A modal logic for games with lies

Bruno Teheux

University of Luxembourg

The RÉNYI – ULAM game

A searching game with lies

1. ALICE chooses an element in $\{1, \ldots, M\}$.
2. BOB tries to guess this number by asking Yes/No questions.
3. ALICE is allowed to lie $n - 1$ times in her answers.

BOB tries to guess ALICE's number as fast as possible.

The game is around for more than 50 years

Finding an optimal strategy :

Pelc, A. Searching games with errors - fifty years of coping with liars, *Theoretical Computer Science* 270 (1) : 71-109.

Using logic and algebras to model the states of the games :

Mundici, D. The logic of Ulam's game with lies. In *Knowledge, belief, and strategic interaction*. Cambridge University Press, 1992.

Tools for the algebraic/logical approach

Static model

Łukasiewicz logic   MV-algebras

Dynamic model

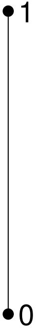
Modal logic   Kripke semantics

Łukasiewicz $(n+1)$-valued logic and MV_n-algebras

$$\mathcal{L} = \{\neg, \rightarrow, 1\}$$

$$\neg x := 1 - x, \qquad x \rightarrow y := \min(1, 1 - x + y)$$

2



$$\phi \in \mathrm{CPL}$$

$$\Longleftrightarrow$$

$$\mathbf{2} \models \phi$$

A is a Boolean algebra

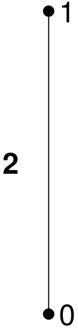
$$\Longleftrightarrow$$

$$\mathbf{A} \models \mathrm{CPL}$$

Łukasiewicz $(n+1)$-valued logic and MV$_n$-algebras

$$\mathcal{L} = \{\neg, \to, 1\}$$

$$\neg x := 1 - x, \qquad x \to y := \min(1, 1 - x + y)$$



$\phi \in$ CPL

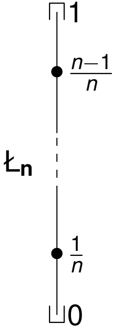
$\iff$

$\mathbf{2} \models \phi$

A is a Boolean algebra

$\iff$

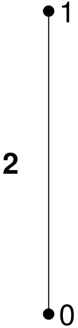
$\mathbf{A} \models$ CPL



Łukasiewicz $(n+1)$-valued logic and MV$_n$-algebras

$$\mathcal{L} = \{\neg, \to, 1\}$$

$$\neg x := 1 - x, \qquad x \to y := \min(1, 1 - x + y)$$



$\phi \in \mathrm{CPL}$

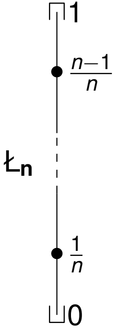
$\iff$

$\mathbf{2} \models \phi$

A is a Boolean algebra

$\iff$

$\mathbf{A} \models \mathrm{CPL}$



$\phi \in \mathrm{PŁ_n}$

$\iff$

$\mathbf{Ł}_n \models \phi$

A is a MV$_n$-algebra

$\iff$

$\mathbf{A} \models \mathrm{PŁ_n}$

Static model

Łukasiewicz logic   MV-algebras

Dynamic model

Modal logic   Kripke semantics

An algebraic encoding of the states of knowledge

1. Knowledge space $K = Ł_n^M$.
2. A state of knowledge $s \in Ł_n^M$: $s(m)$ is the 'distance' between m and the set of numbers that can be discarded.
3. A question Q is a subset of $\{1, \ldots, M\}$.
4. The *positive answer* to Q is the map $f_Q \colon M \to \{\frac{n-1}{n}, 1\}$ defined by

$$f_Q(m) = 1 \iff m \in Q$$

The *negative answer* to Q is the map $f_{M \setminus Q}$.

$$
\begin{aligned}
x \oplus y &:= (\neg x \to y) \\
&= \min(1, x+y) && \text{in } Ł_n
\end{aligned}
$$

$$
\begin{aligned}
x \odot y &:= \neg(\neg x \oplus \neg y) \\
&= \max(0, x+y-1) && \text{in } Ł_n
\end{aligned}
$$

Proposition. If ALICE answers positively to Q at state of knowledge s then the new state of knowledge is

$$s' := s \odot f_Q.$$

Static model

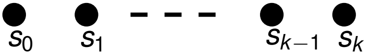
Łukasiewicz logic   MV-algebras

Dynamic model

Modal logic   Kripke semantics

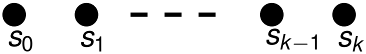
A dynamic model for every instance of the game

The model only talks about *states* of an instance of the game.

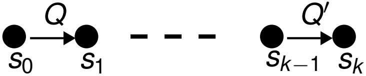


A dynamic model for every instance of the game

The model only talks about *states* of an instance of the game.

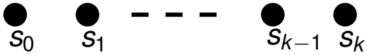


We want a language to talk about *whole instances* of the game.

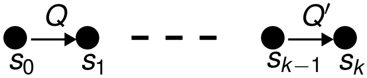


A dynamic model for every instance of the game

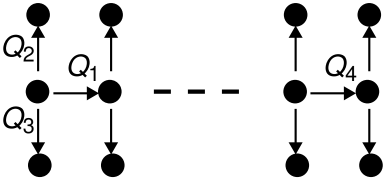
The model only talks about *states* of an instance of the game.



We want a language to talk about *whole instances* of the game.



We want a language to talk about *all instances of any game*.



Static model

Łukasiewicz logic   MV-algebras

Dynamic model

Modal logic   Kripke semantics

We need KRIPKE models with many-valued worlds

$\Pi_0 = \{\text{atomic programs}\}$
$\text{Prop} = \{\text{propositional variables}\}$

Definition. A *(dynamic $n+1$-valued)* KRIPKE *model* is

$$\mathcal{M} = \langle W, R_{\mathbf{.}}, \mathrm{Val} \rangle$$

where

- $W \neq \varnothing$,
- $R_{\mathbf{.}}$ maps any $a \in \Pi_0$ to $R_a \subseteq W \times W$,
- $\mathrm{Val}(u, p) \in \text{Ł}_n$ for any $u \in W$ and any $p \in \text{Prop}$.

The RÉNYI - ULAM game has a KRIPKE model

Language :

- $\text{Prop} = \{p_m \mid m \in M\}$ where

  $p_m \equiv$ how m is far from the set of rejected elements.

- $\Pi_0 = \{m \mid m \in M\}$.

Model :

- $W = Ł_n^M$ is the knowledge space.
- $(s, t) \in R_m$ if $t = s \odot f_{\{m\}}$
- $\mathrm{Val}(s, p_m) = s(m)$.

A modal language for the Kripke models

Programs $\alpha \in \Pi$ and formulas $\phi \in$ Form are defined by

$$\begin{array}{ll} \text{Formulas} & \phi ::= p \mid 0 \mid \phi \to \phi \mid \neg\phi \mid [\alpha]\phi \\ \text{Programs} & \alpha ::= a \mid \phi? \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \end{array}$$

where $p \in$ Prop and $a \in \Pi_0$.

Word	Reading
$\alpha;\beta$	α followed by β
$\alpha \cup \beta$	α or β
α^*	any number of execution of α
$\phi?$	test ϕ
$[\alpha]$	after any execution of α

Interpreting formulas in Kripke Models

$\mathrm{Val}(\cdot,\cdot)$ and $R_{\cdot}$ are extended by induction :

- In a truth functional way for $\neg$ and $\rightarrow$,
- $\mathrm{Val}(u,[\alpha]\psi) := \bigwedge\{\mathrm{Val}(v,\psi) \mid (u,v) \in R_\alpha\}$,
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$ and $R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$,
- $R_{\phi?} = \{(u,u) \mid \mathrm{Val}(u,\phi) = 1\}$,
- $R_{\alpha^*} := (R_\alpha)^* = \bigcup_{k\in\omega} R_\alpha^k$.

Definition. $\mathcal{M}, u \models \phi$ if $\mathrm{Val}(u,\phi) = 1$ and $\mathcal{M} \models \phi$ if $\mathcal{M}, u \models \phi$ for every $u \in W$.

Interpreting formulas in Kripke Models

$\mathrm{Val}(\cdot,\cdot)$ and $R_{\cdot}$ are extended by induction :

- In a truth functional way for $\neg$ and $\rightarrow$,
- $\mathrm{Val}(u,[\alpha]\psi) := \bigwedge\{\mathrm{Val}(v,\psi) \mid (u,v) \in R_\alpha\}$,
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$ and $R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$,
- $R_{\phi?} = \{(u,u) \mid \mathrm{Val}(u,\phi) = 1\}$,
- $R_{\alpha^*} := (R_\alpha)^* = \bigcup_{k\in\omega} R_\alpha^k$.

Definition. $\mathcal{M}, u \models \phi$ if $\mathrm{Val}(u,\phi) = 1$ and $\mathcal{M} \models \phi$ if $\mathcal{M}, u \models \phi$ for every $u \in W$.

$$T_n := \{\phi \mid \mathcal{M} \models \phi \text{ for every Kripke model } \mathcal{M}\}$$

Find an axiomatization of T_n.

Static model

Łukasiewicz logic   MV-algebras

Dynamic model

Modal logic   Kripke semantics

Modal extensions of Łukasiewicz $(n+1)$-valued logic

$$\mathcal{L}_\Box = \{\Box, \neg, \to, 1\}, \qquad \mathcal{M} = \langle W, R, \mathrm{Val} \rangle$$

Theorem. The set $\mathrm{K}_n := \{\phi \in \mathrm{Form}_{\mathcal{L}_\Box} \mid \phi \text{ is a tautology } \}$ is the smallest subset of $\mathrm{Form}_{\mathcal{L}_\Box}$ that

- contains tautologies of Łukasiewicz $(n+1)$-valued logic
- contains (K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- contains

  $$\Box(\phi \star \phi) \leftrightarrow (\Box\phi) \star (\Box\phi)$$

  for $\star \in \{\odot, \oplus\}$
- is closed under MP, substitution and generalization.

Static model

Łukasiewicz logic   MV-algebras

Dynamic model

Modal logic   Kripke semantics

$T_n := \{\phi \mid \mathcal{M} \models \phi$ for every Kripke model $\mathcal{M}\}$

There are three ingredients in the axiomatization

Definition. PDL_n is the smallest set of formulas that contains formulas in Ax_1, Ax_2, Ax_3 and closed for the rules in Ru_1, Ru_2.

Łukasiewicz $n+1$-valued logic	
Ax_1	Axiomatization
Ru_1	MP, uniform substitution

Crisp modal $n+1$-valued logic	
Ax_2	$[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$, $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$, $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$,
Ru_2	$\phi \;/\; [\alpha]\phi$

Program constructions	
Ax_3	$[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$ $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$, $[q?]p \leftrightarrow (\neg q^n \vee p)$ $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$, $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p$, $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p$.

Theorem (Completeness).

$$T_n \ = \ \text{PDL}_n$$

Remark. If $n = 1$, it boils down to PDL (introduced by FISCHER and LADNER in 1979).

Focus on the induction axiom

The formula

$$(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p$$

means

> 'if after an undetermined number of executions of α the truth value of p cannot decrease after a new execution of α, then the truth value of p cannot decrease after any undetermined number of executions of α'.

Focus on the homogeneity axioms

Proposition. The axioms

$$[\alpha](\phi \star \phi) \leftrightarrow ([\alpha]\phi \star [\alpha]\phi), \qquad \star \in \{\odot, \oplus\},$$

can be replaced by *n* axioms that state equivalence between

‘the truth value of $[\alpha]\phi$ is at least $\frac{i}{n}$’

and

‘after any execution of α, the truth-value of ϕ is at least $\frac{i}{n}$’

About the expressive power
of the many-valued modal language.

The ability to distinguish between frames

$$\mathcal{L}_{\Box} = \{\Box, \neg, \to, 1\}$$

Frame : $\mathfrak{F} = \langle W, R \rangle$

$\mathrm{Mod}_n(\Phi) := \{\mathfrak{F} \mid \mathcal{M} \models \Phi$ for any $(n+1)$-valued $\mathcal{M}$ based on $\mathfrak{F}\}$

Definition. A class $\mathcal{C}$ of frames is *Ł$_n$-definable* if there is $\Phi \subseteq \mathrm{Form}_{\mathcal{L}_{\Box}}$ such that

$$\mathcal{C} = \mathrm{Mod}_n(\Phi).$$

The ability to distinguish between frames

$$\mathcal{L}_\Box = \{\Box, \neg, \rightarrow, 1\}$$

Frame : $\mathfrak{F} = \langle W, R \rangle$

$\mathrm{Mod}_n(\Phi) := \{\mathfrak{F} \mid \mathcal{M} \models \Phi$ for any $(n+1)$-valued $\mathcal{M}$ based on $\mathfrak{F}\}$

Definition. A class $\mathcal{C}$ of frames is *Ł$_n$-definable* if there is $\Phi \subseteq \mathrm{Form}_{\mathcal{L}_\Box}$ such that

$$\mathcal{C} = \mathrm{Mod}_n(\Phi).$$

Proposition. If $\mathcal{C}$ is Ł$_1$-definable then it is Ł$_n$-definable for every $n \geq 1$.

Enriching the signature of frames

Theorem. If $\mathcal{C}$ contains ultrapowers of its elements, then

$\mathcal{C}$ is Ł$_n$-definable if and only if $\mathcal{C}$ is Ł$_1$-definable.

The many-valued modal language is not adapted for the signature of frames.

Enriching the signature of frames

Theorem. If $\mathcal{C}$ contains ultrapowers of its elements, then

$$\mathcal{C} \text{ is Ł}_n\text{-definable if and only if } \mathcal{C} \text{ is Ł}_1\text{-definable.}$$

The many-valued modal language is not adapted for the signature of frames.

Definition. An *Ł$_n$-frame* is a structure

$$\mathfrak{F} = \langle W, R, \{r_m \mid m \text{ is a divisor of } n\}\rangle,$$

where $r_m \subseteq W$ for any m. A model $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ is based on $\mathfrak{F}$ if

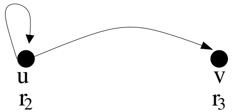
$$u \in r_m \implies \mathrm{Val}(u, \phi) \in \text{Ł}_m$$

Ł$_n$-frames

$$\mathfrak{F} = \langle W, R, \{r_m \mid m \text{ is a divisor of } n\}\rangle,$$

$$u \in r_m \implies \mathrm{Val}(u, \phi) \in Ł_m$$

Example. (Forbidden situation)



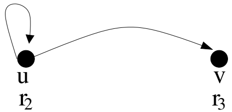
If $\mathrm{Val}(u, p) = 1$ and $\mathrm{Val}(v, p) = 1/3$ then $\mathrm{Val}(u, \Box p) = 1/3 \notin Ł_2$

Ł$_n$-frames

$$\mathfrak{F} = \langle W, R, \{r_m \mid m \text{ is a divisor of } n\}\rangle,$$

$$u \in r_m \implies \mathrm{Val}(u, \phi) \in Ł_m$$

Example. (Forbidden situation)



If $\mathrm{Val}(u, p) = 1$ and $\mathrm{Val}(v, p) = 1/3$ then $\mathrm{Val}(u, \Box p) = 1/3 \notin Ł_2$

Additional conditions on Ł$_n$-frames :

- $r_m \cap r_k = r_{\mathrm{gcd}(m,k)}$
- $u \in r_m \Longrightarrow Ru \subseteq r_m$

Goldblatt - Thomason theorem

$\mathcal{C} \cup \{\mathfrak{F}\}$ = class of Ł$_n$-frames

$\mathrm{Mod}(\Phi) := \{\mathfrak{F} \mid \mathcal{M} \models \Phi$ for any $(n+1)$-valued $\mathcal{M}$ based on $\mathfrak{F}\}$

Definition. A class $\mathcal{C}$ of Ł$_n$-frames is *definable* if there is $\Phi \subseteq \mathrm{Form}_{\mathcal{L}_\Box}$ such that

$$\mathcal{C} = \mathrm{Mod}(\Phi).$$

Goldblatt - Thomason theorem

$\mathcal{C} \cup \{\mathfrak{F}\}$ = class of Ł$_n$-frames

$\mathrm{Mod}(\Phi) := \{\mathfrak{F} \mid \mathcal{M} \models \Phi$ for any $(n+1)$-valued $\mathcal{M}$ based on $\mathfrak{F}\}$

Definition. A class $\mathcal{C}$ of Ł$_n$-frames is *definable* if there is $\Phi \subseteq \mathrm{Form}_{\mathcal{L}_\Box}$ such that

$$\mathcal{C} = \mathrm{Mod}(\Phi).$$

Theorem. If $\mathcal{C}$ is closed under ultrapowers, then the following conditions are equivalent.

- $\mathcal{C}$ is definable
- $\mathcal{C}$ is closed under Ł$_n$-generated subframes, Ł$_n$-bounded morphisms, disjoint unions and reflects canonical extensions.

What to do next ?

1. Shows that PDL_n can actually help in stating many-valued program specifications.
2. There is an epistemic interpretation of PDL. Can it be generalized to the $n + 1$-valued realm ?
3. What happens if KRIPKE models are not crisp.
4. Can coalgebras explain why PDL and PDL_n are so related ?