# DOMINATING SURFACE GROUP REPRESENTATIONS AND DEFORMING CLOSED ANTI-DE SITTER 3-MANIFOLDS

NICOLAS THOLOZAN

Abstract. Let $S$ be a closed oriented surface of negative Euler characteristic and $M$ a complete contractible Riemannian manifold. A Fuchsian representation $j : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2)$ *strictly dominates* a representation $\rho : \pi_1(S) \to \mathrm{Isom}(M)$ if there exists a $(j, \rho)$-equivariant map from $\mathbb{H}^2$ to $M$ that is $\lambda$-Lipschitz for some $\lambda < 1$. In a previous paper by Deroin–Tholozan, the authors construct a map $\mathbf{\Psi}_\rho$ from the Teichmüller space $\mathcal{T}(S)$ of the surface $S$ to itself and prove that, when $M$ has sectional curvature $\leq -1$, the image of $\mathbf{\Psi}_\rho$ lies (almost always) in the domain $\mathrm{Dom}(\rho)$ of Fuchsian representations stricly dominating $\rho$. Here we prove that $\mathbf{\Psi}_\rho : \mathcal{T}(S) \to \mathrm{Dom}(\rho)$ is a homeomorphism. As a consequence, we are able to describe the topology of the space of pairs of representations $(j, \rho)$ from $\pi_1(S)$ to $\mathrm{Isom}^+(\mathbb{H}^2)$ with $j$ Fuchsian strictly dominating $\rho$. In particular, we obtain that its connected components are classified by the Euler class of $\rho$. The link with anti-de Sitter geometry comes from a theorem of Kassel stating that those pairs parametrize deformation spaces of anti-de Sitter structures on closed 3-manifolds.

## Introduction

0.1. **Closed AdS 3-manifolds.** An *anti-de Sitter* (AdS) manifold is a smooth manifold equipped with a Lorentz metric of constant negative sectional curvature. In dimension 3, those manifolds are locally modelled on $\mathrm{PSL}(2, \mathbb{R})$ with its Killing metric, whose isometry group identifies to a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ acting by left and right translations, i.e.

$$(g_1, g_2) \cdot x = g_1 x g_2^{-1}$$

for all $(g_1, g_2) \in \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and all $x \in \mathrm{PSL}(2, \mathbb{R})$.

The connection between closed anti-de Sitter 3-manifolds and surface group representations was established by the following theorem of Kulkarni and Raymond:

**Theorem** (Kulkarni–Raymond, [15]). *Every closed complete anti-de Sitter 3-manifold has, up to finite cover, the form*

$$j \times \rho(\Gamma) \backslash \mathrm{PSL}(2, \mathbb{R}) \ ,$$

*where $\Gamma$ is the fundamental group of a closed surface of negative Euler characteristic and $j$ and $\rho$ two representations of $\Gamma$ into $\mathrm{PSL}(2, \mathbb{R})$, one of which is Fuchsian (i.e. discrete and faithful).*

*Remark* 0.1. Contrary to the Riemannian setting, a Lorentz metric on a closed manifold may not be complete. However, Klingler proved later [14], generalizing Carrière's theorem [2], that closed Lorentz manifolds of constant curvature are complete. Hence the completeness assumption in Kulkarni–Raymond's work can be removed.

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A pair of representations $(j,\rho)$ such that $j\times\rho(\Gamma)$ does act properly discontinuously and cocompactly on $\mathrm{PSL}(2,\mathbb{R})$ is called an *admissible pair*. The first examples of such pairs are when (up to switching the factors) $j$ is Fuchsian and $\rho$ takes values in $\mathrm{SO}(2)$. The resulting AdS 3-manifold is then called *standard*. Goldman constructed in [9] the first examples of non-standard AdS 3-manifolds by deforming standard ones. Later, Salein [17] found a sufficient properness criterion for the action of $j\times\rho(\Gamma)$, allowing him to construct admissible pairs with $\rho$ of any non-extremal Euler class (and thus non-standard AdS 3-manifolds that cannot be deformed into standard ones). Finally, Kassel proved in her thesis that Salein's criterion is also necessary, leading to the following characterization of admissible pairs. Recall that $\mathrm{PSL}(2,\mathbb{R})$ is the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$.

**Theorem** (Salein [17], Kassel [13])**.** *Let $S$ be a closed surface of negative Euler claracteristic and $j,\rho$ two representations of $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, with $j$ Fuchsian. Then the pair $(j,\rho)$ is admissible if an only if there exists a $(j,\rho)$-equivariant map from $\mathbb{H}^2$ to $\mathbb{H}^2$ that is $\lambda$-Lipschitz for some $\lambda<1$.*

The primary purpose of this paper is to describe the space of pairs $(j,\rho)$ statisfying this admissibility criterion, leading to a description of the "deformation space" of anti-de Sitter 3-manifolds.

0.2. **Dominated representations.** Kassel's criterion for admissibility raises many questions that may turn out to be interesting beyond the scope of 3-dimensional AdS geometry. Consider more generally a closed surface $S$ of negative Euler characteristic, and a contractible Riemannian manifold $(M,g_M)$. Let $j:\pi_1(S)\to\mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian representation, and $\rho$ a representation of $\pi_1(S)$ into $\mathrm{Isom}(M,g_M)$. Since $M$ is contractible, there always exists a smooth $(j,\rho)$-equivariant map from $\mathbb{H}^2$ to $M$. Since $S$ is compact, this map is Lipschitz. We can thus define the *minimal Lipschitz constant*

$$\mathrm{Lip}(j,\rho)=\inf\left\{\lambda\in\mathbb{R}_+\mid \exists f:\mathbb{H}^2\to M\ (j,\rho)\text{-equivariant and }\lambda\text{-Lispchitz}\right\}\ .$$

We will say that $j$ *strictly dominates* $\rho$ if $\mathrm{Lip}(j,\rho)<1$.

Let $\mathrm{Rep}(S,G)$ denote the quotient of the space of representations of $\pi_1(S)$ into a group $G$ by the conjugation action of $G$. This quotient is a (not necessarily Hausdorff) topological space. Let $\mathcal{T}(S)$ denote the *Teichmüller space* of $S$, which we see as the connected component of $\mathrm{Rep}(S,\mathrm{PSL}(2,\mathbb{R})$ containing Fuchsian representations of maximal Euler class. The function Lip is clearly invariant by conjugation of $j$ and $\rho$. It can thus be seen as map from $\mathcal{T}(S)\times\mathrm{Rep}(S,\mathrm{Isom}(M))$ to $\mathbb{R}_+$. The following proposition is proved by Guéritaud–Kassel in [10][1].

**Proposition 0.2** (Guéritaud–Kassel)**.** *The function*

$$\mathrm{Lip}:\mathcal{T}(S)\times\mathrm{Rep}(S,\mathrm{Isom}(M))\to\mathbb{R}_+$$

*is continuous.*

Given a representation $\rho:\pi_1(S)\to\mathrm{Isom}(M)$, it is natural to ask whether $\rho$ can be dominated by a Fuchsian representation and, if so, what the domain of Fuchsian representations dominating $\rho$ looks like. The first question was answered by Deroin and the author in [5] when $(M,g_M)$ is a Riemannian $\mathrm{CAT}(-1)$ space. This applies for instance when $M$ is the

[1] In this paper the context is slightly different. Their proof was adapted in our context in the author's thesis ([20, section 3.1]).

symmetric space of a simple Lie group of real rank 1 (with a suitable normalization of the metric), and in particular for representations in $\mathrm{PSL}(2,\mathbb{R})$. In that case, it was obtained independently and with other methods by Guéritaud–Kassel–Wolff [11].

**Theorem** (Deroin–Tholozan)**.** *Let $S$ be a closed connected oriented surface of negative Euler characteristic, $(M, g_M)$ a Riemannian $\mathrm{CAT}(-1)$ space and $\rho$ a representation of $\pi_1(S)$ into $\mathrm{Isom}(M)$. Then either $\rho$ is Fuchsian in restriction to some stable totally geodesic 2-plane of curvature $-1$ embedded in $M$, or there exists a Fuchsian representation that strictly dominates $\rho$.*

*Remark* 0.3. Note that, by a simple volume argument (see [21], proposition 2.1), a Fuchsian representation cannot be strictly dominated. Thus the theorem is optimal.

Let us denote by $\mathrm{Dom}(\rho)$ the domain of $\mathcal{T}(S)$ consisting of representations of Euler class $-\chi(S)$ strictly dominating a given representation $\rho$, and by $\mathrm{Dom}(S, \mathrm{Isom}(M))$ the domain of pairs $(j, \rho) \in \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M))$ such that $j$ strictly dominates $\rho$. By continuity of the function Lip, $\mathrm{Dom}(\rho)$ and $\mathrm{Dom}(S, \mathrm{Isom}(M))$ are open domains. To prove their theorem, Deroin and the author consider in [5] a certain map $\mathbf{\Psi}_\rho : \mathcal{T}(S) \to \mathcal{T}(S)$ and show that its image lies in $\mathrm{Dom}(\rho)$ (see section 2.1). Here, we prove that this map is a homeomorphism from $\mathcal{T}(S)$ to $\mathrm{Dom}(\rho)$ and that it varies continuously with $\rho$. We thus obtain the following:

**Theorem 1.** *Let $S$ be a closed oriented surface of negative Euler characteristic and $(M, g_M)$ a Riemannian $\mathrm{CAT}(-1)$ space. Then the domain $\mathrm{Dom}(S, \mathrm{Isom}(M))$ is homeomorphic to*

$$\mathcal{T}(S) \times \mathrm{Rep}_{nf}(S, \mathrm{Isom}(M)) \ ,$$

*where $\mathrm{Rep}_{nf}(S, \mathrm{Isom}(M))$ denotes the domain of $\mathrm{Rep}(S, \mathrm{Isom}(M))$ consisting of representations that are not Fuchsian in restriction to some totally geodesic plane of curvature $-1$.*

*What's more, this homeomorphism is fiberwise, meaning that it restricts to a homeomorphism from $\mathcal{T}(S) \times \{\rho\}$ to $\mathrm{Dom}(\rho) \times \{\rho\}$ for any $\rho \in \mathrm{Rep}_{nf}(S, \mathrm{Isom}(M, g_M))$.*

Specializing to the case where $M$ is the hyperbolic plane $\mathbb{H}^2$, we obtain a topological description of the space of admissible pairs, and in particular, we classify its connected components. Recall that, by the work of Goldman, the space $\mathrm{Rep}(S, \mathrm{PSL}(2,\mathbb{R}))$ has $2|\chi(S)| + 1$ connected components, classified by the Euler class [8]. Moreover, Fuchsian representations are exactly the representations of Euler class $\pm\chi(S)$. Let us denote by $\mathrm{Rep}_k(S, \mathrm{PSL}(2,\mathbb{R}))$ the connected component of representations of Euler class $k$. The precise topology of $\mathrm{Rep}_k(S, \mathrm{PSL}(2,\mathbb{R}))$ has been discribed by Hitchin [12].

**Corollary 2.** *The space $\mathrm{Dom}(S, \mathrm{PSL}(2,\mathbb{R}))$ of admissible pairs is homeomorphic to*

$$\mathcal{T}(S) \times \bigsqcup_{\chi(S) < k < -\chi(S)} \mathrm{Rep}_k(S, \mathrm{PSL}(2,\mathbb{R})) \ .$$

*In particular, it has $2|\chi(S)| - 1$ connected components, classified by the Euler class of the non-Fuchsian representation in each pair.*

This answers Question 2.2 of the survey *Some open questions in anti-de Sitter geometry* [1].

0.3. **Application to Thurston's asymetric distance.** Note that the map $\mathbf{\Psi}_\rho$ depends non trivially on the choice of a normalization of the metric on $M$. Fix a metric $g_0$ on $M$ of sectional curvature $\leq -1$ and a constant $\alpha \geq 1$. Then the metric $\frac{1}{\alpha^2} g_0$ still has sectional curvature $\leq -1$. To mark the dependence of the function Lip on the metric on the target, we denote by

$\mathrm{Lip}_g(j,\rho)$ the minimal Lipschitz constant of a $(j,\rho)$-equivariant map from $(\mathbb{H}^2, g_P)$ to $(M,g)$. Then we clearly have $\mathrm{Lip}_{\frac{1}{\alpha^2}g_0} = \frac{1}{\alpha}\mathrm{Lip}_{g_0}$. Hence, if we apply Theorem 1 to $(M, \frac{1}{\alpha^2}g_0)$, we obtain a description of the space of pairs $(j,\rho) \in \mathcal{T}(S) \times \mathrm{Rep}(\pi_1(S), G)$ such that $\mathrm{Lip}_{g_0}(j,\rho) < \alpha$. For $\alpha > 1$, the curvature of $(M, \frac{1}{\alpha^2}g_0)$ is eveywhere $< -1$. Therefore, there is no totally geodesic plane of curvature $-1$ in $(M, \frac{1}{\alpha^2}g_0)$ and we obtain the following:

**Corollary 3.** *Let $S$ be a closed oriented surface of negative Euler characteristic, $(M, g_M)$ a Riemannian $\mathrm{CAT}(-1)$ space and $\alpha$ a constant bigger than $1$. Then the domain*

$$\left\{(j,\rho) \in \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) \mid \mathrm{Lip}_{g_M}(j,\rho) < \alpha\right\}$$

*is fiberwise homeomorphic to*

$$\mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) \ .$$

When applied to $M = (\mathbb{H}^2, g_P)$ (where $g_P$ denotes the Poincaré metric), we obtain a description of left open balls for Thurston's *asymmetric distance* on $\mathcal{T}(S)$. Recall that this distance, introduced in [21], is defined by

$$d_{Th}(j,j') = \log\left(\mathrm{Lip}_{g_P}(j,j')\right)$$

for $j$ and $j'$ any two representations of Euler class $-\chi(S)$. The function $d_{Th}$ is continuous, positive whenever $j$ and $j'$ are distinct, and satisfies the triangular inequality. However, it is not symmetric.

Fix a point $j_0$ in $\mathcal{T}(S)$ and a constant $C > 0$. Then, using the convexity of length functions on $\mathcal{T}(S)$ [25] – together with an alternative definition of the asymmetric distance – one can see that the domain

$$\{j \in \mathcal{T}(S) \mid d_{Th}(j_0, j) < C\}$$

is an open convex domain of $\mathcal{T}(S)$ for the Weil–Petersson metric. In particular, it is homeomorphic to a ball of dimension $-3\chi(S)$. Since the distance is asymmetric, it is not clear whether the same holds for

$$\{j \in \mathcal{T}(S) \mid d_{Th}(j, j_0) < C\} \ .$$

However, since

$$d_{Th}(j, j_0) < C \Leftrightarrow \mathrm{Lip}(j, j_0) < e^C \ ,$$

we can apply corollary 3 and obtain:

**Corollary 4.** *Let $S$ be a closed oriented surface of negative Euler characteristic and $j_0$ a point in $\mathcal{T}(S)$. Then for any $C > 0$, the domain*

$$\{j \in \mathcal{T}(S) \mid d_{Th}(j, j_0) < C\}$$

*is homeomorphic to $\mathcal{T}(S)$. In other words, left open balls for Thurston's asymetric distance on $\mathcal{T}(S)$ are contractible.*

0.4. **Admissible pairs and anti-de Sitter 3-manifolds.** We conclude this introduction by a few remarks on the relation between the space of admissible pairs and the deformation space of anti-de Sitter 3-manifolds. Some more details are given in the author's thesis [20, section 4.4].

**Topology of the quotients.** Let $\mathrm{Adm}_k(S)$ denote the space of admissible pairs $(j, \rho)$ with $j$ Fuchsian of positive Euler class and $\rho$ of Euler class $k$, modulo the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ by conjugation. If $(j, \rho)$ is in $\mathrm{Adm}_k(S)$, then the quotient

$$j \times \rho(\pi_1(S))\backslash\mathrm{PSL}(2,\mathbb{R})$$

is a circle bundle over $S$ of Euler class $|\chi(S)| - k$. In particular, different values of $k$ lead to non homeomorphic anti-de Sitter 3-manifolds.

**Deformation space.** Let $E$ be a circle bundle over $S$ of Euler class $|\chi(S)| - k$, with $\chi(S) < k < -\chi(S)$. The *deformation space* of anti-de Sitter structures on $E$ is the space of anti-de Sitter metrics on $E$ modulo isotopy. All these anti-de Sitter structures arise from identifying $E$ with a quotient of a *finite cover* of $\mathrm{PSL}(2,\mathbb{R})$. It follows from Corollary 2 that the connected components of this deformation space are essentially classified by the degree of this cover. Among these connected components, the one corresponding to quotients of $\mathrm{PSL}(2,\mathbb{R})$ is homeomorphic to an abelian covering of $\mathrm{Adm}_k(S) \simeq \mathcal{T}(S) \times \mathrm{Rep}_k(S)$. (This abelian covering comes from the fact that the group of isotopy classes of bundle isomorphisms of $E$ is $\mathbb{Z}^{2g}$, where $g$ is the genus of $S$.)

**AdS circle bundles of higher Euler class.** A circle bundle over $S$ of Euler class $c \geq 2|\chi(S)|$ also admits anti-de Sitter structures. However, it is known since the work of Kulkarni–Raymond [15] that these structures are all standard. To construct them, one can lift the action on the left of a Fuchsian representation to the covering of degree $|\chi(S)|$ of $\mathrm{PSL}(2,\mathbb{R})$, and then quotient on the right by a cyclic subgroup of order $c$.

**Some deformation spaces are not Hausdorff.** Note that $\mathrm{Rep}_0(S)$ and thus $\mathrm{Adm}_0(S)$ is not Hausdorff, because the orbit under conjugation of a non-abelian parabolic representation (i.e. fixing a point at infinity in $\mathbb{H}^2$) is not closed. This actually reflects an irregularity of some deformation space: if $(E, g)$ is an anti-de Sitter 3-manifold isometric to

$$j \times \rho(\pi_1(S))\backslash\mathrm{PSL}(2,\mathbb{R})$$

with $\rho$ parabolic and non-abelian, then there exists a sequence $\varphi_n$ of isotopies of $E$ such that $\varphi_n^* g$ converges to an anti-de Sitter metric on $E$ that is not isometric to $g$. This phenomenon could not happen if the metric $g$ were Riemannian.

**Moduli space.** The isomorphism that we will construct between $\mathcal{T}(S) \times \mathrm{Rep}_k(S, \mathrm{PSL}(2,\mathbb{R}))$ and $\mathrm{Adm}_k(S)$ is naturally equivariant with respect to the diagonal action of the mapping class group of $S$. The quotient by this diagonal action parametrizes a connected component of the *moduli space* of anti-de Sitter 3-manifolds homeomorphic to a circle bundle over $S$ of Euler class $|\chi(S)| - k$.

**Reversing the orientation of $\rho$.** There is a curious duality between closed anti-de Sitter 3-manifolds. Let $\sigma : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ denote the conjugation by an orientation reversing isometry of $\mathbb{H}^2$. If $(j, \rho)$ is an admissible pair, then obviously $(j, \sigma \circ \rho)$ is also an admissible pair. However, since $\mathbf{eu}(\sigma \circ \rho) = -\mathbf{eu}(\rho)$, the anti-de Sitter 3-manifolds

$$j \times \rho(\pi_1(S))\backslash\mathrm{PSL}(2,\mathbb{R})$$

and

$$j \times (\sigma \circ \rho)(\pi_1(S))\backslash \mathrm{PSL}(2,\mathbb{R})$$

are not homeomorphic when $\mathbf{eu}(\rho) \neq 0$.

0.5. **Structure of the article and strategy of the proof.** The article is organized as follows. In the next section we recall some fundamental results about harmonic maps from a surface. In particular, we recall the Corlette–Labourie theorem of existence of equivariant harmonic maps and the Sampson–Hitchin–Wolf parametrization of the Teichmüller space by means of quadratic differentials. In the second section, we start by using those theorems to construct the map $\mathbf{\Psi}_\rho$ studied in [5], and we prove that $\mathbf{\Psi}_\rho$ is a homeomorphism from $\mathcal{T}(S)$ to $\mathrm{Dom}(\rho)$. To do so, we make explicit the inverse of the map $\mathbf{\Psi}_\rho$. It will appear that reverse images of a point $j$ in $\mathcal{T}(S)$ by $\mathbf{\Psi}_\rho$ are exactly critical points of a certain functional $\mathbf{F}_{j,\rho}$. We will prove that when $j$ is in $\mathrm{Dom}(\rho)$, the functional $\mathbf{F}_{j,\rho}$ is proper and admits a unique critical point which is a global minimum. Hence the map $\mathbf{\Psi}_\rho$ is bijective. What's more, the functionals $\mathbf{F}_{j,\rho}$ vary continuously with $(j,\rho)$, and so does their unique minimum. This will prove the continuity of $\mathbf{\Psi}_\rho^{-1}$ and its continuous dependance in $\rho$.

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## 1. Representations of surface groups, harmonic maps, and functionals on Teichmüller space

In this section, we introduce briefly the tools from the theory of harmonic maps that we will need later. We refer to [4] for a more thorough survey.

1.1. **Existence theorems.** Recall that the *energy density* of a non-negative symmetric 2-form $h$ on a Riemannian manifold $(X,g)$ is the function on $X$ defined as

$$e_g(h) = \frac{1}{2}\mathrm{Tr}(A(h)) \ ,$$

where $A(h)$ is the unique field of endomorphisms of the tangent bundle such that for all $x \in X$ and all $u,v \in T_xX$,

$$h_x(u,v) = g_x(u, A(h)_x v) \ .$$

If $f$ is a map between two Riemannian manifolds $(X,g_X)$ and $(Y,g_Y)$, then its energy density is the energy density on $X$ of the pull-back metric:

$$e_{g_X}(f) = e_{g_X}(f^* g_Y) \ .$$

This energy density can be integrated against the volume form $vol_{g_X}$ associated to $g_X$, giving the *total energy* of $f$:

$$E_{g_X}(f) = \int_X e_{g_X}(f) vol_{g_X} \ .$$

The map $f$ is called *harmonic* if it is, in some sense, a critical point of the total energy. For instance, harmonic maps from $\mathbb{R}$ to a Riemannian manifold are geodesics and harmonic maps from a Riemannian manifold to $\mathbb{R}$ are harmonic functions. In general, a map $f$ is harmonic if it satisfies a certain partial differential equation that can be expressed as the vanishing of the *tension field*. Here we will be satisfied with existence results and some fundamental properties of these maps.

The first existence result, due to Eells and Sampson [7], deals with harmonic maps between compact manifolds. Their paper contains a thorough study of the analytic aspects of harmonicity, which allows to extend their theorem in several cases. The one we will be interested in is an equivariant version.

Consider $(X, g_X)$ a closed Riemannian manifold, $(\widetilde{X}, \widetilde{g}_X)$ its universal cover, $(M, g_M)$ another Riemannian manifold and $\rho$ a representation of $\pi_1(X)$ into $\mathrm{Isom}(M)$. A map $f : \widetilde{X} \to M$ is called *$\rho$-equivariant* if for all $x \in \widetilde{X}$ and all $\gamma \in \pi_1(X)$, we have

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x).$$

Given such a map, the symmetric 2-form $f^* g_M$ on $\widetilde{X}$ is invariant under the action of $\pi_1(X)$ and thus induces a symmetric 2-form on $X$. We define the energy density and the total energy of $f$ as the energy density and the total energy of this symmetric 2-form.

The equivariant version of Eells–Sampson's theorem is due to Donaldson [6] and Corlette [3] in the specific case where $M$ is a symmetric space of non-compact type, and to Labourie [16] in the more general case of a complete simply connected Riemannian manifold of non-positive curvature. We state it in the particular case where $(M, g_M)$ is a Riemannian $\mathrm{CAT}(-1)$ space (i.e. a complete simply connected Riemannian manifold of sectional curvature $\leq -1$). Recall that a $\mathrm{CAT}(-1)$ space is in particular *Gromov hyperbolic*. It thus admits a *boundary at infinity*, such that every isometry of $M$ extends to a homeomorphism of the boudary.

**Theorem** (Donaldson, Corlette, Labourie)**.** *Let $(X, g_X)$ be a closed Riemannian manifold, $\widetilde{X}$ its universal cover, and $(M, g_M)$ a Riemannian $\mathrm{CAT}(-1)$ space. Let $\rho$ be a representation of $\pi_1(X)$ into $\mathrm{Isom}(M)$. Assume that $\rho$ does not fix a point in $\partial_\infty M$. Then there exists a unique harmonic map from $(\widetilde{X}, g_{\widetilde{X}})$ to $(M, g_M)$ that is $\rho$-equivariant. This map minimizes the energy among all such equivariant maps.*

1.1.1. *Dealing with representations fixing a point at infinity.* In the sequel, representations fixing a point in $\partial_\infty M$ will be called *parabolic representations*. When $M$ is the symmetric space of some rank 1 Lie group $G$, those are exactly the representations taking values into some parabolic subgroup of $G$. For some of these representations, the theorem of Corlette and Labourie does not apply, and we must say a few words in order to be able to include them in the proof later.

The main problem with a parabolic representation $\rho$ is that a sequence of equi-Lipschitz $\rho$-equivariant maps from $\widetilde{X}$ to $M$ may not converge. To deal with them, one can "replace" them by representations into the group of translations of the real line. More precisely, if $p$ is a fixed point of $\rho$ in $\partial_\infty M$, consider $B_p$ a Buseman function centered at $p$ and set

$$\begin{array}{rccl} m_\rho : & \pi_1(X) & \to & \mathbb{R} \\ & \gamma & \mapsto & B_p(\gamma \cdot x) - B_p(x) \end{array}$$

for some base point $x$. Then $m_\rho$ is a homomorphism into the group of translations of the real line, and for any Fuchsian representation $j$, one has

$$\mathrm{Lip}(j, \rho) = \mathrm{Lip}(j, m_\rho) \ .$$

Moreover, if $f_n$ is a sequence of smooth $\rho$-equivariant maps from $\widetilde{X}$ to $M$ whose energy converges to the infimum of the energies of all such equivariant maps, then the symmetric 2-form $f_n^* g_M$ converges in $L^2$-norm to

$$f^* \mathrm{d}x^2 ,$$

where $f$ is a $m_\rho$-equivariant harmonic function from $\widetilde{X}$ to $\mathbb{R}$ and $\mathrm{d}x^2$ is the canonical metric on $\mathbb{R}$. (The function $f$ is obtained by integrating the unique harmonic 1-form on $X$ whose periods are given by $m_\rho$, and is unique up to a translation.)

In some sense, this $m_\rho$-equivariant function $f$ is the natural extension of the notion of $\rho$-equivariant harmonic maps to parabolic representations. In particular, when $\rho$ fixes two points in $\partial_\infty M$, it stabilizes a unique geodesic $c : \mathbb{R} \to M$; in that case, $\rho$-equivariant harmonic maps exist and have the form

$$c \circ f .$$

1.2. **Harmonic maps from a surface, Hopf differential, and Teichmüller space.** From now on we will restrict to harmonic maps from a Riemann surface. In that case, the energy of a map only depends on the conformal class of the Riemannian metric on the base. For the same reason, harmonicity is invariant under a conformal change of the metric.

1.2.1. *Hopf differential.* Let $S$ be an oriented surface equipped with a Riemannian metric $g$. Let $(M, g_M)$ be a Riemannian manifold, and $f : S \to M$ a smooth map. The conformal class of $g$ induces a complex structure on $S$. The symmetric 2-form $f^* g_M$ can thus be uniquely decomposed into a $(1,1)$ part, a $(2,0)$ part and a $(0,2)$ part. One can check that the $(1,1)$ part is $e_g(f)g$, and we thus have

$$f^* g_M = e_g(f) g + \Phi_f + \bar{\Phi}_f ,$$

where $\Phi_f$ is a *quadratic differential* (i.e. a section of the square of the canonical bundle of $(S,[g])$ called the *Hopf differential* of $f$.

The following proposition is classical.

**Proposition 1.1.** *If the map $f$ is harmonic, then its Hopf differential is holomorphic. The converse is true if $M$ is also a surface.*

1.2.2. *The Teichmüller space.* Let $S$ be a closed oriented surface of negative Euler characteristic. Recall that the Teichmüller space of $S$ can be seen alternatively as the space of complex structures on $S$ modulo isotopy, the space of hyperbolic structures on $S$ modulo isotopy or the space of Fuchsian representations of positive Euler class modulo conjugation. Throughout the paper, a point in $\mathcal{T}(S)$ is denoted alternately by the letter $X$ when we think of it as the surface $S$ equipped with a complex structure, or by the letter $j$ when we think of it as a Fuchsian representation.

It is well known that the Teichmüller space is a manifold diffeomorphic $\mathbb{R}^{-3\chi(S)}$ and that it carries a complex structure. Consider two points $X_1$ and $X_2$ in $\mathcal{T}(S)$ corresponding to two hyperbolic metrics $g_1$ and $g_2$ on $S$. Then, by Eells–Sampson's theorem, there is a unique harmonic map $f_{g_1,g_2} : (S, g_1) \to (S, g_2)$ homotopic to the identity map. Schoen–Yau's theorem [19] (also proved by Sampson [18] in that specific case) states that this map is a diffeomorphism.

Of course, the map $f_{g_1,g_2}$ depends on the choice of $g_1$ and $g_2$ up to isotopy. Actually, fixing $g_1$ or $g_2$, one can choose the other metric so that the identity map itself is harmonic (by replacing $g_2$ by $f^*_{g_1,g_2}g_2$ or $g_1$ by $f^*_{g_1,g_2}g_1$). On the other hand, the total energy of $f_{g_1,g_2}$ is invariant under changing one of the metrics by isotopy, and thus gives a well defined functional

$$\mathbf{E} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}_+ \ .$$

Besides, the Hopf differential of $f_{g_1,g_2}$ is invariant under isotopic changes of $g_2$, and if $h$ is a diffeomorphism of $S$, one has

$$\Phi_{f_{h^*g_1,g_2}} = h^*\Phi_{f_{g_1,g_2}} \ .$$

The Hopf differential thus induces a well defined map

$$\mathbf{\Phi} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathrm{QD}\mathcal{T}(S) \ ,$$

where $\mathrm{QD}\mathcal{T}(S)$ denotes the complex bundle of holomorphic quadratic differentials on $\mathcal{T}(S)$, that is, $\mathrm{QD}_X\mathcal{T}(S)$ is the space of holomorphic quadratic differentials on $X$, which identifies to the cotangent bundle to $\mathcal{T}(S)$.

Sampson [18] proved that the map $\mathbf{\Phi}$ is an injective immersion, and Wolf [24] proved its surjectivity. This was obtained independently by Hitchin [12], as the first construction of a section to the *Hitchin fibration.*

**Theorem** (Sampson, Hitchin, Wolf)**.** *The map* $\mathbf{\Phi} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathrm{QD}\mathcal{T}(S)$ *is a homeomorphism.*

1.3. **Equivariant harmonic maps and functionals on $\mathcal{T}(S)$.** The theorem of Corlette and Labourie allows to extend the maps $\mathbf{E}$ and $\mathbf{\Phi}$ to $\mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M))$. Given a point in $X_0 \in \mathcal{T}(S)$, represented by a hyperbolic metric $g_0$, and a point $\rho$ in $\mathrm{Rep}(\pi_1(S), \mathrm{Isom}(M))$, one can consider $f_{g_0,\rho}$ the unique $\rho$-equivariant harmonic map from $(\widetilde{S}, \widetilde{g}_0)$ to $(M, g_M)$. (If $\rho$ is parabolic, $f_{g_0,\rho}$ will denote instead a $m_\rho$-equivariant harmonic function, see section 1.1.1.)

The energy density and the Hopf differential of $f_{g_0,\rho}$ only depend on the conjugacy class of the representation $\rho$. We thus obtain two well-defined maps

$$\begin{array}{rrcl} \mathbf{E} : & \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) & \to & \mathbb{R}_+ \\ & (X_0, \rho) & \mapsto & \int_S e_{g_0}(f_{g_0,\rho}) vol_{g_0} \end{array}$$

and

$$\begin{array}{rrcl} \mathbf{\Phi} : & \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) & \to & \mathrm{QD}\mathcal{T}(S) \\ & (X_0, \rho) & \mapsto & \Phi_{f_{g_0,\rho}} \ . \end{array}$$

*Remark* 1.2. If $M = \mathbb{H}^2$, we have $\mathrm{Isom}^+(M) \simeq \mathrm{PSL}(2, \mathbb{R})$. Consider two points $X, X_1 \in \mathcal{T}(S)$, and let $j_1$ be a holonomy representation of a hyperbolic metric $g_1$ corresponding to $X_1$. Then a harmonic map from $X$ to $X_1$ isotopic to the identity lifts to a $j_1$-equivariant harmonic map from $\widetilde{X}$ to $\mathbb{H}^2$. We thus have

$$\mathbf{E}(X, j_1) = \mathbf{E}(X, X_1)$$

and

$$\mathbf{\Phi}(X, j_1) = \mathbf{\Phi}(X, X_1) \ .$$

In other words, the new definition of $\mathbf{E}$ and $\mathbf{\Phi}$ extends the one given in the previous paragraph, via the identification of $\mathcal{T}(S)$ with the component of maximal Euler class in $\mathrm{Rep}(S, \mathrm{PSL}(2, \mathbb{R}))$.

**Proposition 1.3.** *The function* $\mathbf{E}(X, \rho)$ *and the map* $\mathbf{\Phi}(X, \rho)$ *are continuous with respect to* $X$ *and* $\rho$*.*

*Proof.* This is a classical consequence of the ellipticity of the equation defining harmonic maps. Let $(j_n, \rho_n)$ converge to $(j, \rho)$. Then the derivatives of a $(j_n, \rho_n)$-equivariant harmonic map $f_n$ can be uniformly controlled by its total energy. When $\rho$ is not parabolic, one deduces that the sequence $f_n$ converges in $C^1$ topology (up to taking a subsequence) to a $(j, \rho)$-equivariant harmonic map $f$. The proposition easily follows. When $\rho$ fixes a point $p$ in $\partial_\infty M$, one can use the fact that the map $B_p \circ f_n$ converges up to translation to a $m_\rho$-equivariant function. □

We will make crucial use of the following results:

**Proposition 1.4** (see [23]). *Let $(M, g_M)$ be a Riemannian* CAT$(-1)$ *space and $\rho$ a representation of $\pi_1(S)$ into* Isom$(M)$. *Then the functional*

$$\begin{array}{rccc} \mathbf{E}(\,\cdot\,,\rho): & \mathcal{T}(S) & \to & \mathbb{R}_+ \\ & X & \mapsto & \mathbf{E}(X,\rho) \end{array}$$

*is $C^1$ and its differential at a point $X_0 \in \mathcal{T}(S)$ is given by*

$$d\mathbf{E}(\,\cdot\,,\rho)(X_0) = -4\mathbf{\Phi}(X_0,\rho)\ .$$

**Theorem 1.5** (Tromba, [22]). *Consider a point $j$ in $\mathcal{T}(S)$. Then the function*

$$\begin{array}{rccc} \mathbf{E}(\,\cdot\,,j): & \mathcal{T}(S) & \to & \mathbb{R}_+ \\ & X & \mapsto & \mathbf{E}(X,j) \end{array}$$

*is proper.*

## 2. The homeomorphism from $\mathcal{T}(S)$ to $D(\rho)$

We are now in possession of all the tools required to define the map $\mathbf{\Psi}_\rho$ introduced in [5] and to prove that it is a homeomorphism from $\mathcal{T}(S)$ to Dom$(\rho)$.

### 2.1. Construction of the map $\mathbf{\Psi}_\rho$.

Let $S$ be a closed oriented surface of negative Euler characteristic and $\rho$ a representation of $\pi_1(S)$ into the isometry group of a Riemannian CAT$(-1)$ space $(M, g_M)$. Let $X_1$ be a point in $\mathcal{T}(S)$. Then $\mathbf{\Phi}(X_1, \rho)$ is a holomorphic quadratic differential on $X_1$, and the theorem of Sampson–Hitchin–Wolf asserts that there is a unique point $j_2$ in $\mathcal{T}(S)$ such that $\mathbf{\Phi}(X_1, j_2) = \mathbf{\Phi}(X_1, \rho)$. Setting

$$\mathbf{\Psi}_\rho(X_1) = j_2\ ,$$

we obtain a well-defined map

$$\mathbf{\Psi}_\rho : \mathcal{T}(S) \to \mathcal{T}(S)\ .$$

This map only depends on the class of $\rho$ under conjugation by Isom$(M)$. We can thus define a map

$$\begin{array}{rccc} \mathbf{\Psi}: & \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) & \to & \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) \\ & (X,\rho) & \mapsto & (\mathbf{\Psi}_\rho(X), \rho)\ . \end{array}$$

In [5], the following is proved:

**Theorem** (Deroin–Tholozan). *Either $\rho$ preserves a totally geodesic 2-plane of curvature $-1$ in restriction to which it is Fuchsian, or the image of the map $\mathbf{\Psi}_\rho$ lies in* Dom$(\rho)$. *In particular,* Dom$(\rho)$ *is non empty.*

2.2. **Surjectivity of the map $\mathbf{\Psi}_\rho$.** We first prove the following

**Proposition 2.1.** *The map $\mathbf{\Psi}_\rho : \mathcal{T}(S) \to \mathrm{Dom}(\rho)$ is surjective.*

*Proof.* Fix $j_0 \in \mathcal{T}(S)$ and $\rho \in \mathrm{Rep}(S, \mathrm{Isom}(M))$. Let us introduce the functional

$$\begin{array}{rccl} \mathbf{F}_{j_0,\rho} : & \mathcal{T}(S) & \to & \mathbb{R} \\ & X & \mapsto & \mathbf{E}(X, j_0) - \mathbf{E}(X, \rho) \ . \end{array}$$

By proposition 1.4, the map $\mathbf{F}_{j_0,\rho}$ is $C^1$ and its differential is given by

$$\mathrm{d}\mathbf{F}_{j_0,\rho}(X) = -4\mathbf{\Phi}(X, j_0) + 4\mathbf{\Phi}(X, \rho) \ .$$

Hence $X_1$ is a critical point of $\mathbf{F}_{j_0,\rho}$ if and only if $\mathbf{\Phi}(X_1, j_0) = \mathbf{\Phi}(X_1, \rho)$, which means precisely that

$$\mathbf{\Psi}_\rho(X_1) = j_0 \ .$$

Proving that $j_0$ is in the image of $\mathbf{\Psi}_\rho$ is thus equivalent to proving that the map $\mathbf{F}_{j_0,\rho}$ admits a critical point. This will be a consequence of the following lemma:

**Lemma 2.2.** *For $j_0 \in \mathcal{T}(S)$ and $\rho \in \mathrm{Rep}(S, \mathrm{Isom}(M))$, we have the following inequality:*

$$\mathbf{F}_{j_0,\rho} \geq (1 - \mathrm{Lip}(j_0, \rho))\, \mathbf{E}(\,\cdot\,, j_0) \ .$$

*Proof.* Let $f$ be a $(j_0, \rho)$-equivariant Lipschitz map from $\mathbb{H}^2$ to $M$ with Lipschitz constant $\mathrm{Lip}(j_0, \rho) + \varepsilon$. Let $X$ be a point in $\mathcal{T}(S)$ represented by some hyperbolic metric $g$, and let $h$ be the unique $j_0$-equivariant map from $\widetilde{X}$ to $\mathbb{H}^2$. We thus have $E_g(h) = \mathbf{E}(X, j_0)$. Besides, the map $f \circ h$ is a $\rho$-equivariant map from $\widetilde{X}$ to $M$, and we thus have

$$\mathbf{E}(X, \rho) \leq E_g(f \circ h) \ .$$

Since $f$ is $(\mathrm{Lip}(j_0, \rho) + \varepsilon)$-Lipschitz, we have

$$\mathbf{E}(X, \rho) \leq E_g(f \circ h) \leq (\mathrm{Lip}(j_0, \rho) + \varepsilon) E_g(h) = (\mathrm{Lip}(j_0, \rho) + \varepsilon)\mathbf{E}(X, j_0) \ ,$$

from which we get

$$\mathbf{F}_{j_0,\rho}(X) = \mathbf{E}(X, j_0) - \mathbf{E}(X, \rho) \geq (1 - \mathrm{Lip}(j_0, \rho) - \varepsilon)\mathbf{E}(X, j_0).$$

This being true for any $\varepsilon > 0$, we obtain the required inequality. ☐

Now, by Theorem 1.5, the map $X \to \mathbf{E}(X, j_0)$ is proper. Therefore, if $j_0$ is in $\mathrm{Dom}(\rho)$, we have $(1 - \mathrm{Lip}(j_0, \rho)) > 0$ and the function $\mathbf{F}_{j_0,\rho}$ is also proper. Hence $\mathbf{F}_{j_0,\rho}$ admits a minimum. This minimum is a critical point, and thus a pre-image of $j_0$ by the map $\mathbf{\Psi}_\rho$. We obtain that $\mathbf{\Psi}_\rho : \mathcal{T}(S) \to \mathrm{Dom}(\rho)$ is surjective. ☐

We proved that if $j_0$ is in $\mathrm{Dom}(\rho)$, the functional $\mathbf{F}_{j_0,\rho}$ is proper. Note that, conversely, if $\mathbf{F}_{j_0,\rho}$ is proper, then it admits a critical point. Hence $j_0$ is in the image of $\mathbf{\Psi}_\rho$, which implies that $j_0$ lies in $\mathrm{Dom}(\rho)$ by [5]. We thus obtain the following corollary that might be interesting in its own way:

**Corollary 2.3.** *The following are equivalent:*

(*i*) *the map $\mathbf{F}_{j_0,\rho}$ is proper,*
(*ii*) *the map $\mathbf{F}_{j_0,\rho}$ admits a critical point,*
(*iii*) *the representation $j_0$ strictly dominates $\rho$.*

2.3. **Injectivity of the map $\mathbf{\Psi}_\rho$.** To prove injectivity, we need to prove that when $j_0$ is in $\mathrm{Dom}(\rho)$, the critical point of $\mathbf{F}_{j_0,\rho}$ is unique. To do so, we prove that any critical point of $\mathbf{F}_{j_0,\rho}$ is a strict minimum of $\mathbf{F}_{j_0,\rho}$.

Let $X_1$ be a critical point of $\mathbf{F}_{j_0,\rho}$, and $X_2$ another point in $\mathcal{T}(S)$. Choose a hyperbolic metric $g_1$ on $S$ representing $X_1$. Let $g_0$ be the hyperbolic metric of holonomy $j_0$ such that $\mathrm{Id} : (S, g_1) \to (S, g_0)$ is harmonic, and $g_2$ the hyperbolic metric representing $X_2$ such that $\mathrm{Id} : (S, g_2) \to (S, g_0)$ is harmonic. Let $f : (\widetilde{S}, \widetilde{g}_1) \to (M, g_M)$ be a $\rho$-equivariant harmonic map. We have the following decompositions:

$$\begin{aligned} g_0 &= e_{g_1}(g_0) g_1 + \Phi + \bar{\Phi}, \\ f^* g_M &= e_{g_1}(f) g_1 + \Phi + \bar{\Phi}, \\ g_2 &= e_{g_1}(g_2) g_1 + \Psi + \bar{\Psi}, \end{aligned}$$

where $\Phi$ and $\Psi$ are quadratic differentials on $S$, with $\Phi$ holomorphic with respect to the complex structure induced by $g_1$. Note that the same $\Phi$ appears in the decomposition of $g_0$ and $f^* g_M$ because $X_1$ is a critical point of $\mathbf{F}_{j_0,\rho}$, and thus $\mathbf{\Phi}(X_1, j_0) = \mathbf{\Phi}(X_1, \rho)$.

*Remark* 2.4. If $\rho$ is parabolic, the proof is still valid, provided that we replace $f_M^g$ by $f^* \mathrm{d}x^2$, where $f$ is a $m_\rho$-equivariant harmonic function.

**Lemma 2.5.** *We have the following identity:*

$$E_{g_2}(g_0) - E_{g_2}(f^* g_M) = \int_S \frac{1}{\sqrt{1 - \frac{4\|\Psi\|_{g_1}^2}{e_{g_1}(g_2)^2}}} \left( e_{g_1}(g_0) - e_{g_1}(f^* g_M) \right) vol_{g_1} \ .$$

*Proof of lemma 2.5.* This is a rather basic computation that we will carry out in local coordinates. Let $z = x + iy$ be a local complex coordinate with respect to which $g_1$ is conformal. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{C}$

Any symmetric 2-form on $\mathbb{C}$ can be written under the form $\langle \cdot, G \cdot \rangle$, where $G$ is a field of symmetric endomorphisms of $\mathbb{R}^2$ depending on the coordinates $(x, y)$. We will represent such an endomorphism by its matrix in the canonical frame $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. In local coordinates, we can thus write

$$\begin{aligned} g_0 &= \langle \cdot, G_0 \cdot \rangle \ , \\ g_1 &= \langle \cdot, G_1 \cdot \rangle \ , \\ g_2 &= \langle \cdot, G_2 \cdot \rangle \ , \\ f^* g_M &= \langle \cdot, G_f \cdot \rangle \ . \end{aligned}$$

Now, since $g_1$ is conformal with respect to the coordinate $z$, we have $g_1 = \alpha \langle \cdot, \cdot \rangle$ for some positive function $\alpha$, and we can write

$$\begin{aligned} \Phi &= \varphi \, \mathrm{d}z^2 \ , \\ \Psi &= \psi \, \mathrm{d}z^2 \ , \end{aligned}$$

for some complex valued functions $\varphi$ and $\psi$. (Since $\Phi$ is holomorphic, $\varphi$ must be holomorphic, but we won't need it for our computation.)

We can now express $G_0$, $G_1$, $G_2$, $G_f$, $vol_{g_1}$ and $vol_{g_2}$ in terms of $\alpha$, $e_{g_1}(g_0)$, $e_{g_1}(g_2)$, $e_{g_1}(f)$, $\varphi$ and $\psi$. We easily check that

$$G_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \ ,$$

$$G_0 = \begin{pmatrix} \alpha e_{g_1}(g_0) + 2\mathrm{Re}(\varphi) & -2\mathrm{Im}(\varphi) \\ -2\mathrm{Im}(\varphi) & \alpha e_{g_1}(g_0) - 2\mathrm{Re}(\varphi) \end{pmatrix} ,$$

$$G_f = \begin{pmatrix} \alpha e_{g_1}(g_f) + 2\mathrm{Re}(\varphi) & -2\mathrm{Im}(\varphi) \\ -2\mathrm{Im}(\varphi) & \alpha e_{g_1}(g_f) - 2\mathrm{Re}(\varphi) \end{pmatrix} ,$$

$$G_2 = \begin{pmatrix} \alpha e_{g_1}(g_2) + 2\mathrm{Re}(\psi) & -2\mathrm{Im}(\psi) \\ -2\mathrm{Im}(\psi) & \alpha e_{g_1}(g_2) - 2\mathrm{Re}(\psi) \end{pmatrix} ,$$

$$vol_{g_1} = \alpha \mathrm{d}z\mathrm{d}\bar{z} ,$$

$$vol_{g_2} = \sqrt{\det G_2}\mathrm{d}z\mathrm{d}\bar{z} = \sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}\mathrm{d}z\mathrm{d}\bar{z} .$$

We now want to express $e_{g_2}(g_0)$ and $e_{g_2}(f^* g_M)$. To do so, note that we can write

$$\begin{aligned} g_0(\,\cdot\,,\,\cdot\,) &= \langle\,\cdot\,, G_0\,\cdot\,\rangle \\ &= \left\langle\,\cdot\,, G_2(G_2^{-1}G_0)\,\cdot\,\right\rangle \\ &= g_2(\,\cdot\,, G_2^{-1}G_0\,\cdot\,) . \end{aligned}$$

By definition of the energy density, we thus obtain that

$$\begin{aligned} e_{g_2}(g_0) &= \frac{1}{2}\mathrm{Tr}(G_2^{-1}G_0) \\ &= \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\det G_2}\begin{pmatrix} \alpha e_{g_1}(g_2) - 2\mathrm{Re}(\psi) & 2\mathrm{Im}(\psi) \\ 2\,\mathrm{Im}(\psi) & \alpha e_{g_1}(g_2) + 2\mathrm{Re}(\psi) \end{pmatrix} \right. \\ &\quad \left. \times \begin{pmatrix} \alpha e_{g_1}(g_0) + 2\mathrm{Re}(\varphi) & -2\mathrm{Im}(\varphi) \\ -2\,\mathrm{Im}(\varphi) & \alpha e_{g_1}(g_0) - 2\mathrm{Re}(\varphi) \end{pmatrix}\right] \\ &= \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}\left(\alpha^2 e_{g_1}(g_2) e_{g_1}(g_0) - (\varphi\bar{\psi} + \bar{\varphi}\psi)\right) . \end{aligned}$$

Similarly, we get that

$$e_{g_2}(f^* g_M) = \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}\left(\alpha^2 e_{g_1}(g_2) e_{g_1}(f) - (\varphi\bar{\psi} + \bar{\varphi}\psi)\right) .$$

When computing the difference, the terms $\varphi\bar{\psi} + \bar{\varphi}\psi$ simplify. (Here we use the fact that $f^* g_M$ and $g_0$ have the same $(2,0)$-part.) We eventually obtain

$$\begin{aligned} (e_{g_2}(g_0) - e_{g_2}(f^* g_M))\, vol_{g_2} &= \frac{\alpha^2 e_{g_1}(g_2)(e_{g_2}(g_0) - e_{g_2}(f^* g_M))}{\sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}}\,\mathrm{d}z\mathrm{d}\bar{z} \\ &= \frac{e_{g_2}(g_0) - e_{g_2}(f^* g_M)}{\sqrt{1 - 4\frac{\|\Psi\|_{g_1}^2}{e_{g_1}(g_2)^2}}} vol_{g_1} . \end{aligned}$$

Now, the parameters of the last expression are well defined functions on $S$ and the identity is true in any local chart. It is thus true everywhere on $S$ and, when integrating, we obtain lemma 2.5. □

From lemma 2.5, we obtain that

$$E_{g_2}(g_0) - E_{g_2}(f^* g_M) \geq \int_S (e_{g_1}(g_0) - e_{g_1}(f^* g_M)) vol_{g_1} = E_{g_1}(g_0) - E_{g_1}(f^* g_M) = \mathbf{F}_{j_0,\rho}(X_1) \ ,$$

with equality if and only if $\|\Psi\|_{g_1} \equiv 0$, that is, if $g_1$ is conformal to $g_2$.

On the other side, we have $E_{g_2}(g_0) = \mathbf{E}(X_2, j_0)$ (since, by hypothesis, the identity map from $(S, g_2)$ to $(S, g_0)$ is harmonic) and $E_{g_2}(f^* g_M) \geq \mathbf{E}(X_2, \rho)$, from which we deduce that

$$E_{g_2}(g_0) - E_{g_2}(f^* g_M) \leq \mathbf{F}_{j_0,\rho}(X_2) \ .$$

Combining the two inequalities, we obtain that

$$\mathbf{F}_{j_0,\rho}(X_2) \geq \mathbf{F}_{j_0,\rho}(X_1) \ ,$$

with equality if and only if $X_1 = X_2$.

Now, if $X_1$ and $X_2$ are two critical points of $\mathbf{F}_{j_0,\rho}$, then by symmetry we must have $\mathbf{F}_{j_0,\rho}(X_2) = \mathbf{F}_{j_0,\rho}(X_1)$, and therefore $X_2 = X_1$. The functional $\mathbf{F}_{j_0,\rho}$ admits a unique critical point, and $j_0$ admits a unique pre-image by $\mathbf{\Psi}_\rho$. Thus $\mathbf{\Psi}_\rho$ is injective.

2.4. **bi-continuity of $\mathbf{\Psi}$.** Recall that one has, by definition,

$$(X, \mathbf{\Psi}_\rho(X)) = \mathbf{\Phi}^{-1}(X, \mathbf{\Phi}(X, \rho)) \ ,$$

where $\mathbf{\Phi}^{-1}$ denotes the inverse of the map $\mathbf{\Phi} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathrm{QD}\mathcal{T}(S)$, which is a homeomorphism by Sampson–Hitchin–Wolf's theorem. Therefore, by proposition 1.3, the maps $\mathbf{\Psi}_\rho$ and $\mathbf{\Psi}$ are continuous.

Let us now prove that $\mathbf{\Psi}^{-1}$ is continuous. We saw that $\mathbf{\Psi}^{-1}(j, \rho)$ is the unique critical point of a proper function $\mathbf{F}_{j,\rho}$ on $\mathcal{T}(S)$ which depends continuously on $j$ and $\rho$. The continuity of $\mathbf{\Psi}^{-1}$ will follow from the fact that the functions $\mathbf{F}_{j,\rho}$ are in some sense *locally uniformly proper.*

**Definition 2.6.** Let $X$ and $Y$ be two metric spaces, and $(F_y)_{y\in Y}$ a family of continuous functions from $X$ to $\mathbb{R}$ depending continuously on $y$ for the compact-open topology. We say that the family $(F_y)_{y\in Y}$ is *uniformly proper* if for any $C \in \mathbb{R}$, there exists a compact subset $K$ of $X$ such that for all $y \in Y$ and all $x \in X \backslash K$, $F_y(x) > C$.

We say that the family $(F_y)_{y\in Y}$ is *locally uniformly proper* if for any $y_0 \in Y$, there is a neighbourhood $U$ of $y_0$ such that the sub-family $(F_y)_{y\in U}$ is uniformly proper.

**Proposition 2.7.** *Let $X$ and $Y$ be two metric spaces and $(F_y)_{y\in Y}$ a locally uniformly proper family of continuous functions from $X$ to $\mathbb{R}$ depending continuously on $y$ (for the compact-open topology). Assume that each $F_y$ achieves its minimum at a unique point $x_m(y) \in X$. Then the function*

$$y \mapsto x_m(y)$$

*is continuous.*

*Proof.* Let us denote by $m(y) = F_y(x_m(y))$ the minimum value of $F_y$. Fix $y_0 \in Y$. Let $U$ be a neighbourhood of $y_0$ and $K$ a compact subset of $X$ such that for all $y \in U$ and all $x \in X\backslash K$, we have

$$F_y(x) > m(y_0) + 1 \ .$$

For $\varepsilon > 0$, define

$$V_\varepsilon = \{x \in X \mid F_{y_0}(x) < m(y_0) + \varepsilon\} \ .$$

Since $F_{y_0}$ is proper and achieves its minimum at a single point $x_m(y_0)$, the family $(V_\varepsilon)_{\varepsilon>0}$ forms a basis of neighbourhoods of $x_m(y_0)$. Let $U_\varepsilon$ be a neighbourhood of $y_0$ included in $U$ such that for all $y \in U_\varepsilon$ and all $x \in K$,

$$|F_y(x) - F_{y_0}(x)| < \frac{\varepsilon}{2} \ .$$

($U_\varepsilon$ exists because the map $y \mapsto F_y$ is continuous for the compact-open topology.) Since $x_m(y_0)$ is obviously in $K$, we have for all $y \in U_\varepsilon$,

$$F_y(x_m(y_0)) < m(y_0) + \frac{\varepsilon}{2} \ ,$$

hence the minimum value $m(y)$ of $F_y$ is smaller than $m(y_0) + \frac{\varepsilon}{2}$. In particular, for $\varepsilon < 2$, this minimum is achieved in $K$ (since outside $K$, we have $F_y \geq m(y_0) + 1$). We thus have $x_m(y) \in K$, from which we deduce

$$F_{y_0}(x_m(y)) < F_y(x_m(y)) + \frac{\varepsilon}{2} = m(y) + \frac{\varepsilon}{2} < m(y_0) + \varepsilon \ .$$

We have thus proved that $x_m(y) \in V_\varepsilon$ for all $y \in U_\varepsilon$. Since $(V_\varepsilon)_{\varepsilon>0}$ is a basis of neighbourhoods of $x_m(y_0)$, this proves that $y \mapsto x_m(y)$ is continuous at $y_0$. □

To prove the continuity of $\mathbf{\Psi}^{-1}$, we can apply proposition 2.7 to the family $\mathbf{F}_{j,\rho}$ of functions on $\mathcal{T}(S)$ depending on the parameter $(j,\rho) \in \mathrm{Dom}(S, \mathrm{Isom}(M))$. The continuity of $(j,\rho) \mapsto \mathbf{F}_{j,\rho}$ comes from proposition 1.3. The only thing we need to check is thus that the family

$$(\mathbf{F}_{j,\rho})_{(j,\rho)\in \mathrm{Dom}(S,\mathrm{Isom}(M))}$$

is locally uniformly proper. But this follows easily from the continuity of the minimal Lipschitz constant. Indeed, let $(j_0, \rho_0)$ be a point in $\mathrm{Dom}(S, \mathrm{Isom}(M))$. We thus have $\mathrm{Lip}(j_0, \rho_0) < 1$. By continuity of the function Lip, there exists a neighbourhood $U$ of $(j_0, \rho_0)$ and a $\lambda < 1$ such that for all $(j,\rho) \in U$, we have $\mathrm{Lip}(j,\rho) \leq \lambda$. By lemma 2.2, we thus have

$$\mathbf{F}_{j,\rho}(Y) \geq \left(\frac{1-\lambda}{2}\right) \mathbf{E}(Y, j)$$

for all $(j,\rho) \in U$ and all $Y \in \mathcal{T}(S)$. Since the function $Y \mapsto \mathbf{E}(Y,j)$ is proper by theorem 1.5, we obtain that the family $(\mathbf{F}_{j,\rho})$ is uniformly proper on $U$. Hence it is locally uniformly proper.

By proposition 2.7, we deduce that the unique minimum of $\mathbf{F}_{j,\rho}$ varies continuously with $(j,\rho)$. Since this minimum is precisely $\mathbf{\Psi}^{-1}(j,\rho)$, we proved that $\mathbf{\Psi}^{-1}$ is continuous. This concludes the proof of theorem 1.

## References


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University of Luxembourg, Campus Kirchberg, 6, rue Richard Coudenhove-Kalergi, L-1359 LUXEMBOURG
*E-mail address*: nicolas.tholozan@uni.lu