\bullet),\mathcal{F}) \simeq: \mathcal{P}\mathrm{ol}^1(\bar{J}^\infty(R_\bullet)) \ ,$$

we can push the $\mathcal{F}[\mathcal{D}]$-structure on the source forward to the target, thus making ψ an $\mathcal{F}[\mathcal{D}]$-module isomorphism. It is easily seen that the last two $\mathcal{F}$-module isomorphisms in (40) are also isomorphisms of $\mathcal{F}[\mathcal{D}]$-modules. For the next to last isomorphism $\mathcal{C} : F \otimes \Delta \mapsto F\ \mathcal{C}\Delta$, consider any linear differential operator in $\mathcal{F}[\mathcal{D}] \simeq \mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$, e.g., to simplify, the operator $\Psi = G \otimes \theta \circ \theta' \simeq G\ \mathcal{C}\theta \circ \mathcal{C}\theta'$, and verify that $\Psi \cdot \mathcal{C}(F \otimes \Delta) = \mathcal{C}(\Psi \cdot (F \otimes \Delta))$. The last isomorphism is confined to the coordinate setting and is straightforwardly checked. More generally, when writing out the coordinate version of the actions of an operator $\Psi = \partial_x^\alpha \simeq D_x^\alpha$ on the isomorphic module elements of Equation (41), we find

$$\begin{aligned} D_x^{\alpha_1}\nabla\ \bar{D}_x^{\alpha_2} v_\beta \simeq D_x^{\alpha_1}\nabla\ v_{\alpha_2+\beta}(x,\mathfrak{u}) \simeq D_x^{\alpha_1}\nabla\ D_x^{\alpha_2+\beta}\,(v(x,\mathfrak{u})) \simeq \\ D_x^{\alpha_1}\nabla\ \partial_x^{\alpha_2+\beta}\,(v(x)) \simeq D_x^{\alpha_1}\nabla\ \partial_x^{\alpha_2+\beta} \cdot v \ , \end{aligned} \tag{42}$$

respectively, where we omitted all not absolutely necessary indices and where we simply wrote formulas of the type $\partial_x^{\alpha_1} f\ \partial_x^{\alpha_2} g$ instead of the full binomial formula.

Above we introduced the KT resolution obtained from a compatibility complex in terms of horizontal differential operators and expressed it later mainly in the polynomial language. To compare this resolution with our $\mathcal{D}$-geometric definition, we have to use the formal parameter approach

$$\mathsf{KT} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V$$

that we already put forth in Equation (31). In other words, we will apply the identifications (41) and (42). As mentioned above, the Koszul-Tate differential δ_{KT} is fully defined by its values on the polynomial variables $v_\beta^\lambda(j)$, i.e., on the elements $\partial_x^\beta \cdot v^\lambda(j)$ of the free $\mathcal{D}$-module V. For $j = 1$, we get from (36) and the definition of the $\mathcal{D}$-action of $\mathcal{F}$ that

$$\delta_{\mathsf{KT}}(\partial_x^\beta \cdot v^\lambda(1)) = D_x^\beta(\psi_D^\lambda) = \partial_x^\beta \cdot \psi_D^\lambda = \partial_x^\beta \cdot \delta_{\mathsf{KT}}(v^\lambda(1)) \ .$$

In the case $j \in \{2, \ldots, k-1\}$, one obtains

$$\delta_{\mathsf{KT}}(\partial_x^\beta \cdot v^\lambda(j)) = \bar{D}_x^\beta \left(\mathfrak{F}_{\Delta_{j-1}}^\lambda\right) \ .$$

However, the polynomial $\mathfrak{F}_{\Delta_{j-1}}^\lambda$ is of the form ∇v_γ and Equation (42) shows that

$$\partial_x^\beta \cdot (\nabla v_\gamma) = D_x^{\beta_1}\nabla\ \bar{D}_x^{\beta_2} v_\gamma = \bar{D}_x^\beta(\nabla v_\gamma) \ .$$

Finally,

$$\delta_{\mathsf{KT}}(\partial_x^\beta \cdot v^\lambda(j)) = \partial_x^\beta \cdot \mathfrak{F}^\lambda_{\Delta_{j-1}} = \partial_x^\beta \cdot \delta_{\mathsf{KT}}(v^\lambda(j)) \;.$$

More generally, we have

$$\delta_{\mathsf{KT}}(D \cdot v^\lambda(j)) = D \cdot \delta_{\mathsf{KT}}(v^\lambda(j)) \;, \tag{43}$$

for any $D \in \mathcal{D}$, any $j \in \{1, \ldots, k-1\}$, and any $\lambda \in \{1, \ldots, r_j\}$.

Since any element c of the graded $\mathcal{D}$-algebra KT reads non-uniquely as a finite sum

$$c = \sum F\,(D_{\lambda_1}(1) \cdot v^{\lambda_1}(1)) \ldots (D_{\lambda_{k-1}}(k-1) \cdot v^{\lambda_{k-1}}(k-1)) \;,$$

where $F \in \mathcal{F}$ and $D_{\lambda_j}(j) \in \mathcal{D}$, and since the Koszul-Tate differential δ_{KT}, which is well-defined on KT, acts as a graded derivation, *it can be completely computed from its above values on the* $v^{\lambda_j}(j)$.

The following stepwise construction of the differential graded $\mathcal{F}[\mathcal{D}]$-algebra $(\mathsf{KT}, \delta_{\mathsf{KT}})$ is along the lines of the similar construction of $(\mathrm{KT}, \delta_{\mathrm{KT}})$, see Subsections 5.4 and 7.3.1. We will mainly insist on differences and new aspects.

Let $V_1 := \bigoplus_{\lambda=1}^{r_1} \mathcal{D} \cdot v^\lambda(1)$. To endow the graded $\mathcal{D}$-algebra

$$\mathcal{C}_1 := \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_1 \tag{44}$$

with a differential graded $\mathcal{D}$-algebra structure d, we set

$$dv^\lambda(1) := \psi^\lambda_D \in \mathcal{F} \;, \tag{45}$$

extend d to V_1 by $\mathcal{D}$-linearity, and equip $\mathcal{C}_1$ with the differential d given by Equation (148). Then the natural $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $\imath_1 : (\mathcal{F}, 0) \ni F \mapsto F \otimes 1_{\mathcal{O}} \in (\mathcal{C}_1, d)$ is a $\mathrm{RS}\mathcal{D}\mathrm{A}$. It is easily seen that δ_{KT} coincides on $\mathcal{C}_1$ with d, so that the $\mathrm{RS}\mathcal{D}\mathrm{A}$ is actually a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\imath_1 : (\mathcal{F}, 0) \ni F \mapsto F \otimes 1_{\mathcal{O}} \in (\mathcal{C}_1, \delta_{\mathsf{KT}}) \;. \tag{46}$$

Consider now the $\mathcal{D}\mathtt{A}$-morphism $\phi : \mathcal{F} \to C^\infty(\Sigma)$. To define a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_1 : \mathcal{C}_1 \to C^\infty(\Sigma) \;, \tag{47}$$

it suffices to set

$$q_1(v^\lambda(1)) = 0 \in (C^\infty(\Sigma))_1 \cap 0^{-1}(\phi(dv^\lambda(1))) \;, \tag{48}$$

to extend q_1 by $\mathcal{D}$-linearity to V_1, and to define q_1 in degree 0 by $q_1(F) = [F]$ and in degree ≥ 1 by $q_1 = 0$. As for Condition (48), note that $\phi(dv^\lambda(1)) = [\psi^\lambda_D] = 0$, in view of the definition of Σ.

An anew application of the same lemma, with $(\mathcal{F}, 0)$ (resp., V_1) replaced by $(\mathcal{C}_1, \delta_{\mathsf{KT}})$ (resp., $V_2 := \bigoplus_{\lambda=1}^{r_2} \mathcal{D} \cdot v^\lambda(2)$), endows the graded $\mathcal{D}$-algebra

$$\mathcal{C}_2 := \mathcal{C}_1 \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_2 \tag{49}$$

with a DG$\mathcal{D}$A structure d that is again fully defined by

$$\mathrm{d}\, v^\lambda(2) := \mathfrak{F}^\lambda_{\Delta_1} = \Delta^{\lambda\mu}_\gamma(2,1)\ \bar{D}^\gamma_x\, v^\mu(1) = \Delta^{\lambda\mu}_\gamma(2,1)\ \partial^\gamma_x \cdot v^\mu(1) \in (\mathcal{C}_1)_1 \cap \delta^{-1}_{\mathsf{KT}}\{0\}\ ,$$

where we used the notation introduced in (38) and one of the identifications (42). Indeed, we have

$$\delta_{\mathsf{KT}}(\Delta^{\lambda\mu}_\gamma(2,1)\ \partial^\gamma_x \cdot v^\mu(1)) = \Delta^{\lambda\mu}_\gamma(2,1)\ \partial^\gamma_x \cdot \delta_{\mathsf{KT}}(v^\mu(1)) =$$
$$\Delta^{\lambda\mu}_\gamma(2,1)\ \partial^\gamma_x \cdot \psi^\mu_D = \Delta^{\lambda\mu}_\gamma(2,1)\, D^\gamma_x(\psi^\mu_D) = \Delta^\lambda_1(\psi_D) = 0\ .$$

We extend d to V_2 and $\mathcal{C}_2$ in the standard manner. As δ_{KT} is a graded derivation that is $\mathcal{D}$-linear and coincides with d on the generators $v^\lambda(2)$, we get $\mathrm{d} = \delta_{\mathsf{KT}}$ on $\mathcal{C}_2$. Hence, the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\imath_2 : (\mathcal{C}_1, \delta_{\mathsf{KT}}) \ni c \mapsto c \otimes 1_{\mathcal{O}} \in (\mathcal{C}_2, \delta_{\mathsf{KT}}) \tag{50}$$

is a relative Sullivan $\mathcal{D}$-algebra.

We then define a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_2 : \mathcal{C}_2 \to C^\infty(\Sigma) \tag{51}$$

by setting

$$q_2(v^\lambda(2)) = 0 \in (C^\infty(\Sigma))_2 \cap 0^{-1}(q_1(\delta_{\mathsf{KT}}\, v^\lambda(2)))\ ,$$

extending q_2 by $\mathcal{D}$-linearity to V_2 and by defining q_2 in degree 0 by $q_2(F) = [F]$ and in degree ≥ 1 by $q_2 = 0$.

The next application of Lemma 1 in Subsection 7.3.1 starts from the DG$\mathcal{D}$A $(\mathcal{C}_2, \delta_{\mathsf{KT}})$ and the free non-negatively graded $\mathcal{D}$-module $V_3 := \bigoplus_{\lambda=1}^{r_3} \mathcal{D} \cdot v^\lambda(3)$. To equip the G$\mathcal{D}$A

$$\mathcal{C}_3 := \mathcal{C}_2 \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}} V_3 \tag{52}$$

with a DG$\mathcal{D}$A structure $\mathfrak{d}$, we set

$$\mathfrak{d} v^\lambda(3) := \mathfrak{F}^\lambda_{\Delta_2} = \Delta^{\lambda\mu}_\gamma(3,2)\ \bar{D}^\gamma_x\, v^\mu(2) = \Delta^{\lambda\mu}_\gamma(3,2)\ \partial^\gamma_x \cdot v^\mu(2) \in (\mathcal{C}_2)_2 \cap \delta^{-1}_{\mathsf{KT}}\{0\}\ .$$

Indeed,

$$\delta_{\mathsf{KT}}(\Delta^{\lambda\mu}_\gamma(3,2)\ \partial^\gamma_x \cdot v^\mu(2)) = \Delta^{\lambda\mu}_\gamma(3,2)\ \partial^\gamma_x \cdot \delta_{\mathsf{KT}}(v^\mu(2)) =$$
$$\Delta^{\lambda\mu}_\gamma(3,2)\ \partial^\gamma_x \cdot (\Delta^{\mu\nu}_\varepsilon(2,1)\ \partial^\varepsilon_x \cdot v^\nu(1)) \simeq \Delta^{\lambda\mu}_\gamma(3,2)\ \partial^\gamma_x \cdot (\Delta^{\mu\nu}_\varepsilon(2,1)\ D^\varepsilon_x\,(v^\nu(1)(x,\mathfrak{u}))) =$$
$$\Delta^{\lambda\mu}_\gamma(3,2)\ D^\gamma_x\,(\Delta^{\mu\nu}_\varepsilon(2,1)\ D^\varepsilon_x\,(v^\nu(1)(x,\mathfrak{u}))) = \Delta^\lambda_2\,(\Delta_1\, v(1)) = 0\ .$$

It is easily checked that $\mathfrak{d} = \delta_{\mathsf{KT}}$ on $\mathcal{C}_3$: the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\imath_3 : (\mathcal{C}_2, \delta_{\mathsf{KT}}) \ni c \mapsto c \otimes 1_{\mathcal{O}} \in (\mathcal{C}_3, \delta_{\mathsf{KT}}) \tag{53}$$

is a relative Sullivan $\mathcal{D}$-algebra.

Finally, we define a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_3 : \mathcal{C}_3 \to C^\infty(\Sigma)\ , \tag{54}$$

again by setting

$$q_3(v^\lambda(3)) = 0 \in (C^\infty(\Sigma))_3 \cap 0^{-1}(q_2(\delta_{\mathsf{KT}}\, v^\lambda(3)))$$

and defining q_3 in degree 0 by $q_3(F) = [F]$ and in degree ≥ 1 by $q_3 = 0$.

Similarly, we define iteratively, for any $j \in \{4, \ldots, k-1\}$, a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\imath_j : (\mathcal{C}_{j-1}, \delta_{\mathsf{KT}}) \to (\mathcal{C}_j, \delta_{\mathsf{KT}})$$

that is a relative Sullivan $\mathcal{D}$-algebra, using the generators $v^\lambda(j)$ and the compatibility relation $\Delta_{j-1} \circ \Delta_{j-2} = 0$, as well as a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$q_j : \mathcal{C}_j \to C^\infty(\Sigma) ,$$

which vanishes, except in degree 0, where it sends F to $[F]$.

Since $V = V_1 \oplus \ldots \oplus V_{k-1}$, the graded $\mathcal{D}$-algebras $\mathcal{S}_{\mathcal{O}}V = \mathcal{S}_{\mathcal{O}}(V_1 \oplus \ldots \oplus V_{k-1})$ and $\mathcal{S}_{\mathcal{O}}V_1 \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}V_{k-1}$ are isomorphic. Hence, the same holds for the graded $\mathcal{D}$-algebras

$$\mathsf{KT} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}V \quad \text{and} \quad \mathcal{C}_{k-1} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}V_1 \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}V_{k-1} .$$

It follows that $\imath_{k-1} \circ \ldots \circ \imath_1 : (\mathcal{F}, 0) \to (\mathsf{KT}, \delta_{\mathsf{KT}})$ is a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism and thus allows to endow $(\mathsf{KT}, \delta_{\mathsf{KT}})$ with a $\mathrm{DG}\mathcal{F}[\mathcal{D}]\mathrm{A}$-structure – see Example 1 (the same as the one we obtained above).

Theorem 4. *The Koszul-Tate resolution* $(\mathsf{KT}, \delta_{\mathsf{KT}})$ *induced by a compatibility complex is a* $\mathcal{D}$*-geometric Koszul-Tate resolution of the* $\mathcal{D}$*-algebra map* $\phi : \mathcal{F} \to C^\infty(\Sigma)$*, in the sense of Definition 1 (in the smooth setting).*

Proof. See analogous proof in Subsection 5.4. □

7 Appendix

7.1 Non-linear partial differential equations in the jet bundle formalism

The goal of the present subsection is to construct from scratch a number of concepts that are of importance in the Geometry of PDEs. The text is written in the differential geometric setting of smooth vector bundles $\pi : E \to X$ over a smooth manifold, as well as, partially, in the corresponding algebraic context of modules $\mathcal{P}$ over a commutative unital associative $\mathbb{R}$-algebra $\mathcal{O}$. Of course, in case there exists an underlying geometric situation, we have $\mathcal{O} = C^\infty(X)$ and $\mathcal{P} = \Gamma(\pi)$. Additional details can be found, for instance, in [KV98].

7.1.1 Jets and differential operators

Consider a differential equation (DE)

$$\psi(t, \phi(t), d_t\phi, \ldots, d_t^k\phi) \equiv 0 , \tag{55}$$

with self-explaining notation. When defining the k-jet of $\phi(t)$ by

$$j_t^k\phi = (\phi(t), d_t\phi, \dots, d_t^k\phi) \ ,$$

we may rewrite this DE as

$$\psi(t, u, u_1, \dots, u_k)|_{j_t^k\phi} \equiv 0 \ .$$

Here $(t, u, u_1, \dots, u_k)$ are independent variables of the so-called k-jet space. Roughly speaking, the (purely) algebraic equation

$$\psi(t, u, u_1, \dots, u_k) = 0 \tag{56}$$

defines a hypersurface Σ^0 in the k-jet space (or, better, since t plays a distinguished role, a subbundle Σ^0 of the k-jet bundle), and a *solution* of the considered DE is nothing but a function $\phi(t)$ such that the graph of its k-jet is located on Σ^0 ('graph' means here the image of $j^k\phi$). This is one of the key-aspects of the jet bundle approach to partial differential equations (PDE-s) – which will be formalized in the following.

Let $\pi : E \to X$ be a smooth vector bundle of rank $\mathrm{rk}(\pi) = r$ over a smooth n-dimensional manifold. For $k \in \mathbb{N}$, the **k-jet** $j_m^k\phi$ at $m \in X$ of a local smooth section $\phi \in \Gamma(\pi)$ of π that is defined around m (the latter condition will be understood in the following), is the equivalence class of all local sections of π, such that in any trivializing chart $(x, u) = (x^i, u^a)$ of π around m, the local coordinates of these sections coincide at $x(m)$, together with their partial derivatives at $x(m)$ up to order k (it actually suffices that they coincide in one trivializing chart). We define the k-jet set $J^k(\pi)$ of π by

$$J^k(\pi) = \{j_m^k\phi : m \in X, \phi \in \Gamma(\pi)\} \ .$$

The k-jet set is a smooth finite rank vector bundle $\pi_k : J^k(\pi) \to X$ – the **k-jet bundle**. Indeed, any trivializing chart (x^i, u^a) of π induces a trivializing chart (x^i, u^a_α) of π_k, defined by

$$x^i(j_m^k\phi) = x^i(m) \quad \text{and} \quad u^a_\alpha(j_m^k\phi) = \partial_x^\alpha\phi^a|_{x(m)} \ ,$$

where $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq k$. For $k \leq \ell$, there is a 'truncation' vector bundle (epi)morphism $\pi_{k\ell} : J^\ell(\pi) \to J^k(\pi)$, so that $(J^k(\pi), \pi_{k\ell})$ is an inverse system. The limit of this diagram is the **∞-jet space** $\pi_\infty : J^\infty(\pi) \to X$ together with the natural projections $\pi_{k\infty} : J^\infty(\pi) \to J^k(\pi)$. Coordinates (x^i, u^a_α) of $J^\infty(\pi)$ can be obtained from coordinates (x^i, u^a) of π, as above, by defining an infinite number of coordinates u^a_α that correspond to the partial derivatives ∂_x^α of the components $\phi^a = u^a(\phi(x))$ of the sections ϕ of π. We denote the **algebra of smooth functions** of $J^k(\pi)$ by $\mathcal{F}_k = \mathcal{F}_k(\pi)$. The canonical epimorphisms $\pi_{k\ell}$ induce inclusions $\mathcal{F}_k \subset \mathcal{F}_\ell$. The colimit of this direct system is the algebra $\mathcal{F} = \bigcup_k \mathcal{F}_k$ (we will also write $\mathcal{F}(\pi)$, $\mathcal{F}_\infty$, or $\mathcal{F}_\infty(\pi)$) of smooth functions of $J^\infty(\pi)$. It follows that any smooth function of $J^\infty(\pi)$ is a smooth function of some $J^k(\pi)$. Note eventually that $j^k : \Gamma(\pi) \to \Gamma(\pi_k)$ and that $j^\infty : \Gamma(\pi) \to \Gamma(\pi_\infty)$ (in fact, we should, as above, consider the case $k = \infty$ separately, as a limit case; however, here and in the following, we refrain from presenting these details).

We will use jet bundles to define differential operators between sections of vector bundles. Let $\pi' : E' \to X$ be a second vector bundle and take the pullback bundle $\pi_k^*(\pi')$, $k \in \mathbb{N}$, see

$$\begin{tikzcd} \pi_k^* E' \arrow[r, "p"] \arrow[d, "\pi_k^*(\pi')"'] & E' \arrow[d, "\pi'"] \\ J^k(\pi) \arrow[r, "\pi_k"] & X \end{tikzcd}$$

Figure 1: Pullback bundle

Figure 1. Consider now the $\mathcal{F}_k(\pi)$-module of sections $\Gamma(\pi_k^*(\pi'))$. If $\pi' : X \times \mathbb{R} \to X$, the latter can be naturally identified with $\mathcal{F}_k(\pi)$. This justifies the notation $\mathcal{F}_k(\pi, \pi') := \Gamma(\pi_k^*(\pi'))$. We denote the composite of

$$\psi \in \mathcal{F}_k(\pi, \pi') \subset C^\infty(J^k(\pi), \pi_k^* E')$$

and $p \in C^\infty(\pi_k^* E', E')$ also by ψ. Hence, $\psi \in C^\infty(J^k(\pi), E')$, and, for any point $j^k_m \phi \in J^k(\pi)$, we have $\psi(j^k_m \phi) \in E'_m$, i.e., ψ is a fiber bundle morphism $\psi \in \mathtt{FB}(J^k(\pi), E')$. We thus get an isomorphism of $C^\infty(X)$-modules:

$$\Gamma(\pi_k^*(\pi')) = \mathcal{F}_k(\pi, \pi') \simeq \mathtt{FB}(J^k(\pi), E') \; . \tag{57}$$

Since, for every section $\phi \in \Gamma(\pi)$, the composite of

$$j^k \phi \in \Gamma(\pi_k) \subset C^\infty(X, J^k(\pi))$$

and ψ is a section $\psi \circ (j^k \phi) \in \Gamma(\pi')$, we see that $\psi \in \mathcal{F}_k(\pi, \pi')$ implements a map

$$D : \Gamma(\pi) \ni \phi \mapsto D(\phi) = \psi \circ (j^k \phi) \in \Gamma(\pi') \; ,$$

such that the value $D(\phi)|_m$ only depends on $j^k_m \phi$. We therefore say that D is a not necessarily linear differential operator of order k between π and π'.

Definition 2. *A (not necessarily linear)* **differential operator** $D \in \mathrm{DO}_k(\pi, \pi')$ *of order k from π to π' is a map $D : \Gamma(\pi) \to \Gamma(\pi')$ that factors through the k-jet bundle, i.e., that reads*

$$D = \psi_D \circ (j^k -) \; , \tag{58}$$

for some section or fiber bundle morphism $\psi_D \in \mathcal{F}_k(\pi, \pi') \simeq \mathtt{FB}(J^k(\pi), E')$. This morphism, which is visibly unique, is the **representative morphism** *of D.*

In trivializations of π and π' over the same chart (U, x) of X, such a k-th order differential operator reads

$$\psi^b_D(x, \partial^\alpha_x \phi^a) = \psi^b_D(x, u^a_\alpha)|_{j^k_x \phi}, \quad (a \in \{1, \ldots, \mathrm{rk}(\pi)\}, b \in \{1, \ldots, \mathrm{rk}(\pi')\}, |\alpha| \leq k) \; . \tag{59}$$

If both ranks are 1 and we write ψ (resp., t) instead of ψ_D (resp., $x = (x^1, \ldots, x^n)$), we recover

$$\psi(t, \phi(t), d_t \phi, \ldots, d^k_t \phi) = \psi(t, u, u_1, \ldots, u_k)|_{j^k_t \phi} \tag{60}$$

(see beginning of 7.1.1).

The composite of a differential operator $D \in \mathrm{DO}_k(\pi, \pi')$ and a differential operator $D' \in \mathrm{DO}_\ell(\pi', \pi'')$ is a differential operator $D' \circ D \in \mathrm{DO}_{k+\ell}(\pi, \pi'')$.

The set $\mathrm{DO}_k(\pi, \pi')$ is a $C^\infty(X)$-module. There is a canonical **$C^\infty(X)$-module isomorphism**

$$\mathrm{DO}_k(\pi, \pi') \simeq \mathcal{F}_k(\pi, \pi') \simeq \mathtt{FB}(J^k(\pi), E') \ . \tag{61}$$

The natural surjective morphisms $\pi_{k\ell}$, $k \leq \ell$, give rise to inclusions $\mathrm{DO}_k(\pi, \pi') \subset \mathrm{DO}_\ell(\pi, \pi')$, thus leading to an increasing sequence of $C^\infty(X)$-modules. The colimit is the filtered $C^\infty(X)$-module

$$\mathrm{DO}(\pi, \pi') = \bigcup_i \mathrm{DO}_i(\pi, \pi') \tag{62}$$

of all differential operators from π to π' .

If, for $r, r' \in \mathbb{R}$ and $\phi, \phi' \in \Gamma(\pi)$, we have

$$D(r\phi + r'\phi') = r\, D(\phi) + r'\, D(\phi') \ ,$$

the differential operator D is said to be linear. We denote the $C^\infty(X)$-submodule made of the linear differential operators of order k (resp., of all linear differential operators) from π to π' by

$$\mathrm{Diff}_k(\pi, \pi') \subset \mathrm{DO}_k(\pi, \pi') \quad (\text{resp.,} \quad \mathrm{Diff}(\pi, \pi') \subset \mathrm{DO}(\pi, \pi')) \ .$$

In trivializations of π and π' over the same chart (U, x) of X, a linear differential operator D of order k reads

$$\psi^b_D(x, \partial^\alpha_x \phi^a) = \psi^b_D(x, u^a_\alpha)|_{j^k_x \phi}, \quad (a \in \{1, \ldots, \mathrm{rk}(\pi)\}, b \in \{1, \ldots, \mathrm{rk}(\pi')\}, |\alpha| \leq k) \ , \tag{63}$$

where the ψ^b_D are $C^\infty(x(U))$-linear in the derivatives, i.e.,

$$\psi^b_D(x, \partial^\alpha_x \phi^a) = \sum_{|\alpha| \leq k} M^b_{\alpha a}(x) \partial^\alpha_x \phi^a \ .$$

In fact, a differential operator is a *linear* operator $D \in \mathrm{Diff}_k(\pi, \pi')$ if and only if its representative morphism is a *vector* bundle morphism $\psi_D \in \mathtt{VB}(J^k(\pi), E')$ (not only a fiber bundle morphism), i.e., a $C^\infty(X)$-linear map $\psi_D \in \mathrm{Hom}_{C^\infty(X)}(\Gamma(\pi_k), \Gamma(\pi'))$ (denoted by the same symbol). This passage from the vector bundle map to the linear map between sections allows to replace $D(-) = \psi_D \circ (j^k -)$, see (58), by $D(-) = (\psi_D \circ j^k)(-)$. Therefore,

Proposition 4. *A* **linear differential operator** $D \in \mathrm{Diff}_k(\pi, \pi')$ *is an $\mathbb{R}$-linear map $D : \Gamma(\pi) \to \Gamma(\pi')$ that factors through the k-jet bundle, i.e., that reads*

$$D = \psi_D \circ j^k \ , \tag{64}$$

for some (and thus unique) vector bundle or $C^\infty(X)$-module morphism $\psi_D \in \mathtt{VB}(J^k(\pi), E') \simeq \mathrm{Hom}_{C^\infty(X)}(\Gamma(\pi_k), \Gamma(\pi'))$. Hence the isomorphisms of $C^\infty(X)$-modules

$$\mathrm{Diff}_k(\pi, \pi') \simeq \mathtt{VB}(J^k(\pi), E') \simeq \mathrm{Hom}_{C^\infty(X)}(\Gamma(\pi_k), \Gamma(\pi')) \ , \tag{65}$$

and

$$\mathrm{Diff}(\pi,\pi') \simeq \mathtt{VB}(J^\infty(\pi),E') \simeq \mathrm{Hom}_{C^\infty(X)}(\Gamma(\pi_\infty),\Gamma(\pi'))\ . \tag{66}$$

We close the present section with the remark that, in the case $\pi=\pi'=\mathrm{pr}_1 : X\times\mathbb{R}\to X$, the differential operators $\mathrm{Diff}(\pi,\pi')$ act on functions $C^\infty(X)$, and that we then write $\mathcal{D}(X)$ instead of $\mathrm{Diff}(\mathrm{pr}_1,\mathrm{pr}_1)$; in other words:

Remark 5. *We denote by $\mathcal{D}(X)$ the associative unital $\mathbb{R}$-algebra of linear differential operators acting on functions $C^\infty(X)$ of a smooth manifold X.*

7.1.2 Partial differential equations and their prolongations

A second fundamental feature is that one prefers replacing the original system of PDE-s by an enlarged system, its prolongation, which also takes into account the differential consequences of the original one. More precisely, if $\phi(t)$ satisfies the original DE (55), we have, for any $\ell\in\mathbb{N}$,

$$d_t^r(\psi(t,\phi(t),d_t\phi,\ldots,d_t^k\phi)) = (\partial_t+u_1\partial_u+u_2\partial_{u_1}+\ldots)^r\psi(t,u,u_1,\ldots,u_k)|_{j_t^{k+\ell}\phi} =:$$
$$D_t^r\left(\psi(t,u,u_1,\ldots,u_k)\right)|_{j_t^{k+\ell}\phi}\equiv 0,\ \forall r\le\ell\ . \tag{67}$$

Let us stress that the 'total derivative' D_t or 'horizontal lift' D_t of d_t is actually an infinite sum. The DE (55) and the system of DE-s (67), have clearly the same solutions, so we may focus just as well on (67). The corresponding system of algebraic equations

$$(D_t^r\psi)(t,u,u_1,\ldots,u_k,u_{k+1},\ldots,u_{k+r}) = 0,\ \forall r\le\ell \tag{68}$$

defines a 'surface' Σ^ℓ in the $(k+\ell)$-jet space. *A solution of the original DE (55) is now a function ϕ such that the graph* $\mathrm{gr}(j^{k+\ell}\phi)$ *is a subset of* Σ^ℓ. The 'surface' Σ^ℓ is referred to as the ℓ-th prolongation of the considered DE or differential operator.

To grasp the interest in differential consequences, consider for instance the integration problem $\partial_{x^i}F=f_i$ ($i\in\{1,\ldots,n\}$) in $\mathbb{R}^n$ – where notation is obvious – . The differential consequences of this (system of) PDE(s) include the equations $\partial_{x^j}\partial_{x^i}F=\partial_{x^j}f_i$ ($i,j\in\{1,\ldots,n\}$), hence, they include the compatibility conditions $\partial_{x^j}f_i=\partial_{x^i}f_j$.

In the case $k=\ell=1$, the equation of $\Sigma^0\subset J^1$ (resp., of $\Sigma^1\subset J^2$) is

$$\psi(t,u,u_1)=0 \quad (\text{resp.,}\quad \psi(t,u,u_1)=0 \text{ and } (D_t\psi)(t,u,u_1,u_2)=0)\ ,$$

(see (68)). Hence, Σ^1 is the set of points $j^2_{t_0}\phi\in J^2$ such that $j^1_{t_0}\phi\in\Sigma^0$ and

$$(\partial_t\psi+u_1\partial_u\psi+u_2\partial_{u_1}\psi)|_{j^2_{t_0}\phi} = \partial_t\psi|_{j^1_{t_0}\phi}+d_t\phi|_{t_0}\partial_u\psi|_{j^1_{t_0}\phi}+d_t^2\phi|_{t_0}\partial_{u_1}\psi|_{j^1_{t_0}\phi}=0\ .$$

The last requirement means that the tangent vector $(1,d_t\phi|_{t_0},d_t^2\phi|_{t_0})$ at t_0 of the curve $(t,\phi(t),d_t\phi)\in J^1$ is an element of the vector space

$$T_{j^1_{t_0}\phi}\Sigma^0 : \partial_t\psi|_{j^1_{t_0}\phi}\, t+\partial_u\psi|_{j^1_{t_0}\phi}\, u+\partial_{u_1}\psi|_{j^1_{t_0}\phi}\, u_1=0$$

that is tangent to Σ^0 at $j^1_{t_0}\phi$. Thus,

$$\Sigma^1 = \{j^2_{t_0}\phi \in J^2 : \mathrm{gr}(j^1\phi) \text{ is tangent to } \Sigma^0 \text{ at } j^1_{t_0}\phi\} . \tag{69}$$

Observe that the equations of Σ^0 and Σ^1 show that Σ^ℓ is not necessarily a smooth manifold and that $\pi_{01} : \Sigma^1 \to \Sigma^0$ is not necessarily a smooth fiber bundle.

We now define partial differential equations and their prolongations in a coordinate-free manner.

Definition 3. *A* **partial differential equation** (*resp., a* **linear** *partial differential equation*) *of order* k ($k \geq 0$) *acting on sections* $\phi \in \Gamma(\pi)$ *of a vector bundle* π*, is a smooth fiber (resp., vector) subbundle* $\pi_k : \Sigma^0 \to X$ *of* $J^k(\pi)$*. The* ℓ**-th prolongation** *of* Σ^0 ($0 \leq \ell \leq \infty$) *is the subset*

$$\Sigma^\ell = \{j^{k+\ell}_m\phi \in J^{k+\ell}(\pi) : \mathrm{gr}(j^k\phi) \textit{ is tangent up to order } \ell \textit{ to } \Sigma^0 \textit{ at } j^k_m\phi\} \tag{70}$$

of $J^{k+\ell}(\pi)$. *A (local)* **solution** *of* Σ^0 *is a (local) section* ϕ *of* π *such that* $\mathrm{gr}(j^k\phi) \subset \Sigma^0$.

Note that the definition of the prolongation means that the points $j^{k+\ell}_m\phi$ of Σ^ℓ provide ℓ-th order approximations $\mathrm{gr}(j^k\phi)$ of possible solutions of Σ^0.

Remark 6. *1. In the following we always assume that the considered equation* $\Sigma^0 \subset J^k(\pi)$ *is* **formally integrable** *(see also Subsection 7.1.6), i.e., that*

- *the prolongations* Σ^ℓ *are smooth manifolds* ($0 \leq \ell \leq \infty$)*, and*
- *the maps* $\pi_{k+\ell,k+\ell+1} : \Sigma^{\ell+1} \to \Sigma^\ell$ ($0 \leq \ell < \infty$) *are smooth fiber bundles.*

2. Let us stress as well that it follows from Definition 3 (see also introduction to the present subsection 7.1.2) that ϕ *is a solution of* Σ^0 *if*

$$\mathrm{gr}(j^{k+\ell}\phi) \subset \Sigma^\ell , \tag{71}$$

for some $0 \leq \ell \leq \infty$, *and that, conversely, we have (71) for every* ℓ, *if* ϕ *is a solution.*

A PDE (resp., a linear PDE) Σ^0 of order k in π is **implemented by a differential operator** $D \in \mathrm{DO}_k(\pi, \pi')$ (resp., $D \in \mathrm{Diff}_k(\pi, \pi')$), if $\Sigma^0 = \ker \psi_D$, where $\pi' : E' \to X$ is a vector bundle and where $\psi_D \in \mathtt{FB}(J^k(\pi), E')$ (resp., $\psi_D \in \mathtt{VB}(J^k(\pi), E')$) is the representative morphism of D. In this case, the differential operator $j^\ell \circ D$ is of order $k + \ell$ and acts from π to π'_ℓ. Its decomposition

$$j^\ell \circ D = \psi_{j^\ell \circ D} \circ j^{k+\ell} \tag{72}$$

corresponds to Equation (67). In the sequel we write

$$\psi^\ell_D : J^{k+\ell}(\pi) \to J^\ell(\pi') \tag{73}$$

for the representative morphism $\psi_{j^\ell\circ D}$ of the ℓ-th prolongation $j^\ell \circ D$ of D. It is now clear that

$$\Sigma^\ell = \ker \psi^\ell_D , \tag{74}$$

i.e., that the ℓ-th prolongation of the equation is given by the ℓ-th prolongation of the corresponding differential operator (see Equation (68)).

7.1.3 Cartan distribution

An important aspect of $\pi_k : J^k(\pi) \to X$, $k \geq 0$, is that any of the points $\kappa_k \in J^k(\pi)$ is the value at $\pi_k(\kappa_k) = m$ of a section $j^k\phi \in \Gamma(\pi_k)$ that is implemented by a section $\phi \in \Gamma(\pi)$. This suggests the idea of a possible foliation and, at the infinitesimal level, of distribution. It is thus natural to consider the tangent spaces at κ_k to the n-dimensional manifolds $\mathrm{gr}(j^k\phi)$, $\phi \in \Gamma(\pi)$, that pass through κ_k, i.e., with $j^k_m\phi = \kappa_k$. Such a tangent space is obviously given by

$$\mathrm{im}(T_m j^k\phi) \subset T_{\kappa_k}(J^k(\pi))\,.$$

We now consider the vector subspace $\mathcal{C}^k_{\kappa_k}$ spanned by the preceding images, for all sections ϕ such that $j^k_m\phi = \kappa_k$, with $m = \pi_k(\kappa_k)$. The assignment

$$\mathcal{C}^k : J^k(\pi) \ni \kappa_k \mapsto \mathcal{C}^k_{\kappa_k} \subset T_{\kappa_k}(J^k(\pi)) \tag{75}$$

is the **Cartan distribution** $\mathcal{C}^k = \mathcal{C}^k(\pi)$ of $J^k(\pi)$. If we are in the presence of a PDE $\Sigma^0 \subset J^k(\pi)$ on π, we also define the Cartan distribution $\mathcal{C}^k(\Sigma^0)$ of Σ^0 by

$$\mathcal{C}^k(\Sigma^0) : \Sigma^0 \ni \kappa_k \mapsto \mathcal{C}^k_{\kappa_k} \cap T_{\kappa_k}\Sigma^0 \subset T_{\kappa_k}\Sigma^0\,. \tag{76}$$

In local coordinates (x^i, u^a_α) of $J^k(\pi)$, the parametrization $j^k\phi$ of $\mathrm{gr}(j^k\phi)$ reads $j^k\phi : x \mapsto (x^i, \partial^\alpha_x\phi^a)$, with $i \in \{1,\ldots,n\}, a \in \{1,\ldots,r\}, |\alpha| \leq k$. Hence, the derivative $T_m j^k\phi$ is given by

$$\begin{pmatrix} \mathbb{I}_n \\ (\partial_{x^i}\partial^\alpha_x\phi^a)_{a\alpha,i} \end{pmatrix}\,,$$

so that its image, expressed in the basis $(\partial_{x^i}, \partial_{u^a_\alpha})$ of $T_{\kappa_k}(J^k(\pi))$, is made of the linear combinations of the vectors

$$\partial_{x^i} + \sum_{a=1}^{r}\sum_{|\alpha|\leq k} \partial_{x^i}\partial^\alpha_x\phi^a\ \partial_{u^a_\alpha}, \quad i \in \{1,\ldots,n\} \tag{77}$$

(of course, the coefficients are evaluated at $x = x(m)$ and the base vectors are taken at κ_k). The space $\mathcal{C}^k_{\kappa_k}$ of the Cartan distribution of $J^k(\pi)$ is obtained similarly, except that ϕ runs through the sections that satisfy $j^k_m\phi = \kappa_k$.

For instance, in the case $k = n = r = 1$, the space $\mathcal{C}^1_{\kappa_1}$ is spanned by the vectors that are tangent at κ_1 to the curves $j^1\phi : t \mapsto (t, \phi(t), d_t\phi) \in J^1$, with $j^1_{t_1}\phi = \kappa_1$ (we set $t_1 := \pi_1(\kappa_1)$), i.e., by the vectors

$$(1, d_t\phi|_{t_1}, d^2_t\phi|_{t_1}) \simeq \partial_t|_{\kappa_1} + d_t\phi|_{t_1}\,\partial_u|_{\kappa_1} + d^2_t\phi|_{t_1}\,\partial_{u_1}|_{\kappa_1} =$$

$$(\partial_t + u_1\partial_u + u_2\partial_{u_1})\,|_{(\kappa_1, d^2_t\phi|_{t_1})} = D^{\leq 1}_t|_{(\kappa_1, d^2_t\phi|_{t_1})} = D^{\leq 0}_t|_{\kappa_1} + d^2_t\phi|_{t_1}\partial_{u_1}|_{\kappa_1}\,, \tag{78}$$

or, still, by the vectors

$$D^{\leq 0}_t|_{\kappa_1} \quad \text{and} \quad \partial_{u_1}|_{\kappa_1}\,, \tag{79}$$

since, if ϕ varies, the value $d^2_t\phi|_{t_1}$ runs through $\mathbb{R}$. More generally, we have the

Proposition 5. *Let $\pi : E \to X$ be a vector bundle of rank r over a manifold of dimension n. For any $k \geq 0$ and any $\kappa_k \in J^k(\pi)$, the Cartan space $\mathcal{C}^k_{\kappa_k} = \mathcal{C}^k_{\pi_k}(\pi)$ is generated by the vectors*

$$D^{\leq k-1}_{x^i}|_{\kappa_k} = \partial_{x^i} + \sum_{a=1}^{r} \sum_{|\alpha| \leq k-1} u^a_{i\alpha} \partial_{u^a_\alpha} |_{\kappa_k} \quad \text{and} \quad \partial_{u^a_\alpha}|_{\kappa_k} \ ,$$

$$i \in \{1, \ldots, n\}, a \in \{1, \ldots, r\}, |\alpha| = k \ , \tag{80}$$

where (x^i, u^a_α) is a trivializing chart of $J^k(\pi)$ around $\pi_k(\kappa_k)$. In the limit case $k = \infty$, the Cartan space $\mathcal{C}^\infty_{\kappa_\infty}$ is generated by the total derivatives

$$D_{x^i}|_{\kappa_\infty} \ , \quad i \in \{1, \ldots, n\} \ . \tag{81}$$

Let $\kappa_k \in J^k(\pi)$, $k \geq 1$, and set $\pi_k(\kappa_k) = m$ and $\pi_{k-1,k}(\kappa_k) = \kappa_{k-1}$. In view of (77), the vectors $D^{\leq k-1}_{x^i}|_{\kappa_k}$ span the tangent space $\operatorname{im}(T_m j^{k-1}\phi)$ at κ_{k-1} to the graph $\operatorname{gr}(j^{k-1}\phi)$ of the section $j^{k-1}\phi$ such that $j^k_m\phi = \kappa_k$. Observe that this n-dimensional subspace of $T_{\kappa_{k-1}}J^{k-1}(\pi)$ is completely defined by $j^k_m\phi = \kappa_k$ and does not depend on the considered section ϕ (see also Equation (78)): we denote it by $R^k_{\kappa_k}$ and refer to it as the R**-space** at κ_{k-1} defined by κ_k. Equations (78) and (80) allow to understand that the Cartan space $\mathcal{C}^k_{\kappa_k}$ and the R-space $R^k_{\kappa_k}$ are related by

$$(T_{\kappa_k}\pi_{k-1,k})^{-1}(R^k_{\kappa_k}) = \mathcal{C}^k_{\kappa_k} \ . \tag{82}$$

It is quite obvious that the difference (82) between the R-spaces and the Cartan spaces, i.e., the existence of the extra generators $\partial_{u^a_\alpha}$ ($a \in \{1, \ldots, r\}$, $|\alpha| = k$), makes *the Cartan distribution $\mathcal{C}^k = \mathcal{C}^k(\pi)$ non-integrable.* Indeed, take, to simplify, again the case $k = n = r = 1$. In view of (82), the bracket $[D^{\leq 0}_t, \partial_{u_1}] = [\partial_t + u_1\partial_u, \partial_{u_1}] = -\partial_u$ of local vector fields in $\mathcal{C}^1$ is not located in $\mathcal{C}^1$. We easily understand that this difference disappears at the limit $k = \infty$ and that *the Cartan distribution $\mathcal{C}^\infty = \mathcal{C}^\infty(\pi)$ is n-dimensional and integrable* (indeed $[D_{x^i}, D_{x^j}] = 0$).

Consider now a PDE $\Sigma^0 \subset J^k(\pi)$ of order k on π (as mentioned before, we systematically assume that the considered PDE-s are formally integrable).

Remark 7. *In the sequel, we deal with limits, e.g., infinite prolongations Σ^∞. To simplify notation, we omit the sub- and superscripts ∞, whenever no confusion arises, thus writing Σ (resp., $\kappa, \mathcal{C}, \ldots$) instead of Σ^∞ (resp., $\kappa_\infty, \mathcal{C}^\infty, \ldots$).*

The algebra of functions of the infinite prolongation $\Sigma \subset J^\infty(\pi)$ of Σ^0 is the quotient algebra $\mathcal{F}(\Sigma) = \mathcal{F}(\pi)/I(\Sigma)$, where $I(\Sigma)$ is the ideal of $\mathcal{F}(\pi)$ made of those functions of $J^\infty(\pi)$ that vanish on Σ. If Σ^0 is implemented by a differential operator $D \simeq \psi_D$ (what we assume), the prolongation Σ is locally given by equations $D^\alpha_x \psi^b_D = 0$, where $|\alpha| \geq 0$, $b \in \{1, \ldots, \operatorname{rk}(\pi')\}$, and $\psi^b_D \in \mathcal{F}_k(\pi)$ (see Equations (74) and (68)). Hence, the ideal $I(\Sigma)$ reads

$$I(\Sigma) = \left\{ \sum F_{\alpha b} \, D^\alpha_x \psi^b_D \right\} \ , \tag{83}$$

where the sum is finite and $F_{\alpha b} \in \mathcal{F}(\pi)$. Since $D_{x^i} I(\Sigma) \subset I(\Sigma)$, the total derivatives act on $\mathcal{F}(\Sigma)$ and their restrictions $D_{x^i}|_\Sigma$ are thus vector fields of Σ. It follows that, for any $\kappa \in \Sigma$, we have $D_{x^i}|_\kappa \in T_\kappa\Sigma$, so that

$$\mathcal{C}_\kappa = \mathcal{C}_\kappa(\pi) \subset T_\kappa\Sigma \ . \tag{84}$$

Just as we defined above the Cartan distribution of $\Sigma^0 \subset J^k(\pi)$, we define the Cartan distribution of $\Sigma \subset J^\infty(\pi)$ by

$$\mathcal{C}(\Sigma) : \Sigma \ni \kappa \mapsto \mathcal{C}_\kappa(\Sigma) = \mathcal{C}_\kappa(\pi) \cap T_\kappa \Sigma \ . \tag{85}$$

In view of (84), we get

$$\mathcal{C}(\Sigma) = \mathcal{C}(\pi)|_\Sigma \ . \tag{86}$$

Moreover, not only $\mathcal{C}(\pi)$, but also *the Cartan distribution* $\mathcal{C}(\Sigma) = \mathcal{C}(\pi)|_\Sigma$ *is n-dimensional and integrable.*

From the construction of the Cartan distribution and Remark 6, it is quite clear that:

Proposition 6. *The maximal dimensional (n-dimensional) integral manifolds of the Cartan distribution $\mathcal{C}(\pi)$ (resp., $\mathcal{C}(\Sigma)$) are the graphs $\operatorname{gr}(j^\infty\phi)$ of the infinite jets of the local sections $\phi \in \Gamma_{\text{loc}}(\pi)$ (resp., the local solutions $\phi \in \Gamma_{\text{loc}}(\pi)$ of Σ^0).*

Hence, the set of maximal dimensional integral manifolds in $(\Sigma, \mathcal{C}(\Sigma))$ can be identified with the set of solutions of Σ^0. Since all relevant information about the original PDE Σ^0 is thus encrypted in the pair $(\Sigma, \mathcal{C}(\Sigma))$, the partial differential equation Σ^0 is frequently identified with the 'diffiety' $(\Sigma, \mathcal{C}(\Sigma))$. **Diffieties**, i.e., 'manifolds equipped with a geometric structure' play a basic role in **secondary calculus**, i.e., calculus on the solution space of a PDE, in the sense that all objects of secondary calculus turn out to be cohomology classes of differential complexes growing on diffieties.

7.1.4 Cartan connection

Horizontal vector fields

Since

$$\mathcal{C}(\pi) : J^\infty(\pi) \ni \kappa \mapsto \mathcal{C}_\kappa(\pi) \subset T_\kappa J^\infty(\pi) \ ,$$

where $\mathcal{C}_\kappa(\pi)$ is the tangent space at κ to the graphs $\operatorname{gr}(j^\infty\phi)$ of the *sections* $j^\infty\phi$ that pass through κ at $m = \pi_\infty(\kappa)$, the following statements are rather obvious:

- $T_\kappa\pi_\infty : \mathcal{C}_\kappa(\pi) \to T_m X$ is a vector space isomorphism (it is easily seen that this derivative sends $D_{x^i}|_\kappa$ to $\partial_{x^i}|_{\pi_\infty(\kappa)}$).
- The $\mathcal{F}(\pi)$-module $\mathcal{C}\Theta(\pi) := \Gamma(\mathcal{C}(\pi))$ (resp., $\Theta^v(\pi)$) of sections of the subbundle $\mathcal{C}(\pi) \subset T\,J^\infty(\pi)$ (resp., of π_∞-vertical vector fields of $J^\infty(\pi)$) is a submodule of the $\mathcal{F}(\pi)$-module $\Theta(\pi)$ of vector fields of $J^\infty(\pi)$. More precisely, we have

$$\Theta(\pi) = \mathcal{C}\Theta(\pi) \oplus \Theta^v(\pi) \ . \tag{87}$$

This suggests the idea of connection, i.e., of a $C^\infty(X)$-linear lift (map with the obvious projection property)

$$\mathcal{C} : \Theta(X) \ni \theta \mapsto \mathcal{C}\theta \in \mathcal{C}\Theta(\pi) \ . \tag{88}$$

Indeed, its suffices to set, for any $\kappa \in J^\infty(\pi)$ with projection $\pi_\infty(\kappa) = m$,

$$(\mathcal{C}\theta)_\kappa := (T_\kappa \pi_\infty)^{-1}\theta_m \in \mathcal{C}_\kappa(\pi) \subset T_\kappa J^\infty(\pi)\ . \tag{89}$$

This connection $\mathcal{C}$ on $J^\infty(\pi)$ is the **Cartan connection** induced by the Cartan distribution $\mathcal{C}(\pi)$ on $J^\infty(\pi)$.

As, in trivializing coordinates (x^i, u^a_α) of $J^\infty(\pi)$ over U around $m = \pi_\infty(\kappa)$, the Cartan space $\mathcal{C}_\kappa(\pi)$ is generated by the $D_{x^i}|_\kappa$, the *horizontal vector fields* $H \in \mathcal{C}\Theta(\pi)$ *are locally generated over functions of* $J^\infty(\pi)$ *by the total derivatives* D_{x^i}:

$$H|_{\pi_\infty^{-1}(U)} = \sum_j H^j(x^i, u^a_\alpha) D_{x^j}\ . \tag{90}$$

Since $T_\kappa\pi_\infty(D_{x^j}|_\kappa) = \partial_{x^j}|_m$, *a vector field* $\theta|_U = \sum_j \theta^j(x^i)\partial_{x^j}$ *is lifted to*

$$(\mathcal{C}\theta)|_{\pi_\infty^{-1}(U)} = \sum_j \theta^j(x^i) D_{x^j}\ . \tag{91}$$

Let us also mention, for the sake of completeness, that *a vector field* $T \in \Theta(\pi)$ (*resp., a vertical vector field* $V \in \Theta^v(\pi)$) *locally reads*

$$T|_{\pi_\infty^{-1}(U)} = \sum_j T^j(x^i, u^a_\alpha)\partial_{x^j} + \sum_{b\beta} T^b_\beta(x^i, u^a_\alpha)\partial_{u^b_\beta} \quad (\text{resp.,}\quad V|_{\pi_\infty^{-1}(U)} = \sum_{b\beta} V^b_\beta(x^i, u^a_\alpha)\partial_{u^b_\beta})\ . \tag{92}$$

We are now able to rewrite the definition of a horizontal lift $\mathcal{C}\theta$ in a useful way. If $\theta \in \Theta(X)$ and $F \in \mathcal{F}(\pi)$, and if ϕ is a local section in $\Gamma(\pi)$ that is defined around $m \in X$, we get

$$(j^\infty\phi)^*((\mathcal{C}\theta)F)|_m = ((\mathcal{C}\theta)F)|_{j^\infty_m\phi} = \left((\mathcal{C}\theta)_{j^\infty_m\phi}F\right)|_{j^\infty_m\phi} = \left(((T\pi_\infty)^{-1}\theta_m)F\right)|_{j^\infty_m\phi} =$$

$$\theta_m(F \circ j^\infty\phi)|_m = \theta((j^\infty\phi)^*F)|_m\ .$$

Indeed, the isomorphism $(T\pi_\infty)^{-1}$ sends a partial derivative to the corresponding total derivative. Observe also that, although the function $F \circ j^\infty\phi$ depends on ϕ, its derivative $\theta_m(F \circ j^\infty\phi)|_m$ depends only on $j^\infty_m\phi$. Hence, the

Proposition 7. *For any* $\theta \in \Theta(X)$, $F \in \mathcal{F}(\pi)$, *and* $\phi \in \Gamma_{\rm loc}(\pi)$, *we have*

$$(j^\infty\phi)^*((\mathcal{C}\theta)F) = \theta((j^\infty\phi)^*F)\ . \tag{93}$$

It is clear that we could define the Cartan connection (89) by means of (93), and that Equation (93) is the generalization of Equation (67).

We already explained that $[\mathcal{C}\Theta(\pi), \mathcal{C}\Theta(\pi)] \subset \mathcal{C}\Theta(\pi)$. Moreover, it immediately follows from (93) that $\mathcal{C}[\theta, \theta'] = [\mathcal{C}\theta, \mathcal{C}\theta']$. In other words, *the integrable Cartan distribution of* $J^\infty(\pi)$ *induces a flat Cartan connection on* $J^\infty(\pi) \to X$. Further, the increasing sequence $\mathcal{C}(\Theta(X)) \subset \mathcal{C}\Theta(\pi) \subset \Theta(\pi)$ is a *sequence of Lie subalgebras*. Eventually, if Σ is the infinite prolongation of a PDE on π, we set $\mathcal{C}\Theta(\Sigma) := \Gamma(\mathcal{C}(\Sigma))$, where $\mathcal{C}(\Sigma)$ is the Cartan distribution of Σ. This

$\mathcal{F}(\Sigma)$-module is locally generated by the $D_{x^i}|_\Sigma$. When restricting the lifts $\mathcal{C}\theta$ to Σ, we get a connection $\mathcal{C} : \Theta(X) \to \mathcal{C}\Theta(\Sigma)$, the Cartan connection on Σ, which is flat as well. Hence, *the integrable Cartan distribution of Σ induces a flat Cartan connection on $\Sigma \to X$, which is the restriction of the connection on the infinite jet space.*

Horizontal differential operators

Total differential operators (TDOs)

$$\Psi = \sum_\beta \Psi^\beta(x^i, u^a_\alpha) D^\beta_x \tag{94}$$

are known to be of primary importance in Field Theory. The fundamental property is that *TDOs act not only on $\mathcal{F}(\pi)$, but also on $\mathcal{F}(\Sigma)$.* This is of course due to the fact that total derivatives restrict to (horizontal) vector fields of Σ (see Equation (84)), and is not true for ordinary differential operators

$$\mathbb{T} = \sum_\gamma \mathbb{T}^\gamma(x^i, u^a_\alpha) \ \ldots \circ \partial^{\gamma_j}_{x^j} \circ \ldots \circ \partial^{\gamma_{b\beta}}_{u^b_\beta} \circ \ldots \tag{95}$$

of $J^\infty(\pi)$. An interesting subclass of TDOs are the lifts

$$\mathcal{C}\Delta = \sum_\beta \Delta^\beta(x^i) D^\beta_x \tag{96}$$

of linear differential operators $\Delta = \sum_\beta \Delta^\beta(x^i)\partial^\beta_x$ acting on $C^\infty(X)$. These lifts can be defined exactly as the lifts of base vector fields in (93).

Note first that differential operators act usually not only on functions $C^\infty(X)$ (resp., on $\mathcal{F}(\pi)$ (functions of $J^\infty(\pi)$)), but act between sections $\Gamma(\eta_k)$ (locally: $\mathbb{R}^{r_k}$-valued functions on 'X') of rank r_k vector bundles $\eta_k : E_k \to X$ (resp., between sections $\mathcal{F}(\pi, \eta_k) = \Gamma(\pi^*_\infty(\eta_k))$ (locally: $\mathbb{R}^{r_k}$-valued functions on '$J^\infty(\pi)$') of the bullbacks $\pi^*_\infty(\eta_k) : \pi^*_\infty(E_k) \to J^\infty(\pi)$ of these bundles). Hence, the

Definition 4. *Let $\pi : E \to X$ and $\eta_k : E_k \to X$ ($k \in \{1,2\}$) be vector bundles. The* **lift of a linear differential operator** $\Delta : \Gamma(\eta_1) \to \Gamma(\eta_2)$ *is the linear differential operator $\mathcal{C}\Delta : \mathcal{F}(\pi, \eta_1) \to \mathcal{F}(\pi, \eta_2)$ (of same order) defined by*

$$(j^\infty\phi)^*((\mathcal{C}\Delta)S) = \Delta((j^\infty\phi)^*S)\ , \tag{97}$$

where $S \in \mathcal{F}(\pi, \eta_1)$ and $\phi \in \Gamma_{\mathrm{loc}}(\pi)$.

The difference with lifts

$$\mathcal{C}\theta = \sum_j \theta^j(x^i) D_{x^j} \in \mathcal{C}\Theta(\pi)$$

of vector fields is that the horizontal or $\mathcal{C}$-vector fields $\mathcal{C}\Theta(\pi)$ had been defined before the lifts $\mathcal{C}\theta$. Here, i.e., for lifts $\mathcal{C}\Delta$ of differential operators, we still need to find the proper definition of

$\mathcal{C}$-differential operators $\mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2))$. In view of (90), these $\mathcal{C}$-differential operators should locally be the TDOs

$$\Psi = \sum_\beta \Psi^\beta(x^i, u_\alpha^a) D_x^\beta \ ,$$

see 94. Since, for any $F \in \mathcal{F}(\pi)$ and any $\phi \in \Gamma(\pi)$, this model $\mathcal{C}$-differential operator Ψ satisfies

$$(\Psi F) \circ j^\infty \phi = \sum_\beta (\Psi^\beta \circ j^\infty \phi)\ ((D_x^\beta F) \circ j^\infty \phi) = \sum_\beta (\Psi^\beta \circ j^\infty \phi)\ \partial_x^\beta (F \circ j^\infty \phi) =: \Psi_\phi(F \circ j^\infty \phi) \ ,$$

we have

$$(j^\infty \phi)^*(\Psi F) = \Psi_\phi((j^\infty \phi)^* F) \ ,$$

where the RHS $\Psi_\bullet$ (see its definition) is a not necessarily linear differential operator in $\phi \in \Gamma(\pi)$ with values Ψ_ϕ in linear differential operators on $C^\infty(X)$. This motivates the

Definition 5. *A linear differential operator $\Psi : \mathcal{F}(\pi, \eta_1) \to \mathcal{F}(\pi, \eta_2)$ is a* $\mathcal{C}$**-differential operator** $\Psi \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2))$*, if, for any $\phi \in \Gamma(\pi)$, there exists a linear differential operator $\Psi_\phi : \Gamma(\eta_1) \to \Gamma(\eta_2)$, such that, for any $S \in \mathcal{F}(\pi, \eta_1)$, the equality*

$$(j^\infty \phi)^*(\Psi S) = \Psi_\phi((j^\infty \phi)^* S) \tag{98}$$

holds.

This definition captures correctly our intuition of $\mathcal{C}$-differential operators. Since it is clear from its definition that the lift $\mathcal{C}$ of differential operators respects composition, we have, locally,

$$\sum_\beta \Psi^\beta(x^i, u_\alpha^a) D_x^\beta = \sum_\beta \Psi^\beta(x^i, u_\alpha^a) \mathcal{C}(\partial_x^\beta) \ .$$

It can be shown [KV98] that this result is global:

Proposition 8. *Any $\Psi \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2))$ reads*

$$\Psi = \sum_\beta \Psi^\beta \mathcal{C} \Delta_\beta \ , \tag{99}$$

where the sum is finite, where $\Psi^\beta \in \mathcal{F}(\pi)$, and where $\Delta_\beta \in \mathrm{Diff}(\eta_1, \eta_2)$. In other words, $\mathcal{C}$-differential operators are generated over $\mathcal{F}(\pi)$ by lifts.

Moreover, just as TDOs, $\mathcal{C}$-differential operators can be restricted to the infinite prolongation Σ of a PDE. More precisely [KV98],

Corollary 1. *For any $\mathcal{C}$-differential operator $\Psi : \mathcal{F}(\pi, \eta_1) \to \mathcal{F}(\pi, \eta_2)$ and any infinite prolongation $\Sigma \subset J^\infty(\pi)$, there is a linear differential operator $\Psi_\Sigma : \mathcal{F}(\Sigma, \eta_1) \to \mathcal{F}(\Sigma, \eta_2)$ such that, for every $s \in \mathcal{F}(\pi, \eta_1)$, we have $\Psi_\Sigma(s|_\Sigma) = (\Psi s)|_\Sigma$.*

Finally, we have the important

Corollary 2. *There is a canonical $\mathcal{F}(\pi)$-module isomorphism*

$$\mathcal{C} : \mathcal{F}(\pi) \otimes_{C^\infty(X)} \mathrm{Diff}(\eta_1, \eta_2) \to \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2)) \tag{100}$$

between the linear differential operators with coefficients in the jet space functions and the corresponding $\mathcal{C}$-differential operators. In particular, in the case of the trivial line bundle $\eta_1 = \eta_2$, we get the isomorphism

$$\mathcal{C} : \mathcal{F}(\pi) \otimes_{C^\infty(X)} \mathcal{D}(X) \to \mathcal{C}\mathcal{D}(J^\infty(\pi)) \ . \tag{101}$$

Proof. Observe first that the action of a differential operator $F \otimes \Delta$, with $F \in \mathcal{F}(\pi)$ and $\Delta \in \mathcal{D}(X)$, on a function $f \in C^\infty(X)$ is naturally defined by

$$(F \otimes \Delta)(f) = F\ ((\Delta f) \circ \pi_\infty) \ .$$

The action $(F \otimes \Delta)(s)$, $\Delta \in \mathrm{Diff}(\eta_1, \eta_2)$ and $s \in \Gamma(\eta_1)$, is defined similarly:

$$(F \otimes \Delta)(s) = F\ ((\Delta s) \circ \pi_\infty) \ . \tag{102}$$

The map

$$\mathcal{C} : \mathcal{F}(\pi) \otimes_{C^\infty(X)} \mathrm{Diff}(\eta_1, \eta_2) \ni F \otimes \Delta \mapsto F\ \mathcal{C}\Delta \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2)) \ , \tag{103}$$

is obviously a well-defined and $\mathcal{F}(\pi)$-linear. To prove injectivity, assume that $F\ (\mathcal{C}\Delta)(S) = 0$, for all $S \in \Gamma(\pi_\infty^*(\eta_1))$, in particular, for all $S = s \circ \pi_\infty$, $s \in \Gamma(\eta_1)$. It follows from (97) that

$$(F \circ j^\infty \phi)\ \Delta s = (F\ ((\Delta s) \circ \pi_\infty)) \circ j^\infty \phi = 0 \ ,$$

for all s, ϕ. Eventually, (102) allows to conclude that $F \otimes \Delta = 0$. As for surjectivity, recall that any $\mathcal{C}$-differential operator Ψ reads $\sum_\beta \Psi^\beta \mathcal{C}\Delta_\beta$, and note that $\sum_\beta \Psi^\beta \otimes \Delta_\beta$ is a preimage of Ψ. □

Let us summarize in coordinate language what we achieved so far. Consider a PDE

$$\psi^b(x^i, \partial_x^\alpha \phi^a) \equiv 0\ , \forall b\ ,$$

whose LHS sends sections $\phi = (\phi^a(x))_a \in \Gamma(\pi)$ to sections $\psi = (\psi^b(x))_b := (\psi^b(x^i, \partial_x^\alpha \phi^a))_b \in \Gamma(\eta_1)$. We take into account the linear differential consequences

$$\Delta\ \psi^b(x^i, \partial_x^\alpha \phi^a) := \sum_\beta M^c_{\beta b}(x) \partial_x^\beta\ \ \psi^b(x^i, \partial_x^\alpha \phi^a) \equiv 0\ , \forall c$$

of this equation, where $\Delta \in \mathrm{Diff}(\eta_1, \eta_2)$. The latter condition can be rewritten in the form

$$(\mathcal{C}\Delta)\ \psi^b(x^i, u^a_\alpha)\ |_{j^\infty_x \phi} = \sum_\beta M^c_{\beta b}(x) D_x^\beta\ \ \psi^b(x^i, u^a_\alpha)\ |_{j^\infty_x \phi} \equiv 0\ , \forall c\ ,$$

thus leading to a $\mathcal{C}$-differential operator $\mathcal{C}\Delta \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\eta_1), \pi_\infty^*(\eta_2))$. Just as the value

$$\psi^b(x^i, \partial_x^\alpha \phi^a)\ |_m$$

at $m \in X$ (in fact we mean here the coordinates of m; the same notational abuse will be tolerated in the sequel) of the image of $\phi = (\phi^a(x))_a \in \Gamma(\pi)$ by a differential operator in $\mathrm{DO}_k(\pi, \eta_1)$ only depends on the values $\partial_x^\alpha \phi^a|_m$ of the coefficients of the 'Taylor expansion' of ϕ at m up to order k, the value

$$\sum_\beta N^c_{\beta b}(x^i, u^a_\alpha) D^\beta_x \; \psi^b(x^i, u^a_\alpha) \mid_\kappa$$

at $\kappa \in J^\infty(\pi)$ of the image of $\psi = (\psi^b(x^i, u^a_\alpha))_b \in \Gamma(\pi^*_\infty(\eta_1))$ by a $\mathcal{C}$-differential operator in $\mathcal{C}\,\mathrm{Diff}_k(\pi^*_\infty(\eta_1), \pi^*_\infty(\eta_2))$ only depends on the values $D^\beta_x \, \psi^b(x^i, u^a_\alpha)|_\kappa$ of the total or horizontal derivatives of ψ at κ up to order k. In fact, the $\mathcal{C}$-differential calculus is similar to the ordinary differential calculus. For $k \in \mathbb{N} \cup \{\infty\}$, the **horizontal k-jet** $\bar{\jmath}^k_\kappa S$ at $\kappa \in J^\infty(\pi)$ of a local section $S \in \Gamma(\pi^*_\infty(\eta_1))$ that is defined around κ is the equivalence class of all such local sections, whose coordinate forms in a trivializing chart (x^i, u^a_α, v^b) around κ coincide at κ, together with their total derivatives at κ up to order k.

Remark 8. *In the following, if $\pi : E \to X$ and $\rho : F \to X$ are two vector bundles, we set $R := \pi^*_\infty(\rho)$ and $\mathcal{R} := \Gamma(R) = \Gamma(\pi^*_\infty(\rho))$.*

The set

$$\bar{J}^k(H_1) = \{\bar{\jmath}^k_\kappa S : \kappa \in J^\infty(\pi), S \in \mathcal{H}_1\}$$

is a vector bundle $H_{1,k} : \bar{J}^k(H_1) \to J^\infty(\pi)$, called the **horizontal k-jet bundle**. A trivializing chart (x^i, u^a_α, v^b) of H_1 induces a trivializing chart $(x^i, u^a_\alpha, v^b_\beta)$ of $H_{1,k}$ given by

$$x^i(\bar{\jmath}^k_\kappa S) = x^i(\kappa), u^a_\alpha(\bar{\jmath}^k_\kappa S) = u^a_\alpha(\kappa), v^b_\beta(\bar{\jmath}^k_\kappa S) = D^\beta_x S^b|_\kappa \; . \tag{104}$$

As already suggested here above, the $\mathcal{C}$-differential or horizontal differential operators

$$\Psi \in \mathcal{C}\,\mathrm{Diff}_k(H_1, H_2)$$

are those

$$\Psi \in \mathrm{Hom}_\mathbb{R}(\mathcal{H}_1, \mathcal{H}_2)$$

that factor through the horizontal k-jet bundle $\bar{J}^k(H_1)$, i.e., that read $\Psi = \psi \circ \bar{\jmath}^k$, for some (and thus unique) vector bundle map

$$\psi \in \mathtt{VB}(H_{1,k}\,, H_2) \simeq \mathrm{Hom}_{\mathcal{F}(\pi)}(\Gamma(\bar{J}^k(H_1)), \mathcal{H}_2) \; .$$

Actually, the whole theory of jet bundles can be transferred to horizontal jet bundles [Ver02]. Indeed, it follows from what has been said that, in the coordinate setting, horizontal jet bundles are just jet bundles with extra coordinates u^a_α in the base.

7.1.5 Classical and higher symmetries I and II

Classical symmetries I

The concept of symmetry is of fundamental importance in many fields of Science and deserves special attention. The notion is quite straightforward – at least in elementary situations. For instance, when thinking about an axial symmetry of a plane domain S, we get a permutation p such that $p(S) = S$. Similarly, a **symmetry of an equation** $\Sigma^0 \subset J^k(\pi)$ should be a fiber bundle automorphism (or, just a diffeomorphism) ψ of $J^k(\pi)$ such that

$$\psi(\Sigma^0) = \Sigma^0 \,. \tag{105}$$

However, since the essential structure of $J^k(\pi)$ is the Cartan distribution $\mathcal{C}^k$ (i.e., the infinitesimal object that encodes jet prolongations of sections), it seems natural to ask that a symmetry respect the Cartan distribution (or, better, that its tangent map does).

In the following, we focus on automorphisms of $J^k(\pi)$ that respect $\mathcal{C}^k$, thus omitting Condition (105) at the first set-out. We refer to such automorphisms as **Lie automorphisms** of π_k. In particular, we may ask whether it is possible to build a Lie automorphism of π_k as a prolongation of an automorphism of π.

Prolongations of diffeomorphisms and vector fields

It is easily seen that, if $\Psi = (\psi_0, \psi)$ is a *fiber bundle automorphism* of $\pi : E \to X$, we can prolong it to a fiber bundle automorphism $j^\ell\Psi := (\psi_0, j^\ell\psi)$ of $\pi_\ell : J^\ell(\pi) \to X$. It actually suffices to recall that $\psi\phi\psi_0^{-1} \in \Gamma(\pi)$, for any $\phi \in \Gamma(\pi)$ (as elsewhere in this text, we do not insist here on the possibility that ϕ might be defined only locally), and to consider the well-defined fiber bundle automorphism

$$j^\ell\psi : J^\ell(\pi) \ni j^\ell_m\phi \mapsto j^\ell_{\psi_0(m)}(\psi\phi\psi_0^{-1}) \in J^\ell(\pi) \,.$$

It is easily seen that the lift $j^\ell\Psi$ is a Lie automorphism, i.e., that, for any $\kappa_\ell \in J^\ell(\pi)$, the inclusion

$$(T_{\kappa_\ell} j^\ell\psi)(\mathcal{C}^\ell_{\kappa_\ell}) \subset \mathcal{C}^\ell_{j^\ell_{\kappa_\ell}\psi} \tag{106}$$

holds. Indeed, if $\kappa_\ell = j^\ell_m\phi$ and if $(T_m\, j^\ell\phi)(v_m)$ $(v_m \in T_mX)$ denotes an element of $\mathcal{C}^\ell_{\kappa_l}$, we have

$$(T_{\kappa_\ell}\, j^\ell\psi)(T_m\, j^\ell\phi)(v_m) = T_{\psi_0(m)}(j^\ell(\psi\phi\psi_0^{-1}))(T_m\psi_0\, v_m) \in \mathcal{C}^\ell_{j^\ell_{\kappa_\ell}\psi} \,.$$

Let us still mention that the prolongation $j^\ell\psi : J^\ell(\pi) \to J^\ell(\pi)$ of $\psi : J^0(\pi) \to J^0(\pi)$ is really a lifting, in the sense that $\pi_{0\,\ell} \circ j^\ell\psi = \psi \circ \pi_{0\,\ell}$.

Instead of considering finite automorphisms or diffeomorphisms, we can take an interest in infinitesimal ones, i.e, in vector fields. Note that a vector field $\Xi \in \Theta(\pi_0)$, i.e., a field of $\pi : E \to X$ (we avoid writing $\Theta(\pi)$, since this notation is used instead of the more precise $\Theta(\pi_\infty)$), is a *π-projectable vector field* if and only if $T\pi\,\Xi_e = \xi_{\pi(e)}$, for all $e \in E$, i.e., if and only if there is a vector field $\xi \in \Theta(X)$ that is π-related to Ξ. It is well-known that this means that π intertwines the flows ψ_t^Ξ and ψ_t^ξ, i.e., that $\pi\circ\psi_t^\Xi=\psi_t^\xi\circ\pi$ (assume for simplicity that the flows are globally defined). In other words, $\Psi_t^\Xi=(\psi_t^\xi,\psi_t^\Xi)$ is a 1-parameter group of fiber bundle isomorphisms of $\pi:E\to X$, and it can thus be prolonged to a 1-parameter group of Lie isomorphisms $j^\ell\Psi_t^\Xi=(\psi_t^\xi,j^\ell\psi_t^\Xi)$ of $\pi_\ell:J^\ell(\pi)\to X$. The latter implements a vector field $j^\ell\Xi\in\Theta(\pi_\ell)$ – the ℓ-jet prolongation of the projectable vector field $\Xi\in\Theta(\pi_0)$ – . In other words, the lift $j^\ell\Xi$ is given by

$$(j^\ell\Xi)_{j^\ell_m\phi}=d_t|_{t=0}j^\ell_{\psi_t^\xi(m)}(\psi_t^\Xi\phi\psi_{-t}^\xi)\;,$$

and the flow of the prolongation $j^\ell\Xi$ of Ξ is the prolongation $j^\ell\psi_t^\Xi$ of the flow of Ξ, and it is thus made of Lie isomorphisms. The explicit coordinate computation of the lift of

$$\Xi=\sum_j A^j(x^i)\partial_{x^j}+\sum_b B^b(x^i,u^a)\partial_{u^b}=\sum_j A^j(\partial_{x^j}+u^b_j\partial_{u^b})+\sum_b(B^b-A^ju^b_j)\partial_{u^b}\tag{107}$$

leads to

$$j^\ell\Xi=\sum_j A^jD^{\le\ell-1}_{x^j}+\sum_b\sum_{|\beta|\le\ell-1}D^\beta_x(B^b-A^ju^b_j)\partial_{u^b_\beta}\tag{108}$$

[Kru73]. Note that the first term (resp., second term) of the lift is obtained by extending the total derivatives $D^{\le0}_{x^j}$ in (107) to $D^{\le\ell-1}_{x^j}$ (resp., by adding new terms whose coefficients are the corresponding total derivatives of the coefficients in (107)).

Hence, any *fiber bundle automorphism* of π (resp., any *projectable vector field* of π) can be prolonged to a fiber bundle automorphism of π_ℓ (resp., a vector field of π_ℓ) that respects (whose flow respects) the Cartan distribution $\mathcal{C}^\ell$. The result can be generalized to arbitrary diffeomorphisms $\psi:J^0(\pi)\to J^0(\pi)$ (resp., vector fields $\Xi\in\Theta(\pi_0)$). More precisely, any diffeomorphism (resp., vector field) of π can be lifted to a diffeomorphism (resp., vector field) of π_ℓ that (whose flow) respects the Cartan distribution. We refer to such distribution respecting diffeomorphisms and vector fields as **Lie transformations** and **Lie fields**, respectively (in the case of $J^0(\pi)$, any vector in T_eE is tangent to a section, so $\mathcal{C}^0_e=T_eE$, and Lie transformations (resp., Lie fields) are just diffeomorphisms (resp., vector fields)). The lift to π_ℓ of an arbitrary vector field of π_0, i.e., of

$$\Xi=\sum_j A^j(x^i,u^a)\partial_{x^j}+\sum_b B^b(x^i,u^a)\partial_{u^b}=\sum_j A^j(\partial_{x^j}+u^b_j\partial_{u^b})+\sum_b(B^b-A^ju^b_j)\partial_{u^b}\tag{109}$$

is locally given by the same formula (108) as before [Vit11]. Even more generally, any Lie transformation (resp., Lie field) of π_k can be lifted to a Lie transformation (resp., Lie field) of any $\pi_{k+\ell}$. Conversely, any Lie transformation (resp., any Lie field) of π_ℓ is the lift of a diffeomorphism (resp., a vector field) of π, at least if $\operatorname{rk}(\pi)>1$, [KV98], [Vit11].

Classical symmetries II

In view of what has been said above, a **symmetry of an equation** $\Sigma^0\subset J^k(\pi)$ is a Lie transformation ψ of $J^k(\pi)$ such that $\psi(\Sigma^0)=\Sigma^0$. As mentioned before, we do in this text usually not insist on possible local characters. For instance, we could consider here local symmetries of $\Sigma^0 \subset J^k(\pi)$, i.e., Lie transformations ψ of an open subset $\mathcal{U} \subset J^k(\pi)$ such that $\psi(\mathcal{U} \cap \Sigma^0) = \mathcal{U} \cap \Sigma^0$. Also the notion of **infinitesimal symmetry of an equation** $\Sigma^0 \subset J^k(\pi)$ is now clear. It is a Lie field τ of $J^k(\pi)$ that is tangent to Σ^0, i.e., such that $\tau_\kappa \in T_\kappa \Sigma^0$, for all $\kappa \in \Sigma^0$.

Higher symmetries I

Let us recall that we systematically assume that the considered equations are formally integrable. Just as a Lie transformation (resp., a Lie field) of $J^k(\pi)$ lifts to a Lie transformation (resp., a Lie field) of any $J^{k+\ell}(\pi)$, a symmetry (resp., an infinitesimal symmetry) of $\Sigma^0 \subset J^k(\pi)$ lifts to a symmetry (resp., an infinitesimal symmetry) of any $\Sigma^\ell \subset J^{k+\ell}(\pi)$ (the converse is true as well) [KV98, Prop. 3.23]. Hence, a symmetry (resp., an infinitesimal symmetry) of Σ^0 induces a symmetry (resp., an infinitesimal symmetry) of $\Sigma := \Sigma^\infty$. To avoid diffeomorphisms of infinite dimensional spaces, *we consider in the following only infinitesimal symmetries and call them just symmetries*. Further, we will study not only the symmetries of Σ that are implemented by symmetries of Σ^0 (such induced symmetries are Lie fields, i.e., the derivatives of the diffeomorphisms obtained from their flows respect the Cartan distribution), but 'all symmetries' of Σ (such 'higher symmetries' will respect the Cartan distribution in a generalized sense).

Recall that a symmetry of $\Sigma = \Sigma^\infty$ is a vector field $T \in \Theta(\pi)$ of $J^\infty(\pi)$ that is tangent to Σ and that is Lie. A **higher symmetry of** Σ (or simply a symmetry of Σ whenever no confusion is possible) is a vector field $T \in \Theta(\pi)$ that is tangent to Σ and respects the Cartan distribution $\mathcal{C} = \mathcal{C}(\pi)$ of $J^\infty(\pi)$, not in the preceding sense that the derivatives of its flow respect $\mathcal{C}$, but in the sense that

$$[T, \mathcal{C}\Theta(\pi)] \subset \mathcal{C}\Theta(\pi) , \tag{110}$$

where $\mathcal{C}\Theta(\pi) = \Gamma(\mathcal{C}(\pi))$ is the space of Cartan fields.

Symmetries of the Cartan distribution

Just as above, where we omitted first Condition (105), we will forget now temporarily the tangency condition, and study infinite jet space vector fields that satisfy the Cartan condition (110). These fields will be called in the following **symmetries of** $\mathcal{C}$. In view of the Jacobi identity, the space $\Theta_{\mathcal{C}}(\pi)$ of symmetries of $\mathcal{C}$ is a Lie $\mathbb{R}$-subalgebra of $\Theta(\pi)$. Since $\mathcal{C}$ is integrable, Cartan fields $\mathcal{C}\Theta(\pi)$ are **trivial**ly **symmetries of** $\mathcal{C}$, and, by definition, they thus form a Lie ideal of $\Theta_{\mathcal{C}}(\pi)$. The quotient

$$\mathrm{sym}(\pi) := \Theta_{\mathcal{C}}(\pi)/\mathcal{C}\Theta(\pi)$$

is the Lie algebra of **proper symmetries of** $\mathcal{C}$. In view of the Cartan connection (87), we have the direct sum decomposition

$$\Theta_{\mathcal{C}}(\pi) = \mathcal{C}\Theta(\pi) \oplus \mathrm{E}\Theta(\pi) , \tag{111}$$

where

$$\mathrm{E}\Theta(\pi) = \{T \in \Theta^v(\pi) : [T, \mathcal{C}\Theta(\pi)] \subset \mathcal{C}\Theta(\pi)\} . \tag{112}$$

It follows that

$$\mathrm{sym}(\pi) \simeq \mathrm{E}\Theta(\pi) , \tag{113}$$

i.e., that any proper symmetry of $\mathcal{C}$ is naturally represented by a vertical symmetry, or, still, by an **evolutionary vector field**.

Vertical vector fields $V \in \Theta^v(\pi)$ are characterized by the property $T\pi_\infty(V) = 0$, i.e., by the property $V(f) = 0$, for all $f \in C^\infty(X)$. Indeed,

$$V(f) = V(f \circ \pi_\infty) = df(T\pi_\infty(V)) = (T\pi_\infty(V))(df) .$$

If $V \in \Theta^v(\pi)$, we get

$$[V, \mathcal{C}\theta](f) = V(\mathcal{C}\theta(f)) - \mathcal{C}\theta(V(f)) = V(\theta(f)) = 0 ,$$

for any $\theta \in \Theta(X)$ and any $f \in C^\infty(X)$, so that $[V, \mathcal{C}(\Theta(X))] \subset \Theta^v(\pi)$. On the other hand, since *Cartan fields* $\mathcal{C}\Theta(\pi)$ *are generated over* $\mathcal{F}$ *by lifts* $\mathcal{C}(\Theta(X))$, the symmetry or evolutionary condition $[T, \mathcal{C}\Theta(\pi)] \subset \mathcal{C}\Theta(\pi)$ is equivalent to $[T, \mathcal{C}(\Theta(X))] \subset \mathcal{C}\Theta(\pi)$, for all $T \in \Theta(\pi)$. Hence, for $V \in \Theta^v(\pi)$, the evolutionary condition is equivalent to

$$[V, \mathcal{C}(\Theta(X))] \subset \Theta^v(\pi) \cap \mathcal{C}\Theta(\pi) = \{0\} .$$

Since lifts $\mathcal{C}(\Theta(X))$ are locally generated over $C^\infty(X)$ by total derivatives, the symmetry or evolutionary condition reads, locally and for vertical fields V, $[V, D_{x^i}] = 0$, or, still, $[V, D_{x^i}](u^a_\alpha) = 0$, i.e., since $D_{x^i} = \partial_{x^i} + u^b_{i\beta}\partial_{u^b_\beta}$ so that $D_{x^i}u^a_\alpha = u^a_{i\alpha}$,

$$V^a_{i\alpha} = V(u^a_{i\alpha}) = V(D_{x^i}u^a_\alpha) = D_{x^i}(V(u^a_\alpha)) = D_{x^i}V^a_\alpha .$$

In other words, $V \in \Theta^v(\pi)$ *is a local symmetry or evolutionary field if and only if its coefficients satisfy*

$$V^a_{i\alpha} = D_{x^i}V^a_\alpha . \tag{114}$$

This shows that evolutionary vector fields $V \in \mathrm{E}\Theta(\pi)$ are completely determined (locally, by their coefficients V^a, i.e., globally) by their restriction $V|_{\mathcal{F}_0} \in \mathrm{Der}^v(\mathcal{F}_0, \mathcal{F})$.

Hence, there is a 1:1 correspondence between $\mathrm{E}\Theta(\pi)$ and $\mathrm{Der}^v(\mathcal{F}_0, \mathcal{F})$. It is worth to further elaborate on this idea. Let $\mathfrak{X} \in \mathrm{Der}(\mathcal{F}_0, \mathcal{F})$. Locally, this is a vector field $\mathfrak{X}$ of $J^0(\pi)$ with coefficients in functions of $J^\infty(\pi)$:

$$\mathfrak{X} = \sum_j A^j(x^i, u^a_\alpha)\partial_{x^j} + \sum_b B^b(x^i, u^a_\alpha)\partial_{u^b} = \sum_j A^j(\partial_{x^j} + u^b_j\partial_{u^b}) + \sum_b (B^b - A^j u^b_j)\partial_{u^b} . \tag{115}$$

Such a field can of course be prolonged to a field of $J^\infty(\pi)$ in the way specified by formula (108), exactly as in the particular cases (107) and (109) – except that $\ell = \infty$ here. The prolonged vector field (108) is the sum of a term in $\mathcal{C}\Theta(\pi)$ (horizontal fields are locally generated over $\mathcal{F}$ by total derivatives) and a term in $\mathrm{E}\Theta(\pi)$ (see Equation (114)). In particular, if we start from $\mathcal{X} \in \mathrm{Der}^v(\mathcal{F}_0, \mathcal{F})$, i.e., locally, from

$$\mathcal{X} = \sum_b B^b(x^i, u^a_\alpha)\,\partial_{u^b} , \tag{116}$$

we obtain the *evolutionary vector field*

$$\delta_{\mathcal{X}} = \sum_{b,\beta} D_x^\beta B^b\, \partial_{u_\beta^b} \in \mathrm{E}\Theta(\pi)\ . \tag{117}$$

Note that a local vertical derivation (116) is the same as a local section $B = (B^b(x^i, u_\alpha^a))_b$ of $\pi_\infty^*(\pi)$. The point is that this *isomorphism*

$$\mathrm{Der}^v(\mathcal{F}_0, \mathcal{F}) \simeq \Gamma(\pi_\infty^*(\pi)) = \mathcal{F}(\pi, \pi) =: \varkappa(\pi) \tag{118}$$

holds globally and that the local evolutionary fields (117), computed from the global $\mathcal{X} \in \mathrm{Der}^v(\mathcal{F}_0, \mathcal{F})$, can be glued to provide a global evolutionary field $\delta_{\mathcal{X}} \in \mathrm{E}\Theta(\pi)$.

It is noteworthy that the *1:1 correspondence*

$$\delta : \varkappa(\pi) \ni \mathcal{X} \mapsto \delta_{\mathcal{X}} \in \mathrm{E}\Theta(\pi) \tag{119}$$

allows to push the $\mathcal{F}(\pi)$-module structure of $\varkappa(\pi)$ forward to $\mathrm{E}\Theta(\pi)$ (this multiplication is different (!) from that of vector fields of π_∞ by functions of π_∞) and to pull the Lie algebra structure of $\mathrm{E}\Theta(\pi)$ back to $\varkappa(\pi)$.

Eventually, the 1:1 correspondence δ allows introducing a linearization of a not necessarily linear differential operator $D \in \mathrm{DO}(\pi, \pi') \simeq \psi_D \in \mathcal{F}(\pi, \pi')$ between two vector bundles π and π'. For any $\mathcal{X} \in \varkappa(\pi)$, one can extend the action on $\mathcal{F}(\pi)$ of $\delta_{\mathcal{X}} \in \mathrm{E}\Theta(\pi)$ to an action on $\mathcal{F}(\pi, \pi')$. Locally, this claim is obvious – the point is that the extended action is actually a global one. The operator

$$\ell_D : \varkappa(\pi) \ni \mathcal{X} \mapsto \ell_D \mathcal{X} := \delta_{\mathcal{X}} \psi_D \in \mathcal{F}(\pi, \pi') \tag{120}$$

is the so-called **universal linearization operator** of D. In view of (117), we have

$$\ell_D \mathcal{X} = \delta_{\mathcal{X}} \psi_D = \sum_{b,\beta} \partial_{u_\beta^b} \psi_D\, D_x^\beta \mathcal{X}^b\ . \tag{121}$$

In fact, the partial derivatives $\partial_{u_\beta^b}$ ($b \in \{1, \ldots, \mathrm{rk}(\pi)\}$) act on the components ψ_D^a ($a \in \{1, \ldots, \mathrm{rk}(\pi')\}$) of ψ_D. In other words, the *coordinate expression of the linearization operator* is

$$\ell_D = \sum_\beta \left(\partial_{u_\beta^b} \psi_D^a\right)_{a,b} D_x^\beta\ , \tag{122}$$

where a (resp., b) refers to the row (resp., column). The linearization of any (not necessarily linear) differential operator

$$D \in \mathrm{DO}(\pi, \pi')$$

is a (*linear*) *horizontal differential operator*

$$\ell_D \in \mathcal{C}\,\mathrm{Diff}(\pi_\infty^*(\pi), \pi_\infty^*(\pi'))\ . \tag{123}$$

Observe also that the coefficients $\partial_{u_\beta^b} \psi_D$ of the linearization of $D \simeq \psi_D$ or of $\ker \psi_D = \Sigma^0$ are coefficients of the equation of the tangent space of Σ^0.

Higher symmetries II

To upgrade an evolutionary vector field $V \in \mathrm{E}\Theta(\pi)$ of $J^\infty(\pi)$ to a **symmetry of** Σ^0 (a proper generalized symmetry of the equation Σ^0), we must (see classical symmetries of Σ^0) still add the requirement that $V_\kappa \in T_\kappa J^\infty(\pi)$ be tangent to the prolongation $\Sigma \subset J^\infty(\pi)$ when $\kappa \in \Sigma$: $V_\kappa \in T_\kappa \Sigma$, for all $\kappa \in \Sigma$. In other words, the considered evolutionary field is a symmetry of the equation Σ^0 if and only if it acts on functions $\mathcal{F}(\Sigma)$ of the infinite prolongation Σ of Σ^0. The space of all symmetries of Σ^0 is a Lie $\mathbb{R}$-algebra that we denote by $\mathrm{E}\Theta(\Sigma)$.

To finish this review of symmetries, we ask what classical and higher symmetries mean locally, in coordinates, in the case the considered formally integrable equation Σ^0 is implemented by a differential operator $D \simeq \psi_D$, i.e., $\Sigma^0 = \ker \psi_D$.

Let first $\tau \in \Theta(\pi_k)$ be a Lie field that is tangent to Σ^0. This Lie field is (if $\mathrm{rk}(\pi) > 1$) the lift $\tau = j^k \Xi$ of a vector field $\Xi \in \Theta(\pi_0)$. Further, the tangency property means locally that, for any $\kappa_k \in \Sigma^0$, we have

$$L_{j^k\Xi}\psi_D|_{\kappa_k} \simeq \frac{1}{h}\left(\psi_D(\kappa_k + h\tau_{\kappa_k}) - \psi_D(\kappa_k)\right) = 0 \,. \tag{124}$$

This is exactly the concept of infinitesimal symmetry used in Physics (it means that the infinitesimal transformation induced by Ξ transforms a solution into a solution up to terms of order ≥ 2 in the infinitesimal parameter).

Consider now $\mathcal{X} \in \varkappa(\pi)$, as well as the corresponding proper symmetry $\delta_\mathcal{X} \in \mathrm{E}\Theta(\pi)$ of $\mathcal{C}$. As mentioned, this field is a symmetry $\delta_\mathcal{X} \in \mathrm{E}\Theta(\Sigma)$ of Σ^0 if and only if it acts on $\mathcal{F}(\Sigma) = \mathcal{F}(\pi)/I(\Sigma)$, where $I(\Sigma)$ is the ideal made of those functions of $\mathcal{F}(\pi)$ that vanish on Σ. Let U run through an open cover of $J^\infty(\pi)$ by coordinate patches. Locally $I(\Sigma)$ is given by

$$I(\Sigma)|_U = \left\{\sum F_{\alpha a}\, D_x^\alpha \psi_D^a\right\} \,,$$

where the sum is finite and the coefficients are functions in $\mathcal{F}(\pi)$ defined on U. Hence, the symmetry condition $\delta_\mathcal{X} I(\Sigma) \subset I(\Sigma)$ means that, for any U of the considered cover, we have

$$0 = (\delta_\mathcal{X} I(\Sigma))|_{U\cap\Sigma} = (\delta_\mathcal{X} \sum F_{\alpha a}\, D_x^\alpha \psi_D^a)|_{U\cap\Sigma} = \sum F_{\alpha a}|_{U\cap\Sigma}(\delta_\mathcal{X} D_x^\alpha \psi_D^a)|_{U\cap\Sigma} =$$

$$\sum F_{\alpha a}|_{U\cap\Sigma}(D_x^\alpha\, \delta_\mathcal{X} \psi_D^a)|_{U\cap\Sigma} = \sum F_{\alpha a}|_{U\cap\Sigma}\, D_x^\alpha\, (\delta_\mathcal{X} \psi_D^a)|_{U\cap\Sigma} \,,$$

where we used (114) and the fact that horizontal differential operators restrict to $\mathcal{F}(\Sigma)$. Eventually, if Σ^0 is, as assumed, implemented by D, the Σ^0-symmetry condition for $\delta_\mathcal{X}$ is

$$(\delta_\mathcal{X}\psi_D)|_\Sigma = 0 \,, \tag{125}$$

or, still,

$$(\ell_D \mathcal{X})|_\Sigma = \ell_D|_\Sigma \mathcal{X}|_\Sigma = 0 \,, \tag{126}$$

since ℓ_D is a horizontal differential operator and can thus be restricted. In other words, if we denote the restrictions of the linearization ℓ_D (resp., of the generating section $\mathcal{X}$) by ℓ_Σ (resp., $\mathcal{X}_\Sigma$), we get the

Proposition 9. *Let Σ^0 be a formally integrable PDE in π, implemented by a differential operator and with infinite prolongation Σ. An evolutionary vector field $\delta_{\mathcal{X}}$ generated by $\mathcal{X} \in \varkappa(\pi)$ is a symmetry $\delta_{\mathcal{X}} \in \mathrm{E}\Theta(\Sigma)$ of Σ^0 under the necessary and sufficient condition that*

$$\mathcal{X}_\Sigma \in \ker \ell_\Sigma \ . \tag{127}$$

7.1.6 Compatibility complex, formal exactness, formal integrability

Compatibility complex and formal exactness

An overdetermined system is a system of linear equations that are not independent, so that the existence of a solution is subject to *compatibility conditions*.

The simplest example of an **overdetermined system** is a system of linear equations $LX = C$, where $L \in \mathrm{gl}(p \times n, \mathbb{R})$, $X \in \mathbb{R}^n$, and $C \in \mathbb{R}^p$, whose rank $\rho(L) \neq p$. This means that, between the (LHSs of the) equations, i.e., between the rows $L_{i\star}$ of L, there do exist non-trivial linear relations. In the following, we assume for simplicity that there is exactly one such relation, $L_{p\star} = \sum_{j=1}^{p-1} \lambda_j L_{j\star}$, with $\lambda_j \in \mathbb{R}$. This existence of **non-trivial linear relations between the equations** is equivalent to the existence of a non-zero linear operator, in the considered case, a **non-zero linear operator** $\Lambda = (\lambda_1, \ldots, \lambda_{p-1}, -1) \in \mathrm{gl}(1 \times p, \mathbb{R})$**, such that** $\Lambda \circ L = 0$. Hence, the existence of a solution X requires that C satisfies the *compatibility condition* $C \in \ker \Lambda$, i.e., $C_p = \sum_{j=1}^{p-1} \lambda_j C_j$. In this case, the **original system reduces** to $L'X = C'$, with self-explaining notation, and, in view of our assumption, we have $\rho(L') = p-1$. Of course, a **homogeneous system** always reduces. The most general *solution* then depends on $n - (p-1) \geq 0$ parameters, so that $C \in \mathrm{im}\, L$ and the complex

$$\mathbb{R}^n \xrightarrow{L} \mathbb{R}^p \xrightarrow{\Lambda} \mathbb{R}$$

is *exact*.

Another basic example is integration in $\mathbb{R}^n$, which corresponds to the system of linear PDEs $\mathrm{d}_0 f = \omega$, where $\mathrm{d}_0 : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ is the de Rham differential. The **non-trivial linear partial differential relations**

$$\partial_{x^j}\partial_{x^i} f - \partial_{x^i}\partial_{x^j} f = 0 \tag{128}$$

between the PDEs can be equivalently written **as** $\mathrm{d}_1\, \mathrm{d}_0 = 0$**, where the non-zero linear partial differential operator** d_1 is the de Rham operator on 1-forms:

$$C^\infty(\mathbb{R}^n) \xrightarrow{\mathrm{d}_0} \Omega^1(\mathbb{R}^n) \xrightarrow{\mathrm{d}_1} \Omega^2(\mathbb{R}^n) \ .$$

The existence of a solution implies that the *compatibility condition* $\omega \in \ker \mathrm{d}_1$ holds. Since the complex is *exact*, we then have $\omega \in \mathrm{im}\, \mathrm{d}_0$, i.e., the considered PDE admits a *solution*.

More generally, let $D \in \mathrm{Diff}(\pi, \pi')$ be a linear differential operator between smooth sections of vector bundles $\pi : E \to X$ and $\pi' : E' \to X$ over a manifold X. The linear (homogeneous)

PDE implemented by $D \simeq \psi_D$ is called **overdetermined**, if there exists a non-zero linear differential operator $\Delta \in \mathrm{Diff}(\pi', \pi'')$, such that

$$\Gamma(\pi) \xrightarrow{D} \Gamma(\pi') \xrightarrow{\Delta} \Gamma(\pi'')$$

is a complex (of $C^\infty(X)$-modules). We then say that Δ is a **compatibility operator** for D, if the pair (Δ, π'') is universal in the obvious sense.

Just as the original operator D can be overdetermined (non-trivial linear differential relations between the corresponding equations – compatibility operator), a compatibility operator Δ can itself be overdetermined (relations between the relations – new compatibility operator). This then leads to a compatibility complex of the original operator D :

$$\Gamma(\pi) \xrightarrow{D} \Gamma(\pi') \xrightarrow{\Delta_1} \Gamma(\pi'') \xrightarrow{\Delta_2} \Gamma(\pi''') \xrightarrow{\Delta_3} \dots$$

In fact, any $D \in \mathrm{Diff}_k(\pi, \pi')$ admits a compatibility complex in the abelian category $\mathtt{Mod}(\mathcal{O})$ of modules over $\mathcal{O} = C^\infty(X)$, but not necessarily in the non-abelian category $\mathtt{rC}^\infty\mathtt{VB}(X)$ of finite rank smooth vector bundles over X. Indeed, for any $k_1 \in \mathbb{N}$, the algebraized k_1-prolongation $\psi_D^{k_1} \in \mathrm{Hom}_{\mathcal{O}}(\Gamma(\pi_{k+k_1}), \Gamma(\pi'_{k_1}))$ of D admits a cokernel $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Gamma(\pi'_{k_1}), \mathcal{P}_2)$ in $\mathtt{Mod}(\mathcal{O})$, which represents a differential operator $\Delta_1 \in \mathrm{Diff}_{k_1}(\pi', \mathcal{P}_2)$. Since ψ is the cokernel of $\psi_D^{k_1}$, the operator Δ_1 satisfies $\Delta_1 \circ D = \psi \circ j^{k_1} \circ D = \psi \circ \psi_D^{k_1} \circ j^{k+k_1} = 0$. In fact Δ_1 is universal and is thus a compatibility operator of D. When turning the crank again and again, we obtain a compatibility complex of D:

$$\Gamma(\pi) \xrightarrow{D} \Gamma(\pi') \xrightarrow{\Delta_1} \mathcal{P}_2 \xrightarrow{\Delta_2} \mathcal{P}_3 \xrightarrow{\Delta_3} \dots \tag{129}$$

Here we actually use the algebraic approach – in the frame of $\mathcal{O}$-modules – to differential operators, see for instance [KV98], [GKP13b], [GKP13a]. However, the $\mathcal{O}$-modules $\mathcal{P}_2, \mathcal{P}_3, \dots$ are not necessarily projective of finite rank, i.e., they are not necessarily modules $\Gamma(\pi''), \Gamma(\pi'''), \dots$ of sections of vector bundles.

In the following, we stay within the setting of algebraic differential operators and consider a diagram of the type we just used to construct a compatibility operator:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}_{i-1} \arrow[r, "\Delta_{i-1}"] \arrow[d, "j^{k_{i-1}+k_i+\ell}"'] & \mathcal{P}_i \arrow[r, "\Delta_i"] \arrow[d, "j^{k_i+\ell}"'] & \mathcal{P}_{i+1} \arrow[r] \arrow[d, "j^\ell"'] & \cdots \\
\cdots \arrow[r] & \mathcal{J}^{k_{i-1}+k_i+\ell}(\mathcal{P}_{i-1}) \arrow[r, "\psi^{k_i+\ell}_{\Delta_{i-1}}"] & \mathcal{J}^{k_i+\ell}(\mathcal{P}_i) \arrow[r, "\psi^\ell_{\Delta_i}"] & \mathcal{J}^\ell(\mathcal{P}_{i+1}) \arrow[r] & \cdots
\end{tikzcd} \tag{130}$$

Here $\mathcal{P}_{i-1}, \mathcal{P}_i, \mathcal{P}_{i+1}$ are $\mathcal{O}$-modules, $\Delta_{i-1} \in \mathrm{Diff}_{k_{i-1}}(\mathcal{P}_{i-1}, \mathcal{P}_i)$, $\Delta_i \in \mathrm{Diff}_{k_i}(\mathcal{P}_i, \mathcal{P}_{i+1})$, $\ell \in \mathbb{N}$, and $\mathcal{J}^k(\mathcal{P})$ is the algebraic counterpart of $\Gamma(J^k(P))$, where $P \to X$ is a vector bundle and $J^k(P)$ is the ordinary k-jet bundle ('algebraic counterpart' means that, in the geometric case $\mathcal{P} = \Gamma(P)$, we have $\mathcal{J}^k(\mathcal{P}) = \Gamma(J^k(P))$).

The bottom row of (130) is made of prolonged algebraized operators, or, still, prolonged formal operators (acting on formal derivatives). The study of formal operators is referred to as the **formal theory**.

It is clear (see above) that one of the main questions in the context of compatibility complexes is exactness (exactness of the top row in (130)), i.e., 'the question whether the considered equation admits a solution whenever the compatibility condition is satisfied'. The question of exactness can of course also be considered in the (simpler) formal theory (exactness of the bottom row).

More precisely, a compatibility complex (top row) is called **formally exact**, if the corresponding formal complex (bottom row) is exact, for any $\ell \in \mathbb{N}$. In this case, the main task is to look for criteria for (true) exactness of the original (top row) complex.

We will not investigate the latter problem. On the other hand, it is important to know that [KV98], for any sufficiently large $k_1 \in \mathbb{N}$, the compatibility complex (129) is formally exact, for any operator D. We actually have the

Proposition 10. *Any linear differential operator $D \in \operatorname{Diff}(\pi, \pi')$ admits a formally exact compatibility complex. The same is true for any horizontal linear differential operator $D \in \mathcal{C}\operatorname{Diff}(\pi^*_\infty(\eta), \pi^*_\infty(\eta'))$.*

Formal integrability

Let us now briefly comment on formal integrability of a linear partial differential equation Σ^0 or linear differential operator D.

The first observation is that the category $\mathtt{rC}^\infty\mathtt{VB}(X)$ is not Abelian. Indeed, kernels, like e.g., $\Sigma^\ell = \ker \psi^\ell_D$, are not necessarily vector bundles over X. The reason is that, if $\psi : E \to E'$ is a map of vector bundles over X, the rank $\rho(\psi_m)$ of the linear map $\psi_m : E_m \to E'_m$ may vary with $m \in X$. Then, the kernel $\ker \psi := \coprod_{m\in X} \ker \psi_m$ is a bundle of vector spaces of varying dimension $\operatorname{rk}(E) - \rho(\psi_m)$. However, if the rank $\rho(\psi)$ is constant, it is easily seen that the kernel $\ker \psi$ is a vector bundle over X. Therefore, it is natural to ask that $D \simeq \psi_D$ be **regular**, i.e., that the rank $\rho(\psi^\ell_D)$ be constant, for any $\ell \in \mathbb{N}$, or, still, that $\Sigma^\ell = \ker \psi^\ell_D$ be a vector bundle over X, for any $\ell \in \mathbb{N}$.

The second remark is that, if D is of order k, the prolongation Σ^ℓ is the kernel in $J^{k+\ell}(E)$ of the differential consequences ψ^ℓ_D up to order ℓ of the equation $\psi_D = 0$. It follows that any solution in $J^{k+\ell+1}(E)$ of the system $\psi^{\ell+1}_D = 0$ (differential consequences up to order $\ell + 1$) projects by $\pi_{k+\ell,k+\ell+1}$ to a solution in $J^{k+\ell}(E)$ of the system $\psi^\ell_D = 0$ (differential consequences up to order ℓ):

$$\pi_{k+\ell,k+\ell+1}\Sigma^{\ell+1} \subset \Sigma^\ell \ .$$

On the other hand, any family $j^{k+\ell}_m\phi$ $(m \in X)$ of solutions of $\psi^\ell_D = 0$ can be extended to a family $j^{k+\ell+1}_m\phi$ $(m \in X)$ of solutions of $\psi^{\ell+1}_D = 0$. Of course, the best situation is when any solution of $\psi^\ell_D = 0$ can be extended to a solution of $\psi^{\ell+1}_D = 0$, i.e., when

$$\pi_{k+\ell,k+\ell+1}\Sigma^{\ell+1} = \Sigma^\ell \ .$$

We thus understand that the **existence of extended formal solutions**, i.e., formal integrability, is a simplifying requirement.

Actually we say that a linear differential operator $D \simeq \psi_D$ is **formally integrable**, if it is regular and if extended formal solutions do exist, i.e., more precisely, if Σ^ℓ is a vector bundle, for all $\ell \in \mathbb{N}$, and the vector bundle map $\pi_{k+\ell,k+\ell+1} : \Sigma^{\ell+1} \to \Sigma^\ell$ is surjective, for all $\ell \in \mathbb{N}$. In the present text, all partial differential equations Σ^0, even those that are not implemented by a differential operator, are assumed to be **formally integrable in the sense of Remark 6** [KV98].

7.2 Remarks on gauge theories

Much of what will be said in this text about regular irreducible gauge theories can be better understood with the Koszul resolution of a regular surface and some aspects of electromagnetism in mind. In the following, we use without reference results and notation of Subsection 7.1.

7.2.1 Koszul resolution of a regular surface

Let Σ be an embedded p-dimensional submanifold of $\mathbb{R}^n$. This means that, for each $x \in \Sigma$, there is an open neighborhood $\Omega \subset \mathbb{R}^n$ such that $\Sigma \cap \Omega$ is described by a regular cartesian equation $f \in C^\infty(\Omega, \mathbb{R}^{n-p})$. By 'regular' we mean that **the equations** $f^\mathfrak{a} \in C^\infty(\Omega, \mathbb{R})$ **are independent**, i.e., that the rank $\rho(\partial_x f)$ is equal to $n-p$, for all $x \in \Sigma \cap \Omega$. Assume for simplicity that the first $n-p$ columns of the Jacobian matrix are independent and use the decomposition $x = (x', x'') \in \mathbb{R}^{n-p} \times \mathbb{R}^p$. Then, locally, in the neighborhood of Σ, we have $f = f(x', x'') \Leftrightarrow x' = x'(f, x'')$. It follows that, locally, in the new coordinates (f, x''), the equation of Σ is $f = 0$, or, still, $f^\mathfrak{a} = 0$, for all $\mathfrak{a}$.

To avoid obscuration by technicalities, we often ignore in the sequel such local aspects (thus following [Bar10], which is our main reference for the Koszul-Tate resolution of shell functions in a regular irreducible gauge theory).

One of the fundamental consequences of regularity is the structure of the ideal $I(\Sigma)$ made of those smooth functions $C^\infty(\mathbb{R}^n)$ that vanish on Σ. It is clear that any linear combination $F = \sum_\mathfrak{a} F_\mathfrak{a} f^\mathfrak{a}$, $F_\mathfrak{a} \in C^\infty(\mathbb{R}^n)$, of the equations belongs to $I(\Sigma)$. Conversely, if $F \in I(\Sigma)$, we get, working in the new coordinates (f, x''),

$$F(f, x'') = \int_0^1 d_t \left(F(tf, x'')\right) \mathrm{d}\,t = \sum_\mathfrak{a} f^\mathfrak{a} \int_0^1 \left(\partial_{f^\mathfrak{a}} F\right)(tf, x'') \,\mathrm{d}\,t \;=: \sum_\mathfrak{a} F_\mathfrak{a} f^\mathfrak{a} \;.$$

We are now prepared to recall the construction of the **Koszul resolution** of the function algebra $C^\infty(\Sigma)$ of

$$\Sigma : f^\mathfrak{a} = 0,\; \forall \mathfrak{a} \in \{1, \ldots, n-p\}\;, \tag{131}$$

where the $f^\mathfrak{a}$ are the first coordinates of an appropriate coordinate system (f, x'') of $\mathbb{R}^n$. The Koszul resolution of $C^\infty(\Sigma)$ is then the chain complex made of the free Grassmann algebra

$$\mathrm{K} = C^\infty(\mathbb{R}^n) \otimes \mathcal{S}[\phi^{\mathfrak{a}*}]$$

on $n-p$ odd generators $\phi^{\mathfrak{a}*}$ – associated to the equations (131) – and of the Koszul differential

$$\delta_{\mathrm{K}} = \sum_{\mathfrak{a}} f^{\mathfrak{a}}\, \partial_{\phi^{\mathfrak{a}*}} \ . \tag{132}$$

Of course, the claim that this complex is a resolution of $C^\infty(\Sigma)$ means that the homology of $(\mathrm{K}, \delta_{\mathrm{K}})$ is given by

$$H_0(\mathrm{K}) = C^\infty(\Sigma) \quad \text{and} \quad H_k(\mathrm{K}) = 0,\ \forall k > 0\ . \tag{133}$$

At least the result concerning the 0-homology space is quite clear. Indeed, in degree 0, the cycles are the functions in $C^\infty(\mathbb{R}^n)$ and the boundaries are the elements of

$$\delta_{\mathrm{K}}\{\sum_{\mathfrak{b}} F_{\mathfrak{b}}\, \phi^{\mathfrak{b}*}\} = \{\sum_{\mathfrak{a}} F_{\mathfrak{a}}\, f^{\mathfrak{a}}\} = I(\Sigma)\ ,$$

so that $H_0(\mathrm{K}) = C^\infty(\Sigma)$.

7.2.2 Electromagnetism - an Abelian gauge theory

In Minkowski space $\mathbb{R}^{3,1}$, and with respect to any intertial observer or coordinate system, the behavior of the electromagnetic field $(\vec{E}, \vec{B}) = (\vec{E}(x,y,z,t),\ \vec{B}(x,y,z,t))$ is governed by Maxwell's equations, which read in the vacuum,

$$\vec{\nabla}\cdot\vec{E} = 0,\ \vec{\nabla}\cdot\vec{B} = 0,\ \vec{\nabla}\wedge\vec{E} = -\partial_t\vec{B},\ \vec{\nabla}\wedge\vec{B} = \frac{1}{c^2}\partial_t\vec{E}, \tag{134}$$

where c is the celerity of light. The second and third equations can be equivalently written as

$$\vec{B} = \vec{\nabla}\wedge\vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}F - \partial_t\vec{A}\ . \tag{135}$$

Here, $\vec{A} = \vec{A}(x,y,z,t)$ and $F = F(x,y,z,t)$ are the vector and scalar potentials, respectively. In the sequel, we use the space-time coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, and $x^4 = ct$. The principle of Special Relativity, as well as experimental facts, show that, if the considered coordinates change, the components

$$\mathcal{A}_1 = A_1, \mathcal{A}_2 = A_2, \mathcal{A}_3 = A_3, \text{ and } \mathcal{A}_4 = \frac{-1}{c}F$$

transform according to the 1-form transformation law

$$\mathcal{A}_\mu = \partial_{x^\mu} x'^\nu \mathcal{A}'_\nu\ ,$$

so that $\mathcal{A} = \mathcal{A}_\mu\, \mathrm{d}\, x^\mu = \mathcal{A}'_\nu\, \mathrm{d}\, x'^\nu$ is a form $\mathcal{A} \in \Omega^1(\mathbb{R}^{3,1})$.

The Minkowski space $\mathbb{R}^{3,1}$ with the flat Minkowski metric is the local model of a Lorentzian 4-manifold X. When working in a local chart domain U of X, we usually view $\mathcal{A}$ as a form

$$\mathcal{A} \in \Omega^1(U) \otimes \mathfrak{g},$$

valued in $\mathfrak{g} = u(1) = i\mathbb{R}$. Since $\mathfrak{g}$ is the Lie algebra of the unitary group $G = U(1) = S^1$, the potential $\mathcal{A}$ is a local connection 1-form in a trivialization (U, Φ) of a principal G-bundle

L over X. It is easily seen that the freedom of choice concerning the (clearly not unique antiderivative (see (135)) or) potential $\mathcal{A}$ is given by

$$\mathcal{B}_\mu = \mathcal{A}_\mu + \partial_{x^\mu}\theta \ , \tag{136}$$

where θ is an arbitrary function. Therefore, if the considered trivialization (U, Φ) or observer changes to (U', Φ'), what corresponds to a smooth transformation

$$\mathfrak{t} : U \cap U' \to G \ , \tag{137}$$

the form $\mathcal{A}$ will change to $\mathcal{B}$. However, since the matrix $\mathfrak{t}$ is a number in the present case, the relation (136) between $\mathcal{A}$ and $\mathcal{B}$ is exactly (the coordinate form of) the transformation law

$$\mathcal{B} = \mathfrak{t}^{-1}\mathcal{A}\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t} \tag{138}$$

of the local connection 1-form of a connection 1-form ω of the principle bundle L. The function θ chosen by a given observer, or even the observer itself, is called a **gauge**, and the transformation (136) of this gauge is a **gauge transformation**. In Mathematics, an observer or his trivialization are often regarded as a gauge, a transformation like (137) is referred to as a gauge transformation, and Equation (138) is the transformation law – under gauge transformation – for local connection 1-forms.

If we use the preceding conclusion that the electromagnetic potential is nothing but a connection $\omega \in \Omega^1(L) \otimes \mathfrak{g}$ on a G-bundle $L \to X$ over a Lorentzian manifold (X, g), as a principle of electromagnetism, a number of known results come automatically. Indeed, a short computation shows that the local form

$$\mathcal{F} \in \Omega^2(U) \otimes \mathfrak{g}$$

of the curvature Ω of ω, which is here given by $\mathcal{F} = \mathrm{d}\,\mathcal{A}$, i.e., in coordinates, by

$$\mathcal{F}_{\mu\nu} = \partial_{x^\mu}\mathcal{A}_\nu - \partial_{x^\nu}\mathcal{A}_\mu \ ,$$

is exactly the electromagnetic tensor. Hence, under a gauge transformation, the electromagnetic tensor changes according to the transformation law

$$\mathcal{F}' = \mathfrak{t}^{-1}\mathcal{F}\mathfrak{t}$$

for local curvature 2-forms. Since, as mentioned, $\mathfrak{t}$ is a number here, we get $\mathcal{F}' = \mathcal{F}$, i.e., we see that the electromagnetic tensor is gauge invariant, or, still, that the electromagnetic field is a physical observable. Moreover, the obvious equation $\mathrm{d}\,\mathcal{F} = \mathrm{d}^2\,\mathcal{A} = 0$ straightforwardly leads to

$$\partial_{x^\lambda}\mathcal{F}_{\mu\nu} + \partial_{x^\mu}\mathcal{F}_{\nu\lambda} + \partial_{x^\nu}\mathcal{F}_{\lambda\mu} = 0 \ ,$$

which is easily seen to be equivalent to the Maxwell equations (135). Hence, these Maxwell equations follow automatically from general properties of connections and are thus of geometric nature.

The two remaining equations can be found, in a trivialization (U, Φ) of L over a local orthonormal coordinate chart $(U, (x^1, x^2, x^3, x^4))$ of (X, g), as the dynamical equations of the fundamental potential field $\mathcal{A}$, via variational calculus. The indices of the components of local tensor fields, in particular those of the components $\mathcal{A}_\mu$ (resp., $\mathcal{F}_{\mu\nu}$) of $\mathcal{A}$ (resp., $\mathcal{F}$), can be lifted by means of the 'metric' g – which in the considered coordinates is given by the diagonal matrix $(1, 1, 1, -1)$. Take now the Lagrangian $\mathcal{L}$ defined by

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} .$$

The corresponding Euler-Lagrange equations read

$$\partial_{x^\nu}\mathcal{F}^{\nu\mu} = 0 , \quad \text{or} \quad D_{x^\nu}\mathcal{F}^{\nu\mu} = 0 ,$$

depending on whether we view $\mathcal{L}$ as a function of $\mathbb{R}^{3,1}$, or, since it is essentially given by $\partial_{x^\mu}\mathcal{A}_\nu \simeq \mathcal{A}_{\nu;\,\mu}$, as a function of the first jet bundle of $T^*\mathbb{R}^{3,1}$. These equations are equivalent to the first and fourth Maxwell equations, which are thus dynamical ones.

Electromagnetism is a prototypical example of a (an Abelian) gauge theory (since its structure or symmetry group G is Abelian).

7.2.3 Regular irreducible gauge theories

In **field theory**, fields are sections $\phi \in \Gamma(\pi)$ of a vector bundle $\pi : E \to X$. Since we consider here gauge theories from the standpoint of Physics, we work systematically in a trivialization of E (fiber coordinates $u = (u^1, \dots, u^r)$ – we will sometimes write u^a instead of u) over a coordinate patch of X (coordinates $x = (x^1, \dots, x^n)$), or we just assume that $E = \mathbb{R}^n \times \mathbb{R}^r$. The dynamics of the considered field theory is given by a distinguished functional S acting on compactly supported sections $\phi \in \Gamma(\pi)$,

$$S[\phi] = \int_X \mathcal{L}(x^i, u^a_\alpha)|_{j^{k-1}\phi}\, \mathrm{d}\, x \in \mathbb{R} ,$$

where the **Lagrangian** $\mathcal{L}$ is a function $\mathcal{L} \in \mathcal{F}(\pi_{k-1})$ of the $(k-1)-$jet bundle of π (jet bundle coordinates (x^i, u^a_α)) such that $\mathcal{L}(x^i, 0) = 0$ (it suffices to set $\widetilde{F}(x^i, u^a_\alpha) := F(x^i, u^a_\alpha) - F(x^i, 0)$, for any $F \in \mathcal{F}$, to see that $\mathcal{F} = C^\infty(X) \oplus \widetilde{\mathcal{F}}$, where the functions in $\widetilde{\mathcal{F}}$ vanish on the zero section). Equivalently, we may use the corresponding **Euler-Lagrange equations**

$$\delta_{u^a}\mathcal{L}|_{j^k\phi} = (-D_x)^\alpha \partial_{u^a_\alpha}\mathcal{L}|_{j^k\phi} = 0 , \tag{139}$$

where δ_{u^a} is the algebraized Euler-Lagrange operator, see Subsection 7.1.

The extended algebraized Euler-Lagrange equations

$$D^\alpha_x \delta_{u^a}\mathcal{L} = 0 \tag{140}$$

define the constraint surface Σ in the infinite jet space $J^\infty(\pi)$. The solutions ϕ of the original Euler-Lagrange equations (139) are those compactly supported sections $\phi \in \Gamma(\pi)$ that satisfy the condition $(j^\infty\phi)(X) \subset \Sigma$. If a function $f \in \mathcal{F}(\pi)$ of $J^\infty(\pi)$ vanishes on Σ, i.e., if $f \in I(\Sigma)$, we write $f \approx 0$.

As for any system of linear equations, we may find linear relations between the considered equations (140), i.e., relations of the type

$$N^a_\alpha D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0 \;, \tag{141}$$

with $N^a_\alpha \in \mathcal{F}(\pi)$. It is easy to write such relations, if we use coefficients in $I(\Sigma)$, i.e., that vanish on the 'shell' Σ. Indeed, for any functions $n^{[ab]} \in \mathcal{F}(\pi)$ (that are antisymmetric in a, b), we have the linear relation $n^{[ab]}\partial_{u^b}\mathcal{L}\;\partial_{u^a}\mathcal{L} \equiv 0$ between the equations $\partial_{u^a}\mathcal{L} = 0$. What we actually have in mind are non-trivial linear relations, i.e., relations of the type (141), but with at least one coefficient $N^a_\alpha \notin I(\Sigma)$ (on-shell reducibility). We refer to such relations as **non-trivial Noether identities**.

A deep result, which is already present in elementary Mechanics, is the 1:1 correspondence between, roughly speaking, 'symmetries of the action' (resp., 'gauge symmetries') and conserved currents (resp., Noether identities). It motivates the definition of a **gauge theory** as a field theory (see above) with non-trivial Noether identities.

The efficient investigation of gauge theories is subject to some regularity conditions that we now describe. More precisely, the **regularity conditions for 'irreducible' gauge theories** can be formulated as follows:

Assumption 1. For any $\ell \in \mathbb{N}$, the LHSs $D^\alpha_x \delta_{u^a}\mathcal{L}$ of the equations of Σ, up to order $k + \ell$ (i.e., since $\mathcal{L} \in \mathcal{F}(\pi_{k-1})$, we consider derivatives D^α_x up to order ℓ), can be separated into **two packages** $E_\mathfrak{a}$ and E_Δ (of course, the ranges of (α, a) and of $(\mathfrak{a}, \Delta)$ are the same) (we could even only ask that the $D^\alpha_x \delta_{u^a}\mathcal{L}$ and the $(E_\mathfrak{a}, E_\Delta)$ be related by an invertible matrix, i.e., that

$$D^\alpha_x \delta_{u^a}\mathcal{L} = M^{\alpha\mathfrak{a}}_a E_\mathfrak{a} + M^{\alpha\Delta}_a E_\Delta \;,$$

where the matrix $M = (M^{\alpha\mathfrak{a}}_a, M^{\alpha\Delta}_a)$, with row index (α, a), is invertible; however, to simplify, we ignore this matrix in the following, just as we ignore, as mentioned before, a number of local aspects).

Assumption 2. The functions $E_\mathfrak{a} \in \mathcal{F}(\pi_{k+\ell})$ are independent. This is the actual **regularity condition** (see Subsection 7.2.1). In other words, we assume that (locally – but we ignore this restriction) the $E_\mathfrak{a} = E_\mathfrak{a}(x^i, u^a_\alpha)$ can be chosen as the first variables of a new coordinate system $(x^i, E_\mathfrak{a}, u''^a_\alpha)$ in $J^{k+\ell}(\pi)$:

$$(x^i, u'^a_\alpha, u''^a_\alpha) \leftrightarrow (x^i, E_\mathfrak{a}, u''^a_\alpha) \;.$$

Assumption 3. The functions E_Δ are linear consequences of the functions $E_\mathfrak{a}$: $E_\Delta = F^\mathfrak{a}_\Delta E_\mathfrak{a}$, with $F^\mathfrak{a}_\Delta \in \mathcal{F}(\pi_{k+\ell})$. It follows that $E_\Delta = 0$, if $E_\mathfrak{a} = 0$: the $E_\mathfrak{a}$ (resp., E_Δ) are the **independent (resp., dependent) equations**.

To illustrate what has been said, we consider the example of electromagnetism. Depending on whether we interpret the $\mathcal{A}_\mu$ and their derivatives $\partial_{x^\nu}\mathcal{A}_\mu$ as functions of the base $\mathbb{R}^{3,1}$, or, on the contrary, as independent variables $\mathcal{A}_\mu, \mathcal{A}_{\mu;\,\nu}$ in the jet space, we must view the following operators ∂_λ as partial derivatives ∂_{x^λ} or as total derivatives D_{x^λ} .

The non-extended Euler-Lagrange equations read

$$\delta_{\mathcal{A}_\mu}\mathcal{L} = \partial_\nu \mathcal{F}^{\nu\mu} = \partial_\nu \partial^\nu \mathcal{A}^\mu - \partial^\mu \partial_\nu \mathcal{A}^\nu = -\partial_4\partial_4 \mathcal{A}^\mu + \partial_i \partial_i \mathcal{A}^\mu - \partial^\mu \partial_4 \mathcal{A}^4 - \partial^\mu \partial_i \mathcal{A}^i \ , \qquad (142)$$

where $1 \leq \mu, \nu \leq 4$ and $1 \leq i \leq 3$. As a consequence, we get the non-trivial Noether identity

$$\partial_\mu \delta_{\mathcal{A}_\mu}\mathcal{L} = \partial_4 \delta_{\mathcal{A}_4}\mathcal{L} + \partial_i \delta_{\mathcal{A}_i}\mathcal{L} \equiv 0 \ . \qquad (143)$$

Here 'identity' means, depending on the chosen interpretation, that the equality holds for all sections $\mathcal{A}$ and all base points x, or, equivalently, for all points of the jet space. Of course, Identity (143) implies the identities

$$\partial_{(\beta^1,\beta^2,\beta^3,\beta^4)}\partial_\mu \delta_{\mathcal{A}_\mu}\mathcal{L} = \partial_{(\beta^1,\beta^2,\beta^3,\beta^4)}\partial_4 \delta_{\mathcal{A}_4}\mathcal{L} + \partial_{(\beta^1,\beta^2,\beta^3,\beta^4)}\partial_i \delta_{\mathcal{A}_i}\mathcal{L} \equiv 0 \ , \qquad (144)$$

where $\partial_{(\beta^1,\beta^2,\beta^3,\beta^4)}$ means ∂_x^β or D_x^β, depending on the chosen standpoint.

Equation (142) splits into

$$\delta_{\mathcal{A}_j}\mathcal{L} = -\partial_4\partial_4\mathcal{A}_j + \partial_i\partial_i\mathcal{A}_j + \partial_j\partial_4\mathcal{A}_4 - \partial_j\partial_i\mathcal{A}_i = -\mathcal{A}_{j;\,44} + \mathcal{A}_{j;\,ii} + \mathcal{A}_{4;\,4j} - \mathcal{A}_{i;\,ij}$$

and

$$\delta_{\mathcal{A}_4}\mathcal{L} = \partial_4\partial_4\mathcal{A}_4 - \partial_i\partial_i\mathcal{A}_4 - \partial_4\partial_4\mathcal{A}_4 + \partial_4\partial_i\mathcal{A}_i = -\mathcal{A}_{4;\,ii} + \mathcal{A}_{i;\,4i} \ .$$

These non-extended algebraized Euler-Lagrange equations allow us to compute $\mathcal{A}_{j;\,44}$ and $\mathcal{A}_{4;\,11}$ in terms of the other jet space variables and the new coordinates $E_j := \delta_{\mathcal{A}_j}\mathcal{L}$ and $E_4 := \delta_{\mathcal{A}_4}\mathcal{L}$. Hence, E_j, E_4 belong to the first package $E_\mathfrak{a}$ of independent equations that can be chosen as first coordinates of a new system.

However, the derivatives $D_x^\alpha \delta_{\mathcal{A}_\mu}\mathcal{L}$, where $\alpha \neq 0$, are not independent, in view of (144): the $D_x^\beta D_{x^4}\delta_{\mathcal{A}_4}\mathcal{L}$ are dependent equations E_Δ . The challenge resides in the proof that all the other equations $D_x^\alpha \delta_{\mathcal{A}_\mu}\mathcal{L}$ are independent equations $E_\mathfrak{a}$. This is actually a consequence of some geometric facts.

Assumption 4. The dependent equations E_Δ are total derivatives of a finite number of dependent equations $E_\delta = F_\delta^\mathfrak{b} E_\mathfrak{b}$, i.e., there is a finite number of **generators** E_δ **by derivation**: $E_\Delta = D_x^\beta E_\delta$.

In the case of electromagnetism, for instance, there is a unique generator, namely $E_\delta = D_{x^4}\delta_{\mathcal{A}_4}\mathcal{L}$.

Assumption 5. Note that the differences $E_\Delta - F_\Delta^\mathfrak{a} E_\mathfrak{a} \equiv 0$ are non-trivial Noether identities. We assume that, if $E_\Delta = D_x^\beta E_\delta$, the derivative D_x^β of the Noether identity $E_\delta - F_\delta^\mathfrak{b} E_\mathfrak{b} \equiv 0$ is the preceding Noether identity associated to E_Δ . If we write this requirement out, we find an invertibility condition for some matrix, which is called the **irreducibility assumption** of the considered gauge theory.

Observe that the latter hypothesis is satisfied in electromagnetism.

7.2.4 Higher symmetries III

In this subsection, we explain the concepts of symmetry of the Euler-Lagrange equations, symmetry of the action, and gauge symmetry, in the context of a regular irreducible gauge theory. As usual, we denote the coordinates of the considered trivial bundle $\pi : E = \mathbb{R}^n \times \mathbb{R}^r \to X = \mathbb{R}^n$ by (x^i, u^a) and the Lagrangian of the theory by $\mathcal{L}(x^i, u^a_\alpha)$.

As mentioned above, a vector field $\mathfrak{X}$ of $J^0(\pi)$ with coefficients in functions of $J^\infty(\pi)$ (see Equation (115)) can be prolonged to a field of $J^\infty(\pi)$ in the way described by Equation (108) (with $\ell = \infty$). This prolongation $j^\infty \mathfrak{X} \in \Theta(\pi)$ is the sum of a horizontal vector field $A^j D_{x^j} \in \mathcal{C}\Theta(\pi)$ and an evolutionary vector field $\delta_{\mathfrak{X}} \in \mathrm{E}\Theta(\pi)$.

In conformity with the symmetry conditions (124) and (125), which ask that the prolongation of the considered vector field annihilates the algebraized equation on-shell, we say that the **generalized vector field** $\mathfrak{X} \in \mathrm{Der}(\mathcal{F}_0, \mathcal{F})$ is a **symmetry of the Euler-Lagrange equations** $\delta_{u^a}\mathcal{L}|_{j^k\phi} = 0\,, \forall a$, if

$$\delta_{\mathfrak{X}}(\delta_{u^a}\mathcal{L}) \approx 0\ , \forall a\ . \tag{145}$$

As said before, the requirement means that the infinitesimal transformation induced by $\mathfrak{X}$ transforms a solution into a solution up to terms of order ≥ 2 in the infinitesimal parameter.

As for the concept of symmetry of the action, remember first a well-known fact of Lagrangian Mechanics. The gauge transformation (136), or, more precisely, the transformation

$$F' = F - \partial_t\theta, \quad \vec{A}' = \vec{A} + \vec{\nabla}\theta\ ,$$

where θ is a function of time and positions, modifies the generalized electromagnetic potential $U = e(F - \vec{v}\cdot\vec{A})$, where e is the charge and $\vec{v}$ the velocity of the considered particle, and thus leads to different Lagrangians $\mathcal{L}$ and $\mathcal{L}'$. However, it is easily seen that the latter differ by the total derivative $\mathcal{L}' - \mathcal{L} = d_t\,\jmath$ of a function $\jmath$ of time and positions, and that the Euler-Lagrange equations associated to $\mathcal{L}$ and $\mathcal{L}'$, hence, the dynamics, are therefore the same. This observation can be extended to the present field theoretic context. Two Lagrangians $\mathcal{L}, \mathcal{L}' \in \widetilde{\mathcal{F}}$ implement the same Euler-Lagrange equations if and only if they differ by a total divergence:

$$\delta_{u^a}\mathcal{L} = \delta_{u^a}\mathcal{L}', \quad \forall a \quad \Leftrightarrow \quad \mathcal{L}' - \mathcal{L} = D_{x^i}\jmath^i, \quad \jmath^i \in \widetilde{\mathcal{F}}\ .$$

This indicates that two action functionals $S_{\mathcal{L}}$ and $S_{\mathcal{L}'}$, which are defined by Lagrangians $\mathcal{L}$ and $\mathcal{L}'$, coincide (on all compactly supported sections) if and only if the underlying Lagrangians $\mathcal{L}, \mathcal{L}'$ differ by a total divergence. It is thus natural to identify the space of action functionals $S_{\mathcal{L}}$ with the space of classes $[\mathcal{L}]$ of functions $\mathcal{L} \in \widetilde{\mathcal{F}}$ considered up to total divergence. Alternatively, an action can be viewed as a class $[\mathcal{L}\,\mathrm{d}x]$, where $\mathrm{d}x = \mathrm{d}\,x^1 \ldots \mathrm{d}\,x^n$ and where

$$\mathcal{L}\,\mathrm{d}x \simeq \mathcal{L}\,\mathrm{d}x + D_{x^i}\jmath^i\,\mathrm{d}x\ .$$

A **symmetry of the action** is now a generalized vector field $\mathfrak{X}$, such that

$$\delta_{\mathfrak{X}}[\mathcal{L}\,\mathrm{d}x] = [0]\ .$$

This definition only makes sense, if we define how the prolongation $\delta_{\mathfrak{X}}$ acts on the differential form $\mathrm{d}x$ and show that its action on $[\mathcal{L}\,\mathrm{d}x]$ is well-defined. We confine ourselves here to mentioning that the symmetry condition finally reads

$$\delta_{\mathfrak{X}}\mathcal{L} = D_{x^i} j^i \ ,$$

where $j^i \in \mathcal{F}$, i.e., just requires that $\delta_{\mathfrak{X}}\mathcal{L}$ be a total divergence. Moreover, any symmetry of the action is a symmetry of the Euler-Lagrange equations (but the converse is not true).

Eventually, a **gauge symmetry** is a symmetry

$$\mathfrak{X}(f) = A^j(x^i, u^a_\alpha)\partial_{x^j} + B^b(x^i, u^a_\alpha)\partial_{u^b} = A^j(\partial_{x^j} + u^b_j \partial_{u^b}) + (B^b - A^j u^b_j)\partial_{u^b} \tag{146}$$

of the action, whose coefficients

$$A^j = A^j(f) = A^j_\alpha D^\alpha_x f \quad \text{and} \quad B^b = B^b(f) = B^b_\beta D^\beta_x f$$

are the values of some total differential operators on an arbitrary / a varying function $f \in \mathcal{F}$.

Symmetries of the action (resp., symmetries of the action obtained as value of a gauge symmetry on a specific / a fixed function $f \in \mathcal{F}$) are often termed as **global symmetries** (resp., **local symmetries**). Further, we call **symmetry in characteristic form** a symmetry given by a vertical generalized vector field

$$\mathcal{X} = C^b(x^i, u^a_\alpha)\partial_{u^b} \in \mathrm{Der}^v(\mathcal{F}_0, \mathcal{F}) \ .$$

For all types of symmetry (symmetry of the Euler-Lagrange equations, symmetry of the action, or gauge symmetry), any symmetry $\mathfrak{X}$ (see Equation (146)) provides a symmetry

$$\mathcal{X} = (B^b - A^j u^b_j)\partial_{u^b}$$

in characteristic form (note that $\mathcal{X}$ is a symmetry, since $\delta_{\mathfrak{X}} = \delta_{\mathcal{X}}$).

7.2.5 Noether's theorems

Einstein qualified Noether's result as a monument of mathematical thinking. The tight relationship between symmetries and conserved quantities is part of each course in Classical Mechanics. More precisely, Noether's theorems claim that there exists a 1:1 correspondence between (equivalence classes of) symmetries of the action in characteristic form and (equivalence classes of) 'conserved currents', and that there exists a 1:1 correspondence between gauge symmetries in characteristic form and Noether identities.

The latter correspondence is via formal adjoint operators. More precisely, if $N^a_\alpha D^\alpha_x \delta_{u^a}\mathcal{L} \equiv 0$ is a Noether identity, we consider the total differential operator N with components $N^a = N^a_\alpha D^\alpha_x$, and define the corresponding gauge symmetry in characteristic form $\mathcal{X}(f) = C^a(f)\partial_{u^a}$ as the adjoint N^+ of N, i.e., by $C^a(f) = N^{a+}(f) = (-D_x)^\alpha\,(N^a_\alpha f)$. The converse association is similar. It follows that non-trivial Noether identities correspond to non-trivial gauge symmetries in characteristic form.

7.3 Partial differential equations and algebraic $\mathcal{D}$-geometry

7.3.1 Construction of non-split relative Sullivan $\mathcal{D}$-algebras

For convenience, we recall Lemma 1 of [BPP15b] that is needed in the main part of this text.

Lemma 1. *Let $(T, d_T) \in \mathtt{DG}\mathcal{D}\mathtt{A}$, let $(g_j)_{j\in J}$ be a family of symbols of degree $n_j \in \mathbb{N}$, and let $V = \bigoplus_{j\in J} \mathcal{D}\cdot g_j$ be the free non-negatively graded $\mathcal{D}$-module with homogeneous basis $(g_j)_{j\in J}$.*

(i) To endow the graded $\mathcal{D}$-algebra $T \otimes \mathcal{S}V$ with a differential graded $\mathcal{D}$-algebra structure d, it suffices to define

$$dg_j \in T_{n_j-1} \cap d_T^{-1}\{0\} \;, \tag{147}$$

to extend d as $\mathcal{D}$-linear map to V, and to equip $T \otimes \mathcal{S}V$ with the differential d given, for any $t \in T_p$, $v_1 \in V_{n_1}, \ldots, v_k \in V_{n_k}$, by

$$d(t \otimes v_1 \odot \ldots \odot v_k) =$$

$$d_T(t) \otimes v_1 \odot \ldots \odot v_k + (-1)^p \sum_{\ell=1}^{k} (-1)^{n_\ell \sum_{j<\ell} n_j} (t * d(v_\ell)) \otimes v_1 \odot \ldots \widehat{\ell} \ldots \odot v_k \;, \tag{148}$$

where $*$ is the multiplication in T. If J is a well-ordered set, the natural map

$$(T, d_T) \ni t \mapsto t \otimes 1_{\mathcal{O}} \in (T \boxtimes \mathcal{S}V, d)$$

is a RS$\mathcal{D}$A.

(ii) Moreover, if $(B, d_B) \in \mathtt{DG}\mathcal{D}\mathtt{A}$ and $p \in \mathtt{DG}\mathcal{D}\mathtt{A}(T, B)$, it suffices – to define a morphism $q \in \mathtt{DG}\mathcal{D}\mathtt{A}(T \boxtimes \mathcal{S}V, B)$ (where the differential graded $\mathcal{D}$-algebra $(T \boxtimes \mathcal{S}V, d)$ is constructed as described in (i)) – to define

$$q(g_j) \in B_{n_j} \cap d_B^{-1}\{p\, d(g_j)\} \;, \tag{149}$$

to extend q as $\mathcal{D}$-linear map to V, and to define q on $T \otimes \mathcal{S}V$ by

$$q(t \otimes v_1 \odot \ldots \odot v_k) = p(t) \star q(v_1) \star \ldots \star q(v_k) \;, \tag{150}$$

where $\star$ denotes the multiplication in B.

7.3.2 Jet functor

We now give some explanations about the construction of the jet functor

$$\mathcal{J}^\infty : \mathtt{qcCAlg}(\mathcal{O}_X) \to \mathtt{qcCAlg}(\mathcal{D}_X) \;.$$

For simplicity, we assume that the smooth scheme X is a smooth affine algebraic variety, so that we can substitute global sections to sheaves – but the same proof goes through in the general case. We denote by $\mathcal{O}$ (resp., $\mathcal{D}$) the algebra $\mathcal{O}_X(X)$ (resp., $\mathcal{D}_X(X)$).

The functor $\mathcal{J}^\infty$ must be left adjoint to the forgetful functor For, i.e., for $B \in \mathcal{O}\mathtt{A} := \mathtt{CAlg}(\mathcal{O})$ and $A \in \mathcal{D}\mathtt{A} := \mathtt{CAlg}(\mathcal{D})$, we must have

$$\mathrm{Hom}_{\mathcal{D}\mathtt{A}}(\mathcal{J}^\infty B, A) \simeq \mathrm{Hom}_{\mathcal{O}\mathtt{A}}(B, \mathrm{For}\, A) \ , \tag{151}$$

functorially in A, B. The construction of $\mathcal{J}^\infty B$ is quite natural. We start from the $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathcal{O}} B$, and consider the $\mathcal{D}$-algebra $\mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B)$ over $\mathcal{D} \otimes_{\mathcal{O}} B$. Since Equation (151) suggests the existence of an $\mathcal{O}$-algebra morphism $B \to \mathcal{J}^\infty B$, we define $\mathcal{J}^\infty B$ as the quotient of the $\mathcal{D}$-algebra $\mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B)$ by a $\mathcal{D}$-ideal such that the natural inclusion

$$i : B \ni b \mapsto 1 \otimes b \in \mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B)$$

becomes an $\mathcal{O}$-algebra morphism $\pi \circ i : B \to \mathcal{J}^\infty B$ when composed with the natural projection π. Since an $\mathcal{O}$-algebra morphism is an $\mathcal{O}$-linear map (a condition that is automatically verified) that respects the multiplications and the units, we must ensure that

$$\pi(1 \otimes (bb')) = \pi(1 \otimes b) \odot \pi(1 \otimes b') = \pi((1 \otimes b) \odot (1 \otimes b')) \quad \text{and} \quad \pi(1 \otimes 1_B) = \pi(1) \ ,$$

where 1 (resp., 1_B) denotes the unit in $\mathcal{O}$ (resp., B) and where $\odot$ is the symmetric tensor product (we denote the product of two residue classes by the same symbol). Hence, we consider the $\mathcal{D}$-ideal K generated by the elements

$$D \cdot \big((1 \otimes b) \odot (1 \otimes b') - 1 \otimes (bb')\big) \in \mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B) \quad \text{and} \quad D \cdot (1 \otimes 1_B - 1) \in \mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B) \ ,$$

where $D \cdot$ denotes the action by an arbitrary differential operator $D \in \mathcal{D}$.

It now suffices to show that

$$\mathcal{J}^\infty : \mathcal{O}\mathtt{A} \ni B \mapsto \mathcal{J}^\infty B := \mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B)/K \in \mathcal{D}\mathtt{A}$$

possesses the adjointness property (151).

If $f : \mathcal{J}^\infty B \to A$ is a $\mathcal{D}$-algebra morphism, the map

$$\tilde{f} : B \ni b \mapsto f(\pi(1 \otimes b)) \in \mathrm{For}\, A$$

is obviously an $\mathcal{O}$-algebra morphism.

Conversely, let $g : B \to \mathrm{For}\, A$ be an $\mathcal{O}$-algebra morphism. The map

$$\bar{g} : \mathcal{D} \otimes_{\mathcal{O}} B \ni D \otimes b \mapsto D \cdot (g(b)) \in A$$

is a well-defined $\mathcal{D}$-module morphism. Since $\mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B)$ is the free $\mathcal{D}$-algebra over the $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathcal{O}} B$, the $\mathcal{D}$-module morphism $\bar{g}$ can be uniquely extended to a $\mathcal{D}$-algebra morphism $\bar{g} : \mathcal{S}_{\mathcal{O}}(\mathcal{D} \otimes_{\mathcal{O}} B) \to A$. As $\bar{g}$ vanishes on K (note that $\bar{g}(1) = 1_A$, where 1_A is the unit in A), it descends to the quotient $\mathcal{J}^\infty B$. Hence the searched $\mathcal{D}$-algebra morphism $\bar{g} : \mathcal{J}^\infty B \to A$.

Let now $\pi : E \to X$ be a smooth morphism of smooth affine algebraic varieties. The total sections $\mathcal{O}_X^E(X)$ of the pushforward $\mathcal{O}_X^E$ by π of the structure sheaf $\mathcal{O}_E$ of E form an $\mathcal{O}$-algebra, whose image $J := \mathcal{J}^\infty(\mathcal{O}_X^E(X))$ by the jet functor is a $\mathcal{D}$-algebra. This algebra is the $\mathcal{D}$-geometric counterpart of the function algebra $\mathcal{F} = \mathcal{F}(\pi)$ of the infinite jet space $J^\infty(\pi)$ of a smooth vector bundle $\pi : E \to X$.

To gain insight into this statement, consider the example $\pi : E = \mathbb{R}^2 \ni (t, x) \mapsto t \in X = \mathbb{R}$. To compare this differential geometric situation with our former algebraic geometric setting, we define $\mathcal{O} = \mathcal{O}_X(X) := \mathbb{R}[t]$ and $B := \mathcal{O}_X^E(X) = \mathcal{O}_E(E) := \mathbb{R}[t, x]$. It is easily seen that the symmetric algebra $\mathcal{S}_\mathcal{O}(\mathcal{D} \otimes_\mathcal{O} B)$ coincides with the polynomial algebra $\mathbb{R}[t, \partial_t^i \otimes x^j]$, where $i, j \in \mathbb{N}$. When dividing the ideal K out, we obtain

$$J = \mathbb{R}[t, x, \partial_t \otimes x, \partial_t^2 \otimes x, \ldots] .$$

Indeed, the initial generator $\partial_t \otimes x^2$ (resp., $\partial_t \otimes 1_B$), for instance, coincides in the quotient with

$$\partial_t \otimes x^2 = \partial_t \cdot ((1 \otimes x) \odot (1 \otimes x)) \quad (\text{resp., } \partial_t \otimes 1_B = \partial_t \cdot 1) .$$

This generator is thus a polynomial in $\partial_t \otimes x$ and $1 \otimes x \simeq x$ (resp., is thus equal to 0, since ∂_t acts on the element 1 of the $\mathcal{D}$-module $\mathcal{O}$) and can therefore be omitted in the quotient. Hence, the announced result. When setting $x^{(k)} := \partial_t^k \otimes x$, we get

$$J = \mathbb{R}[t, x, x^{(1)}, x^{(2)}, \ldots] ,$$

i.e., we obtain the polynomial function algebra of the jet space $J^\infty(\pi)$.

Observe eventually that the vector field ∂_t acts on a function in J as a derivation, see above, and that by definition $\partial_t \cdot x^{(k)} = x^{(k+1)}$. This means that

$$\partial_t \cdot x^{(k)} = (\partial_t + x^{(1)}\partial_x + x^{(2)}\partial_{x^{(1)}} + \ldots)\, x^{(k)} = D_t\, x^{(k)} ,$$

where D_t is the total derivative. In other words, the action of a differential operator of the base on a function in J coincides with the action of the corresponding total differential operator.

7.3.3 Proof of Proposition 1

Let $\pi : E \to X$ be an affine morphism of schemes (i.e., a locally ringed space morphism $\Pi = (\pi, \pi^\sharp) : (E, \mathcal{O}_E) \to (X, \mathcal{O}_X)$ such that there is an affine cover of X whose preimages by π are affine), in particular a vector bundle. In the following, we consider the sheaf $\mathcal{O}_E \in \texttt{Sh}(E)$ as sheaf $\mathcal{O}_X^E := \pi_* \mathcal{O}_E \in \texttt{Sh}(X)$, where π_* denotes the direct image of sheaves. It is known [Har97] that π_* induces an equivalence of the categories $\texttt{qcMod}(\mathcal{O}_E)$ and $\texttt{qcMod}(\mathcal{O}_X) \cap \texttt{Mod}(\mathcal{O}_X^E)$, with self-explaining notation. It follows that $\mathcal{O}_X^E \in \texttt{qcMod}(\mathcal{O}_X)$. Moreover, $\mathcal{O}_X^E$ is clearly a unital commutative ring and thus an algebra $\mathcal{O}_X^E \in \texttt{qcCAlg}(\mathcal{O}_X)$. Indeed, such an algebra is a commutative monoid in $\texttt{qcMod}(\mathcal{O}_X)$, i.e., it is an object in $\texttt{qcMod}(\mathcal{O}_X)$ that carries an associative unital commutative multiplication, which is a morphism in $\texttt{qcMod}(\mathcal{O}_X)$. These conditions are obviously satisfied for $\mathcal{O}_X^E$. As for $\mathcal{O}_X$-linearity, note that, if $V \subset X$ is open, $f \in \mathcal{O}_X(V)$ and $F \in \mathcal{O}_X^E(V) = \mathcal{O}_E(\pi^{-1}(V))$, the ring morphism $\pi^\sharp : \mathcal{O}_X(V) \to \mathcal{O}_E(\pi^{-1}(V))$ allows to define the $\mathcal{O}_X$-action by $f \cdot F := \pi^\sharp(f) \star F$, where $\star$ is the ring multiplication. Hence, the multiplication $\star$ is $\mathcal{O}_X(V)$-bilinear, i.e.,

$$\star : \mathcal{O}_X^E(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X^E(V) \to \mathcal{O}_X^E(V)$$

is $\mathcal{O}_X(V)$-linear, and this presheaf morphism induces a sheaf morphism $\star : \mathcal{O}_X^E \otimes_{\mathcal{O}_X} \mathcal{O}_X^E \to \mathcal{O}_X^E$.

7.3.4 Differential operators with coefficients in a $\mathcal{D}$-algebra

Let X be a smooth scheme and let $\mathcal{A} \in \mathtt{qcCAlg}(\mathcal{D}_X)$ with multiplication $\star$ (let us recall that $\mathcal{D}_X$ is generated by the sheaf $\mathcal{O}_X$ of functions and the sheaf Θ_X of vector fields). We denote the action on $a \in \mathcal{A}$ by $f \in \mathcal{O}_X$ (resp., $\theta \in \Theta_X$) by $f \cdot a$ (resp., $\nabla_\theta a$). An element $f \in \mathcal{O}_X$ is viewed as element in $\mathcal{A}$ via the identification $f \simeq f \cdot 1_{\mathcal{A}}$. Hence,

$$f \cdot a = f \cdot (1_{\mathcal{A}} \star a) = (f \cdot 1_{\mathcal{A}}) \star a \simeq f \star a \; . \tag{152}$$

The ring $\mathcal{A}[\mathcal{D}_X]$ of **differential operators with coefficients in $\mathcal{A}$** is the $\mathcal{D}_X$-module

$$\mathcal{A}[\mathcal{D}_X] = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

endowed with the associative unital $\mathbb{R}$-algebra structure $\bullet$ defined, for $a, a' \in \mathcal{A}$, $\theta \in \Theta_X$, and $D \in \mathcal{D}_X$, by

$$(a \otimes 1_{\mathcal{O}}) \bullet (a' \otimes D) = (a \star a') \otimes D \tag{153}$$

and

$$(1_{\mathcal{A}} \otimes \theta) \bullet (a' \otimes D) = (\nabla_\theta a') \otimes D + a' \otimes (\theta \circ D) \; . \tag{154}$$

This multiplication is canonically extended to a first factor of the type

$$a \otimes (f \circ \theta \circ \theta') = ((a \star f) \otimes 1_{\mathcal{O}}) \bullet (1_{\mathcal{A}} \otimes \theta) \bullet (1_{\mathcal{A}} \otimes \theta') \; .$$

It is straightforwardly checked that the usual relations like, e.g., $\theta \circ \theta' = \theta' \circ \theta + [\theta, \theta']$, do not lead to any contradiction. Moreover, the embedding

$$\mathcal{A} \ni a \mapsto a \otimes 1_{\mathcal{O}} \in \mathcal{A}[\mathcal{D}_X]$$

is an associative algebra morphism (i.e., $\mathcal{A}$ is a subalgebra of $\mathcal{A}[\mathcal{D}_X]$), whereas the embedding

$$\Theta_X \ni \theta \mapsto 1_{\mathcal{A}} \otimes \theta \in \mathcal{A}[\mathcal{D}_X]$$

is a Lie algebra morphism (i.e., Θ_X is a Lie subalgebra of $\mathcal{A}[\mathcal{D}_X]$). These inclusions satisfy

$$\theta \bullet a - a \bullet \theta = \nabla_\theta a$$

and $f \bullet \theta = f \circ \theta$, and extend to an associative algebra morphism

$$\mathcal{D}_X \ni D \mapsto 1_{\mathcal{A}} \otimes D \in \mathcal{A}[\mathcal{D}_X] \; .$$

Consider now an algebra $\mathcal{A} \in \mathtt{qcCAlg}(\mathcal{D}_X)$, i.e., a commutative monoid in the symmetric monoidal category $(\mathtt{qcMod}(\mathcal{D}_X), \otimes_{\mathcal{O}_X}, \mathcal{O}_X)$. In the following, it is understood that all modules are left modules. An **$\mathcal{A}$-module in the category $\mathtt{qcMod}(\mathcal{D}_X)$** is an object $\mathcal{M} \in \mathtt{qcMod}(\mathcal{D}_X)$ together with an $\mathcal{A}$-action, i.e., a $\mathcal{D}_X$-linear map $\mu : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ that satisfies the usual action conditions. Of course, $\mathcal{D}_X$-linearity is equivalent to $\mathcal{O}_X$- and Θ_X-linearity. Let $m \in \mathcal{M}$ and set $a \triangleleft m := \mu(a \otimes m)$. Since

$$f \cdot (a \otimes m) = (f \cdot a) \otimes m = a \otimes (f \cdot m)$$

(resp.,

$$\nabla_\theta(a \otimes m) = (\nabla_\theta\, a) \otimes m + a \otimes (\nabla_\theta\, m) \;),$$

$\mathcal{O}_X$-linearity (resp., Θ_X-linearity) of μ means that

$$f \cdot (a \triangleleft m) = (f \cdot a) \triangleleft m = a \triangleleft (f \cdot m) \tag{155}$$

(resp.,

$$\nabla_\theta(a \triangleleft m) = (\nabla_\theta\, a) \triangleleft m + a \triangleleft (\nabla_\theta\, m) \;). \tag{156}$$

In view of (152), Condition (155) means exactly that

$$f \cdot m = f \triangleleft m \; . \tag{157}$$

Remark 9. *In the following, it will be understood that $\mathcal{O}_X \subset \mathcal{A}$ and that the $\mathcal{O}_X$-action on $\mathcal{A}$ (resp., on $\mathcal{M}$) coincides with the $\mathcal{A}$-action.*

The compatibility between the $\mathcal{A}$- and $\mathcal{D}_X$-actions of an $\mathcal{A}$-module in the category of $\mathcal{D}_X$-modules then reduces to the condition (156) requiring that vector fields act on $\triangleleft$ as derivations.

The next result can be found for instance in [BD04].

An $\mathcal{A}$-module in the category $\texttt{qcMod}(\mathcal{D}_X)$ is the same as an $\mathcal{A}[\mathcal{D}_X]$**-module that is quasi-coherent as $\mathcal{O}_X$-module**.

Indeed, an $\mathcal{A}[\mathcal{D}_X]$-action $\diamond$ on $\mathcal{M}$ provides an action $a \triangleleft m \simeq (a \otimes 1_{\mathcal{O}}) \diamond m$ and an action $D \triangleright m \simeq (1_{\mathcal{A}} \otimes D) \diamond m$; conversely, an $\mathcal{A}$-action $\triangleleft$ and a $\mathcal{D}_X$-action $\triangleright$ on $\mathcal{M}$ allow to define an action

$$(a \otimes D) \diamond m = ((a \otimes 1_{\mathcal{O}}) \bullet (1_{\mathcal{A}} \otimes D)) \diamond m \simeq (a \bullet D) \diamond m = a \triangleleft (D \triangleright m) \; . \tag{158}$$

More precisely, assume for instance that we are given an $\mathcal{A}$-module in $\texttt{qcMod}(\mathcal{D}_X)$, and define $\diamond$ from $\triangleleft$ and $\triangleright$ as indicated in (158). In view of (155), this action is well-defined on $\mathcal{A}[\mathcal{D}_X] = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and in view of (156), we get, when taking (154) and (158) into account,

$$((1_{\mathcal{A}} \otimes \theta) \bullet (a \otimes 1_{\mathcal{O}})) \diamond m = (1_{\mathcal{A}} \otimes \theta) \diamond ((a \otimes 1_{\mathcal{O}}) \diamond m) \; .$$

The remaining verifications are left to the reader.

Let now $\mathcal{M}, \mathcal{N}$ be two $\mathcal{A}[\mathcal{D}_X]$-modules that are quasi-coherent as $\mathcal{O}_X$-modules, i.e., two $\mathcal{A}$-modules in $\texttt{qcMod}(\mathcal{D}_X)$. A morphism $f : \mathcal{M} \to \mathcal{N}$ is just an $\mathcal{A}$- and $\mathcal{D}_X$-linear map. Hence,

Proposition 11. *Let X be a smooth scheme and let $\mathcal{A} \in \texttt{qcCAlg}(\mathcal{D}_X)$. The category $\texttt{qcMod}(\mathcal{A}[\mathcal{D}_X])$ of $\mathcal{O}_X$-quasi-coherent $\mathcal{A}[\mathcal{D}_X]$-modules and the category $\texttt{Mod}_{\texttt{qcMod}(\mathcal{D}_X)}(\mathcal{A})$ of $\mathcal{A}$-modules in $\texttt{qcMod}(\mathcal{D}_X)$ coincide.*

If $\mathcal{M}, \mathcal{N} \in \texttt{qcMod}(\mathcal{A}[\mathcal{D}_X])$, the $\mathcal{A}$-module $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is a $\mathcal{D}_X$-module for the canonical $\mathcal{O}_X$- and Θ_X-actions; the $\mathcal{A}$-action is $\mathcal{D}_X$-linear, so that $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \in \texttt{qcMod}(\mathcal{A}[\mathcal{D}_X])$. In fact, the category $(\texttt{qcMod}(\mathcal{A}[\mathcal{D}_X]), \otimes_{\mathcal{A}}, \mathcal{A})$ is symmetric monoidal.

7.3.5 DG algebras over differential operators with coefficients in a $\mathcal{D}$-algebra

A commutative monoid $\mathfrak{A}$ in $(\mathtt{qcMod}(\mathcal{A}[\mathcal{D}_X]), \otimes_{\mathcal{A}}, \mathcal{A})$ is a (quasi-coherent associative unital commutative) $\mathcal{A}[\mathcal{D}_X]$-algebra. More precisely, just as a $\mathcal{D}_X$-algebra is an $\mathcal{O}_X$-algebra and a $\mathcal{D}_X$-module such that vector fields Θ_X act as derivations, an $\mathcal{A}[\mathcal{D}_X]$-algebra is an (associative unital commutative) $\mathcal{A}$-algebra and an $\mathcal{A}[\mathcal{D}_X]$-module $\mathfrak{A} \in \mathtt{qcMod}(\mathcal{A}[\mathcal{D}_X])$ such that vector fields Θ_X act as derivations. In other words, an $\mathcal{A}[\mathcal{D}_X]$-algebra is an $\mathcal{A}$-algebra (say with $\mathcal{A}$-action $\triangleleft$ and multiplication $*$) and a $\mathcal{D}_X$-module $\mathfrak{A} \in \mathtt{qcMod}(\mathcal{D}_X)$ such that vector fields act as derivations on $\triangleleft$ and on $*$. Similarly,

Definition 6. *A* **differential non-negatively graded $\mathcal{A}[\mathcal{D}_X]$-algebra** *is a differential graded commutative $\mathcal{A}$-algebra, as well as a differential graded $\mathcal{D}_X$-module $\mathfrak{A}_\bullet \in \mathtt{DG_+qcMod}(\mathcal{D}_X)$, such that vector fields act as derivations on the $\mathcal{A}$-action on $\mathfrak{A}_\bullet$ and on the multiplication of $\mathfrak{A}_\bullet$. A morphism of DG $\mathcal{A}[\mathcal{D}_X]$-algebras is a morphism of DG $\mathcal{D}_X$-modules that is $\mathcal{A}$-linear and respects the multiplications and the units. The category of DG $\mathcal{A}[\mathcal{D}_X]$-algebras and morphisms between them will be denoted by $\mathtt{DG_+qcCAlg}(\mathcal{A}[\mathcal{D}_X])$.*

In other words, a DG $\mathcal{A}[\mathcal{D}_X]$-algebra is a DG $\mathcal{A}$-algebra, as well as a DG $\mathcal{D}_X$-algebra, such that the $\mathcal{A}$-action and the $\mathcal{D}_X$-action are compatible in the sense that vector fields $\Theta_X \subset \mathcal{D}_X$ act on the $\mathcal{A}$-action $\triangleleft$ as derivations.

Example 1. Let $\mathcal{A}$ be, as above, a $\mathcal{D}_X$-algebra. Any DG $\mathcal{D}_X$-algebra morphism $f : \mathcal{A} \to \mathcal{B}_\bullet$ allows to endow $\mathcal{B}_\bullet$ with a DG $\mathcal{A}[\mathcal{D}_X]$-algebra structure, i.e., to view $\mathcal{B}_\bullet$ as an object $\mathcal{B}_\bullet \in \mathtt{DG_+qcCAlg}(\mathcal{A}[\mathcal{D}_X])$. Indeed, it suffices to set

$$a \triangleleft b := f(a) \star_{\mathcal{B}} b \ ,$$

with self-explaining notation. Verifications are straightforward (see also Remark 9). In particular, $\mathcal{A}$ can be interpreted as DG $\mathcal{A}[\mathcal{D}_X]$-algebra with $\mathcal{A}$-action $\triangleleft$ given by the $\mathcal{A}$-multiplication $\star_{\mathcal{A}}$.

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