The A_∞ de Rham theorem and integration of representations up to homotopy

Camilo Arias Abad[*] and Florian Schätz[†]

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Abstract

We use Chen's iterated integrals to integrate representations up to homotopy. That is, we construct an A_∞ functor $\int : \mathcal{R}ep^\infty(A) \to \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A))$ from the representations up to homotopy of a Lie algebroid A to those of its infinity groupoid. This construction extends the usual integration of representations in Lie theory. We discuss several examples including Lie algebras and Poisson manifolds.

The construction is based on an A_∞ version of de Rham's theorem due to Gugenheim [15]. The integration procedure we explain here amounts to extending the construction of parallel transport for superconnections, introduced by Igusa [17] and Block-Smith [5], to the case of certain differential graded manifolds.

Contents

1   Introduction

The purpose of this work is to understand Lie theory for representations up to homotopy. That is, we study the way in which the L_∞ representations of a Lie algebroid are related to the A_∞ representation of the corresponding global object.

Let us first consider the case of ordinary representations. If G is the source simply connected Lie groupoid integrating the Lie algebroid A, then the categories of representations $\mathrm{Rep}(G)$ and $\mathrm{Rep}(A)$ are naturally equivalent. From the point of view of differential graded manifolds, this can be seen as an instance of the fact that flat connections on $\mathcal{M}$ correspond to representations of the fundamental groupoid $\Pi_1(\mathcal{M})$. Indeed, there is a precise sense in which G is the fundamental groupoid of the differential graded manifold $A[1]$ associated to A. Hence, ordinary representations only capture information about the fundamental groupoid of the associated differential graded manifold. On the other hand, representations up to homotopy capture information about the higher homotopy group(oid)s as well.

Igusa [17] and Block-Smith [5] showed that flat $\mathbb{Z}$-graded superconnections on a smooth manifold M correspond to representations up to homotopy of its infinity groupoid, namely, the simplicial set of smooth singular chains on M. Building on this, we will show that for any Lie algebroid A, there is an A_∞ functor $\int$ from the category of representations up to homotopy of A to the category of unital representations up to homotopy of the infinity groupoid of the differential graded manifold $A[1]$ associated to A. Moreover, we show that, ultimately, the A_∞ functor $\int$ relies on an A_∞ version of de Rham's theorem that is due to Gugenheim [15], which is constructed using the theory of Chen's iterated integrals.

It has been observed in [2, 3] that some of the natural representations associated to Lie groups and Lie algebras appear as representations up to homotopy in the case of Lie groupoids and Lie algebroids. These constructions provide many examples to which the A_∞ functor $\int$ can be applied. We hope to come back to those examples, particularly in the case of Poisson manifolds, in a sequel.

The paper is organized as follows. In the remaining part of the introduction we will make some comments on simplicial sets, describe in more detail the question that we study and explain in general terms how the A_∞ functor $\int$ is constructed. In Section §2 we review Chen's iterated integrals and give a proof of the A_∞ de Rham theorem, originally due to Gugenheim [15]. In Section §4 we show that general considerations about A_∞ algebras allow one to construct the integration A_∞ functor $\int$ using the A_∞ de Rham theorem. In Section §5 we discuss some examples to which the construction applies, including smooth manifolds, Lie algebras and Poisson

manifolds. We also show that, when applied to ordinary representations, the A_∞ functor $\int$ specializes to the usual integration of representations in Lie theory. In the Appendix we review the definitions and some general facts concerning representations up to homotopy of Lie algebroids and A_∞-algebras.

1.1 General comments on simplicial sets

It is well known that the theory of simplicial sets allows one to look at some aspects of homotopy theory, higher category theory and Lie theory in a unified way. Let us briefly recall this point of view. We denote by Δ the category of nonempty finite ordered sets, and by $[n]$ the ordered set

$$[n] = \{0 \leq 1 \leq \cdots \leq n\},$$

seen as a category. We are interested in simplicial sets that appear via the following constructions:

Example 1.1. The nerve functor

$$N : \mathsf{Cat} \to \mathsf{Set}_\Delta,$$

from the category of small categories to the category of simplicial sets is defined by

$$N(\mathcal{C})_n := \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}).$$

In this way, one can view the category Cat as a full subcategory of the category Set_Δ, and it is easy to characterize precisely the simplicial sets that are nerves of categories in terms of horn filling conditions.

Example 1.2. We will denote by Δ_n the geometric n-simplex:

$$\Delta_n := \{(t_1, ..., t_n) \in \mathbb{R}^n : 1 \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq 0\}.$$

The sequence of spaces $\Delta_\bullet$, together with the usual face and degeneracy maps, forms a cosimplical space. There is a functor $Sing : \mathsf{Top} \to \mathsf{Set}_\Delta$ from topological spaces to simplicial sets given by the formula

$$Sing(X)_n := \mathrm{Hom}_{\mathsf{Top}}(\Delta_n, X).$$

The simplicial sets obtained by this procedure are always Kan complexes, also called infinity groupoids (see for example [6, 19, 20]). This means that they satisfy some horn filling conditions similar to those that characterize the nerves of categories.

Example 1.3. Let us also consider the spatial realization functor from rational homotopy theory introduced by Sullivan in [28]. This is a functor

$$S : \mathsf{DGCA} \to \mathsf{Set}_\Delta,$$

from the category of differential graded commutative algebras to simplicial sets. It is defined by:

$$S(\Omega)_n = \mathrm{Hom}_{\mathsf{DGCA}}(\Omega, \Omega(\Delta_n)),$$

where $\Omega(\Delta_n)$ denotes the algebra of differential forms on Δ_n.

The homotopy type of a space X can be reconstructed from the simplicial set $Sing(X)$ and, as we mentioned before, any category can be reconstructed from its nerve. Thus, one can think of the theory of simplicial sets as a simultaneous generalization of category theory and homotopy theory. An explanation of the role of simplicial sets in higher category theory can be found in the first chapter of [20].

Given a space X, there is always a map of simplicial sets

$$\pi : Sing(X) \to \mathsf{N}(\Pi_1(X)),$$

which sends a simplex to the sequence of homotopy classes of paths associated to its edges. Thus, the simplicial set $Sing(X)$ is a refinement of the fundamental groupoid, which captures information not only about homotopy classes of paths, but also about the higher homotopy group(oid)s. Because of this, $Sing(X)$ is often denoted by $\Pi_\infty(X)$ and called the infinity groupoid of X.

Observe that the relationship between examples 1.2 and 1.3 is also very close. Given a smooth manifold X, denote by $\mathrm{Sing}(X)$ the simplicial subset of $Sing(X)$, which consists of smooth simplices, and by $\Omega(X)$ the algebra of differential forms on X. Clearly, there is a natural isomorphism of simplicial sets

$$\mathsf{S}(\Omega(X)) \cong \mathrm{Sing}(X).$$

Since the inclusion $\mathrm{Sing}(X) \hookrightarrow Sing(X)$ is a homotopy equivalence, one can think of the spatial realization functor S as a generalization of the singular chain functor.

This point of view is quite useful in Lie theory. First, recall that a Lie algebroid structure on a vector bundle A is the same as a differential on $\Omega(A) = \Gamma(\Lambda A^*)$, which, in turn, is equivalent to equipping the graded manifold $A[1]$ with a cohomological vector field. There are natural identifications

$$\mathrm{Hom}_{\mathsf{Lie-alg}}(TM, A) \cong \mathrm{Hom}_{\mathsf{dg-man}}(T[1]M, A[1]) \cong \mathrm{Hom}_{\mathsf{DGCA}}(\Omega(A), \Omega(M)).$$

With these identifications in mind, it becomes natural to think of the simplicial set $\mathsf{S}(\Omega(A))$ as the space of singular chains in the dg-manifold $A[1]$, and to use the notation

$$S(\Omega(A)) = \mathrm{Sing}(A[1]) = \Pi_\infty(A[1]),$$

which, for simplicity, will be denoted by $\Pi_\infty(A)$ in the following.

If instead of looking at the infinity groupoid of $A[1]$, one considers the fundamental groupoid $\Pi_1(A[1])$, one obtains a candidate for the groupoid integrating the Lie algebroid A. Indeed $\Pi_1(A[1])$ is always a topological groupoid and in case the algebroid is integrable, it is the unique source simply connected Lie groupoid integrating A. See [7, 11, 23, 29] for more details on this construction. The integration of L_∞-algebras can also be interpreted as the computation of higher homotopy groupoids of dg-manifolds, see [13, 16, 23].

1.2 The problem

Our goal is to understand the relation between the global (A_∞) and the infinitesimal (L_∞) version of representations up to homotopy. If A is the Lie algebroid of G, one can differentiate any unital representation up to homotopy $E \in \hat{\mathcal{R}}\mathrm{ep}^\infty(G)$ to obtain a representation up to homotopy of A. This process is explained in [4], and can be summarized as follows:

Theorem . *Suppose that A is the Lie algebroid of G. Then there is a dg-functor*

$$\Psi : \hat{\mathcal{R}}ep^{\infty}(G) \to \mathcal{R}ep^{\infty}(A).$$

The differentiation functor Ψ is constructed by differentiating along the flows of vector fields in all possible directions in order to obtain infinitesimal cochains from global ones. Appropriately, the inverse process of this iterated differentiation is given by Chen's iterated integrals.

One is led to ask whether it is possible to integrate a representation up to homotopy of A to one of G. The answer to this question is: no, as explained in Paragraph 5.6. Indeed, the global counterpart of a representation up to homotopy of A turns out to be a representation up to homotopy of the infinity groupoid $\Pi_{\infty}(A)$ of A, not of its fundamental groupoid G.

Observe that a representation up to homotopy of a groupoid G is essentially – i.e. up to a smoothness condition – an A_{∞} functor to the *dg*-category of *dg*-vector spaces. This data can be expressed purely in terms of the simplicial structure of the nerve $\mathsf{N}G$ of G, and therefore it is sensible to speak about representations up to homotopy of an arbitrary simplicial set.

Alternatively, one can take the adjoint point of view via Cordier's nerve construction [9] to define a representation up to homotopy as a map of simplicial sets, see the Appendix of [5].

Our main result is the following:

Theorem 4.18. *Let A be a Lie algebroid. The assignments*

$$\mathcal{R}ep^{\infty}(A) \ni E \mapsto \int [E] \in \hat{\mathcal{R}}ep^{\infty}(\Pi_{\infty}(A))$$

and

$$\int_{n} : \mathsf{s}\underline{\mathrm{Hom}}(E_1, E_0) \otimes \cdots \otimes \mathsf{s}\underline{\mathrm{Hom}}(E_n, E_{n-1}) \quad \to \quad \mathsf{s}\underline{\mathrm{Hom}}(E_n, E_0),$$
$$\phi_1 \otimes \cdots \otimes \phi_n \quad \mapsto \quad \mathsf{s}\left(\mathsf{Hol}(-, \phi_1, \ldots, \phi_n)\right)$$

define an A_{∞} functor

$$\int : \mathcal{R}ep^{\infty}(A) \to \hat{\mathcal{R}}ep^{\infty}(\Pi_{\infty}(A))$$

between the dg-category of representations up to homotopy of A and the dg-category of unital representations up to homotopy of $\Pi_{\infty}(A)$.

The construction of the A_{∞} functor $\int$ is based on and inspired by the parallel transport of superconnections studied by Igusa [17] and Block-Smith [5]. It seems to us that – beyond the extension to arbitrary Lie algebroids – our contributions are as follows:

We show that the parallel transport from [17, 5] can be derived from the A_{∞} version of de Rham's theorem due to Gugenheim [15]; this allows us to extend parallel transport to an A_{∞} functor.

Moreover, we establish that the integration process lands in the category of *unital* representations up to homotopy. This is significant because, as explained in Paragraph 2.2, unitality implies higher dimensional versions of the glueing rules that express how the holonomies behave under a refinement of the triangulation. Observe that the unitality condition also comes up naturally in the construction of the differentiation functor $\Psi : \hat{\mathcal{R}}ep^{\infty}(G) \to \mathcal{R}ep^{\infty}(A)$ from [4].

2 Simplicial sets and representations up to homotopy

In this section we give the definition of a representation up to homotopy of a simplicial set and discuss some examples. Later on we will be interested in the case where the simplicial set is the ∞-groupoid $\Pi_\infty(A)$ of a Lie algebroid A.

2.1 Definitions

Remark 2.1. Let $X_\bullet$ be a simplicial set with face and degeneracy maps denoted by

$$d_i : X_k \to X_{k-1} \quad \text{and} \quad s_i : X_k \to X_{k+1},$$

respectively. We will use the notation

$$
\begin{aligned}
P_i &:= (d_0)^{k-i} : X_k \to X_i, \\
Q_i &:= d_{i+1} \circ \cdots \circ d_k : X_k \to X_i,
\end{aligned}
$$

for the maps that send a simplex to its i-th back and front face. The i-th vertex of a simplex $\sigma \in X_k$ will be denoted $v_i(\sigma)$, or simply v_i, when no confusion can arise. In terms of the above operations, one can write

$$v_i = (P_0 \circ Q_i)(\sigma).$$

A cochain F of degree k on $X_\bullet$ with values in an algebra A is a map

$$F : X_k \to \mathsf{A}.$$

As usual, the cup product of two cochains F, F' of degree i and j, respectively, is the cochain of degree $i + j$ defined by the formula:

$$(F \cup F')(\sigma) := F(Q_i(\sigma))F'(P_j(\sigma)).$$

Definition 2.2. *A representation up to homotopy of $X_\bullet$ consists of the following data:*

1. *A graded vector space $E_x = \bigoplus_{k \in \mathbb{Z}} E_x^k$, for each zero simplex $x \in X_0$.*

2. *A sequence of operators $\{F_k\}_{k \geq 0}$, where F_k is a k-cochain that assigns to $\sigma \in X_k$ a linear map*

$$F_k(\sigma) \in \operatorname{Hom}^{1-k}(E_{v_k(\sigma)}, E_{v_0(\sigma)}).$$

These operators are required to satisfy the following equations:

$$\sum_{j=1}^{k-1}(-1)^j F_{k-1}(d_j(\sigma)) + \sum_{j=0}^{k}(-1)^{j+1}(F_j \cup F_{k-j})(\sigma) = 0. \tag{1}$$

A representation up to homotopy is called unital if

$$F_1(s_0(\sigma)) = \mathrm{id} \quad and \quad F_k(s_i(\sigma)) = 0 \ for \ k > 1.$$

Remark 2.3. The definition of representation up to homotopy of a simplicial set is such that when applied to the nerve of a category, it is the same as an A_∞-functor to the *dg*-category of differential graded vector spaces. These equations were first considered by Sugawara [27]. The structure of a representation up to homotopy can also be described from the adjoint point of view via Cordier's nerve construction [9, 20], as explained in the appendix of [5]. The case where the simplicial set is the nerve of a Lie groupoid was considered in [3].

Observe that the structure equations of a representation up to homotopy imply that F_0 gives each of the vector spaces E_x the structure of a cochain complex. We will often write ∂ instead of F_0 for this coboundary operator.

Remark 2.4. The representations up to homotopy of $X_\bullet$ form a *dg*-category: Let E, E' be two representations up to homotopy of $X_\bullet$. A degree n morphism $\phi \in \underline{\mathrm{Hom}}^n(E, E')$ is a formal sum

$$\phi = \phi_0 + \phi_1 + \phi_2 + \ldots,$$

where ϕ_k is a k-cochain which assigns to a simplex $\sigma \in X_k$ a linear map

$$\phi_k(\sigma) \in \mathrm{Hom}^{n-k}(E_{v_k(\sigma)}, E'_{v_0(\sigma)}).$$

The differential

$$D : \underline{\mathrm{Hom}}^n(E, E') \to \underline{\mathrm{Hom}}^{n+1}(E, E')$$

is defined by

$$D(\phi) = D(\phi)_0 + D(\phi)_1 + D(\phi)_2 + \ldots,$$

where

$$D(\phi)_k(\sigma) := \sum_{i+j=k}(-1)^{jn}F'_j \cup \phi_i(\sigma) + \sum_{i+j=k}(-1)^{n+j+1}\phi_j \cup F_i(\sigma) + \sum_{j=1}^{k-1}(-1)^{j+n}\phi_{k-1}(d_j(\sigma)). \tag{2}$$

If $\phi' : E' \to E''$ is a morphism of degree m, the composition $\phi' \circ \phi \in \underline{\mathrm{Hom}}^{m+n}(E, E'')$ is given by

$$\phi' \circ \phi = (\phi' \circ \phi)_0 + (\phi' \circ \phi)_1 + (\phi' \circ \phi)_2 + \ldots,$$

where

$$(\phi' \circ \phi)_k := \sum_{i+j=k}(-1)^{jn}(\phi'_j \cup \phi_i).$$

One easily checks that these operators define a *dg*-category – the category of representations up to homotopy of $X_\bullet$ – which we denote by $\mathcal{R}\mathrm{ep}^\infty(X_\bullet)$.

We will be particularly interested in the category of unital representations up to homotopy, which we denote by $\hat{\mathcal{R}}\mathrm{ep}^\infty(X_\bullet)$. This is the sub-*dg*-category of $\mathcal{R}\mathrm{ep}^\infty(X_\bullet)$ whose objects are unital representations up to homotopy and whose morphisms are unital morphisms, that is, morphisms ϕ such that $\phi_k(s_i(\sigma)) = 0$.

2.2 Some examples

Example 2.5. Let G be a group and $\mathsf{N}G$ the nerve of G. A unital representations up to homotopy of $\mathsf{N}G$ on a graded vector space V, which is concentrated in degree zero, is the same as a representation of G.

Example 2.6. Let C be a category and $\mathsf{N}C$ the nerve of C. A representations up to homotopy of $\mathsf{N}C$ is the same as an A_∞-functor from the category $\mathbb{R}C$ to the *dg*-category of complexes of vector spaces. Here $\mathbb{R}C$ denotes the *dg*-category whose objects are those of C and whose morphisms are the linear spans of morphisms of C.

Example 2.7. The most important simplicial set for our purposes is the ∞-groupoid $\Pi_\infty(A)$ of a Lie algebroid A, given by

$$(\Pi_\infty(A))_k := \mathrm{Hom}_{\mathsf{DGCA}}(\Omega(A), \Omega(T\Delta_k)),$$

i.e. the morphisms of differential graded commutative algebras between $\Omega(A)$ and the de Rham algebra of the k-simplex Δ_k. Observe that this is the same as the set of Lie algebroid morphisms from $T\Delta_k$ to A. The structure operations for a representation up to homotopy of $\Pi_\infty(A)$ are rules that assign holonomies to simplices, and the structure equations are compatibility conditions between the holonomies. In the pictures below, a shaded face of a simplex denotes the holonomy assigned to it. For the 1-simplex we obtain

$$\left[\ \partial,\ \rule[0.3ex]{2em}{1.2pt}\ \right]\quad=\quad 0.$$

This says that the holonomy assigned to a path should be a chain map between the chain complexes associated to the endpoints of the paths.

The compatibility for condition for the triangle is

requiring that – even though two homotopic paths are assigned different holonomies in general – any specific homotopy between the two paths induces a homotopy between the corresponding chain maps.

For the tetrahedron one obtains

Example 2.8. The unitality condition on a representation up to homotopy describes the way in which the holonomies change under a refinement of the triangulation. For a degenerate one simplex, the unitality implies that the constant path gets assigned the identity. For a degenerate triangle, one obtains that dividing an interval in two pieces and composing their holonomies

gives the holonomy associated to the original interval. For a tetrahedron, unitality yields the higher dimensional analogue of this glueing rule. We illustrate the degenerate tetrahedron as a triangle with a vertex in the middle, which comes from the degeneracy map. Then, the structure equation gives

$$\text{} \; - \; \text{} \; + \; \text{} \; - \; \text{} \; = \; \left[\; \partial, \; \text{} \; \right] \; = 0,$$

where the right hand side vanishes because the tetrahedron under consideration is degenerate. Thus, we obtain

$$\text{} \; = \; \text{} \; - \; \text{} \; + \; \text{} \; .$$

This relation expresses the holonomy of the big triangle in terms of the holonomies associated to the small triangles and the edges.

3 Differential forms and singular cochains

In this section, we explain how to use Chen's iterated integrals to construct a natural A_∞ quasi-isomorphism

$$\psi : (\Omega(M), -d, \wedge) \to (C(M), \delta, \cup),$$

between the algebra of differential forms and the algebra of smooth singular cochains of a manifold M. This result is originally due to Gugenheim ([15]). Implicitly, this quasi-isomorphism is also present in [5] and [17].

Remark 3.1. Given a differential graded algebra $(A, d, \wedge)$, the bar complex

$$\mathsf{B}(\mathsf{s}A) := \bigoplus_{k \geq 1} (\mathsf{s}A)^{\otimes k}$$

carries a coboundary operator D given by

$$
\begin{aligned}
D(\mathsf{s}a_1 \otimes \cdots \mathsf{s}a_n) \quad := \quad & \sum_{i=1}^{n} (-1)^{[a_1]+\cdots+[a_{i-1}]} \mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{i-1} \otimes \mathsf{s}(da_i) \otimes \mathsf{s}a_{i+1} \otimes \cdots \otimes \mathsf{s}a_n \\
& + \sum_{i=1}^{n-1} (-1)^{[a_1]+\cdots+[a_i]} \mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{i-1} \otimes \mathsf{s}(a_i \wedge a_{i+1}) \otimes \mathsf{s}a_{i+1} \otimes \cdots \otimes \mathsf{s}a_n.
\end{aligned}
$$

The morphism ψ is constructed out of two maps

$$\mathsf{B}(\mathsf{s}\Omega(M)) \xrightarrow{\quad \mathsf{C} \quad} \Omega(\mathsf{P}M) \xrightarrow{\quad \mathsf{S} \quad} \mathsf{s}C(M).$$

9

Here,

$$\mathsf{B}(\mathsf{s}\Omega(M)) \xrightarrow{\quad\mathsf{C}\quad} \Omega(\mathsf{P}M)$$

is given by Chen's iterated integrals, while

$$\Omega(\mathsf{P}M) \xrightarrow{\quad\mathsf{S}\quad} \mathsf{s}C(M)$$

is constructed with the help of a certain family of maps

$$(\Theta_{(k)} : I^{k-1} \to \mathsf{P}\Delta_k)_{k \geq 1}$$

from the cubes to the path spaces of the simplices. The crucial property of this family is that it relates the cellular structures of the cubes to those of the simplices. Such a family was already considered by Chen [8]. We will make use of the explicit family constructed by Igusa in [17].

3.1 Chen's iterated integrals

Here, we explain how Chen's iterated integrals provide a map

$$\mathsf{C} : \mathsf{B}(\mathsf{s}\Omega(M)) \to \Omega(\mathsf{P}M),$$

from the bar complex of the suspension of $\Omega(M)$ to the differential forms on the path space of M. The most important property of C is that it is almost a chain map between $\mathsf{B}(\mathsf{s}\Omega(M))$ – equipped with the bar differential $\overline{D}$ corresponding to the dg-algebra $(\Omega(M), -d, \wedge)$ – and $\Omega(\mathsf{P}M)$ – equipped with the de Rham differential. In fact, C fails to be a chain map only because of two boundary terms, see Theorem 3.10.

Remark 3.2. We use the conventions from [14], except for the definition of the simplices Δ_k, where we follow Igusa ([17]).

Our first task is to introduce the path space $\mathsf{P}M$ of M, as well as differential forms on it.

Remark 3.3. The path space $\mathsf{P}M$ of a smooth manifold M is the topological space $\mathcal{C}^\infty(I, M)$, equipped with the $\mathcal{C}^1$-topology. By definition, a map $f : X \to \mathsf{P}M$ is smooth if

$$f^{\mathrm{ev}} : I \times X \to M, \quad f^{\mathrm{ev}}(t, x) := (f(x))(t)$$

is smooth.

Differential forms on $\mathsf{P}M$ are defined as follows:

i) Denote by $\mathcal{C}^\infty(-, \mathsf{P}M)$ the category whose objects are pairs (X, f), with X a smooth manifold and f a smooth map from X to $\mathsf{P}M$; the morphisms from (X, f) to (Y, g) are all smooth maps $h : X \to Y$ such that $f = g \circ h$.

ii) Let $\mathbb{R}(-)$ be the functor from $\mathcal{C}^\infty(-, \mathsf{P}M)$ to the category of real vector spaces Vect, which maps any object in $\mathcal{C}^\infty(-, \mathsf{P}M)$ to $\mathbb{R}$ and every morphism to the identity.

iii) The functor $\Omega(-)$ from $\mathcal{C}^\infty(-, \mathsf{P}M)$ to Vect is defined via $(X, f) \mapsto \Omega(X)$ and $g \mapsto g^*$.

iv) A differential form on $\mathsf{P}M$ is a natural transformation from $\mathbb{R}(-)$ to $\Omega(-)$.

Put another way: a differential form α on $\mathsf{P}M$ is a natural association of a differential form on X, which we will denote by $f^*\alpha$, to a pair $(X, f : X \to \mathsf{P}M)$. The idea is simply that it suffices to know all the pull backs of a differential form to finite dimensional manifolds in order to know the differential form itself. The wedge product and the de Rham differential are defined by requiring the pullback operation to preserve them.

We call a continuous map $h : \mathsf{P}M \to \mathsf{P}N$ smooth if the composition of any smooth map $f : X \to \mathsf{P}M$ with h is smooth. Smooth maps induce pull back maps $\Omega(\mathsf{P}N) \to \Omega(\mathsf{P}M)$. Moreover, observe that every smooth map $h : M \to N$ induces a smooth map $\mathsf{P}h : \mathsf{P}M \to \mathsf{P}N$ on the path spaces via composition.

Remark 3.4. Let X be a manifold and Δ_k the k-simplex, i.e.

$$\Delta_k := \{(t_1, \ldots, t_k) \in \mathbb{R}^k : 1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0\} \subset \mathbb{R}^k.$$

We define the push forward along the projection

$$\pi : \Delta_k \times X \to X$$

to be the linear map

$$\pi_* : \Omega(\Delta_k \times X) \to \Omega(X)$$

of degree $-k$ determined by setting

$$\pi_* \left(f(t,x)dt^{i_1} \cdots dt^{i_r} dx^{j_1} \cdots dx^{j_s} \right) := \left(\int_{\Delta_k} f(t,x)dt^{i_1} \cdots dt^{i_r} \right) dx^{j_1} \cdots dx^{j_s}.$$

Observe that for X compact and oriented,

$$\int_X \pi_*(\alpha) = \int_{\Delta_k \times X} \alpha$$

holds for all $\alpha \in \Omega(\Delta_k \times X)$.

Lemma 3.5. *The push forward $\pi_* : \Omega(\Delta_k \times X) \to \Omega(X)$ is a morphism of left $\Omega(X)$-modules of degree $-k$, i.e. for every $\alpha \in \Omega(X)$ and every $\beta \in \Omega(\Delta_k \times X)$, we have*

$$\pi_*(\pi^*\alpha \wedge \beta) = (-1)^{|\alpha|k}\alpha \wedge \pi_*(\beta).$$

Furthermore, let $\partial\pi$ be the composition

$$\partial\Delta_k \times X \xrightarrow{\ \iota \times \mathrm{id}\ } \Delta_k \times X \xrightarrow{\ \pi\ } X,$$

where $\partial\Delta_k$ denotes the disjoint union of the codimension one strata of Δ_k.

Then the following formula holds:

$$\pi_* \circ d - (-1)^k d \circ \pi_* = (\partial\pi)_* \circ (\iota \times \mathrm{id})^*.$$

Here, the push forward along $\partial\pi$ is understood as the sum over the push forwards of the individual faces which appear as the connected components of $\partial\Delta_k$.

Remark 3.6. Next we define Chen's iterated integrals for differential forms on a smooth manifold M. This is a degree zero linear map:

$$\mathsf{C} : \mathsf{B}(\mathsf{s}\Omega(M)) := \bigoplus_{k \geq 1} (\mathsf{s}\Omega(M))^{\otimes k} \to \Omega(\mathsf{P}M).$$

Let $\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n$ be an element of the bar complex $\mathsf{B}(\mathsf{s}\Omega(M))$. Given any smooth map $f : I \times X \to M$, we define a differential form on X by the following procedure:

i) Thinking of Δ_n as the compactified configuration space of (unordered) points on the interval I, one obtains an extension of $f : I \times X \to M$ to

$$f_{(n)} : \Delta_n \times X \to M \times \cdots \times M, \quad f_{(n)}(t_1, \ldots, t_n, x) := (f(t_1, x), \ldots, f(t_n, x)).$$

We will sometimes omit the subscript (n) if the dimension of the simplex is clear from the context.

ii) Pull back the differential form a_i to $M \times \cdots \times M$ via the i'th projection map

$$p_i : M^{\times n} \to M$$

and pull back the wedge product $p_1^* a_1 \wedge \cdots \wedge p_n^* a_n$ to $\Delta_n \times X$ via $f_{(n)}$.

iii) To obtain a differential form on X, push the differential form $f_{(n)}^*(p_1^* a_1 \wedge \cdots \wedge p_n^* a_n)$ forward to X along $\pi : \Delta_n \times X \to X$.

iv) Finally, multiply this differential form with the sign

$$(-1)^{\sum_{i=1}^{n} [a_i](n-i)},$$

where $[a]$ denotes the degree of an element $\mathsf{s}a \in \mathsf{s}\Omega(M)$.

Note that this is exactly the sign by which $(\mathsf{s}^{-1} \otimes \cdots \otimes \mathsf{s}^{-1})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ differs from $(a_1 \otimes \cdots \otimes a_n)$.

We denote the resulting differential form on X by $f^*(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_k))$. It is straight forward to check that this construction is natural and therefore defines a differential form on the path space $\mathsf{P}M$. Let us write down the whole construction more compactly:

Definition 3.7. *Chen's map*

$$\mathsf{C} : \mathsf{B}(\mathsf{s}\Omega(M)) \to \Omega(\mathsf{P}M)$$

from the bar complex of the suspension of differential forms to the de Rham complex of the path space $\mathsf{P}M$ is defined by setting

$$f^*(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \mathsf{s}a_n)) := (-1)^{\sum_{i=1}^{n} [a_i](n-i)} \pi_*((f_{(n)}^{ev})^*(p_1^* a_1 \wedge \cdots \wedge p_n^* a_n))$$

for any object (X, f) in the category $\mathcal{C}^{\infty}(-, \mathsf{P}M)$.

Remark 3.8. An important property of Chen's map is that the image of an element $\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n$ of $\mathsf{B}(\mathsf{s}\Omega(M))$, for which one of the factors a_i is a function, is zero. This follows from the observation that in this case the pull back $f^*\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ by any smooth map $f : X \to \mathsf{P}M$ vanishes. Indeed, the differential form

$$(f^{\mathrm{ev}}_{(n)})^*(p_1^*a_1 \wedge \cdots \wedge p_n^*a_n) \in \Omega(\Delta_n \times X)$$

is annihilated by the vector field $\frac{\partial}{\partial t^i}$, where t^i denotes the i-th coordinate on the simplex Δ_n. Hence the push forward of the differential form along $\pi : \Delta_n \times X \to X$ vanishes.

Lemma 3.9. *Chen's map C is natural, i.e. for any smooth map $h : M \to N$ the diagram*

$$
\begin{array}{ccc}
\mathsf{B}(\mathsf{s}\Omega(M)) & \xrightarrow{\ \mathsf{C}\ } & \Omega(\mathsf{P}M) \\
{\scriptstyle \mathsf{B}h}\big\uparrow & & \big\uparrow{\scriptstyle (\mathsf{P}h)^*} \\
\mathsf{B}(\mathsf{s}\Omega(N)) & \xrightarrow{\ \mathsf{C}\ } & \Omega(\mathsf{P}N)
\end{array}
$$

commutes. Here, $\mathsf{B}h$ denotes the linear map which extends

$$\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n \mapsto \mathsf{s}h^*(a_1) \otimes \cdots \otimes \mathsf{s}h^*(a_n).$$

Proof. This is essentially an exercise in unraveling the definitions. The key-observation is that for any smooth map $f : X \to \mathsf{P}M$,

$$(\mathsf{P}h \circ f)^{\mathrm{ev}}_{(n)} = (h \times \cdots \times h) \circ f^{\mathrm{ev}}_{(n)}$$

is satisfied. $\qquad\square$

Theorem 3.10 (Chen). *For any element $\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n$ of the bar complex $\mathsf{B}(\mathsf{s}\Omega(M))$, the following identity holds:*

$$
\begin{aligned}
d(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \ = \ & \mathsf{C}(\overline{D}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) + \\
& ev_1^*(a_1) \wedge \mathsf{C}(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_n) - (-1)^{[a_1]+\cdots+[a_{n-1}]}\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{n-1}) \wedge ev_0^*(a_n).
\end{aligned}
$$

Here:

1. *$\overline{D}$ is the differential corresponding to the differential graded algebra $(\Omega(M), -d, \wedge)$.*

2. *The maps ev_0 and $ev_1 : \mathsf{P}M \to M$ are given by $\gamma \mapsto \gamma(i)$ for $i = 0$ or 1, respectively.*

Proof. It suffices to establish the formula for the pull back of $\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ by any smooth map $f : X \to \mathsf{P}M$. Using Lemma 3.5, one sees that $f^*d\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ is equal to

$$
\begin{aligned}
& (-1)^n (\pi_* d\left(f^*_{(n)}(p_1^*a_1 \wedge \cdots \wedge p_n^*a_n)\right)) + \\
& (-1)^{n+1}((\partial\pi)_*(\iota \times \mathrm{id})^* f^*_{(n)}(p_1^*a_1 \wedge \cdots \wedge p_n^*a_n))
\end{aligned}
$$

times the sign $(-1)^{\sum_{i=1}^n [a_i](n-i)}$. The first term gives

$$\sum_{i=1}^n (-1)^{[a_1]+\cdots+[a_{i-1}]} f^*\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}(-da_i) \otimes \cdots \otimes \mathsf{s}a_n),$$

while the second one yields

$$\sum_{i=1}^{n-1}(-1)^{[a_1]+\cdots+[a_i]}f^*\mathsf{C}(\mathsf{s}a_1\otimes\cdots\otimes\mathsf{s}(a_i\wedge a_{i+1})\otimes\cdots\otimes\mathsf{s}a_n)$$
$$+(\mathrm{ev}_1\circ f)^*a_1\wedge f^*\mathsf{C}(\mathsf{s}a_2\otimes\cdots\otimes\mathsf{s}a_n)$$
$$-(-1)^{[a_1]+\cdots+[a_{n-1}]}f^*\mathsf{C}(\mathsf{s}a_1\otimes\cdots\otimes\mathsf{s}a_{n-1})\wedge(\mathrm{ev}_0\circ f)^*a_n.$$

$\square$

Definition 3.11. *The set*

$$\mathcal{C}_+^\infty(I,\partial I):=\{\phi:I\to I\ smooth\ and\ monotone:\phi(0)=0,\phi(1)=1\}$$

is a monoid under composition and acts on $\mathcal{C}^\infty(X,\mathsf{P}M)$ via reparametrizations, i.e. via

$$\mathcal{C}_+^\infty(I,\partial I)\times\mathcal{C}^\infty(X,\mathsf{P}M)\ \to\ \mathcal{C}^\infty(X,\mathsf{P}M),$$
$$(\phi,f)\ \mapsto\ (\phi\bullet f)^{ev}(t,x):=f^{ev}(\phi(t),x).$$

A differential form $\alpha\in\Omega(\mathsf{P}M)$ is reparametrization invariant if for any smooth map $f:X\to\mathsf{P}M$ and any reparametrization $\phi\in\mathcal{C}_+^\infty(I,\partial I)$, the equation

$$f^*\alpha=(\phi\bullet f)^*\alpha$$

holds. We denote the subcomplex of invariant differential forms by $\Omega_{inv}(\mathsf{P}M)$.

Lemma 3.12. *The image of Chen's map*

$$\mathsf{C}:\mathsf{B}(\mathsf{s}\Omega(M))\to\Omega(\mathsf{P}M)$$

is contained in the subcomplex $\Omega_{inv}(\mathsf{P}M)$ of reparametrization invariant differential forms on $\mathsf{P}M$.

Proof. Pick $f:X\to\mathsf{P}M$ and $\phi\in\mathcal{C}_+^\infty(I,\partial I)$ arbitrary. We have to check that

$$f^*\mathsf{C}(\mathsf{s}a_1\otimes\cdots\otimes\mathsf{s}a_n)=(\phi\bullet f)^*\mathsf{C}(\mathsf{s}a_1\otimes\cdots\otimes\mathsf{s}a_n)$$

holds for any $a_1,\ldots,a_n\in\Omega(M)$.

Using

$$(\phi\bullet f)_{(n)}^{\mathrm{ev}}=f_{(n)}^{\mathrm{ev}}\circ(\phi\times\cdots\times\phi\times\mathrm{id}):\Delta_n\times X\to M^{\times n},$$

one sees that it suffices to prove the equation

$$\pi_*=\pi_*\circ(\phi\times\cdots\times\phi\times\mathrm{id})^*:\Omega(\Delta_n\times X)\to\Omega(X),$$

where π denotes the projection $\Delta_n\times X\to X$. This equation can be deduced inductively from the behavior of one-dimensional integrals under substitution. $\square$

Remark 3.13. Later on, we will use the reparametrization invariance of differential forms in the image of Chen's map C under piecewise linear reparametrizations. In order to handle these kind of reparametrizations correctly, one should introduce the space of piecewise smooth paths $\tilde{\mathsf{P}}M$ of M. By definition, a smooth map from X to $\tilde{\mathsf{P}}M$ is a continuous maps $F^{\mathrm{ev}}:I\times X\to M$ together with a finite partition $([a_i,b_i])$ of I such that the restriction of F^{ev} to $X\times]a_i,b_i[$ is smooth. The set $\mathcal{C}^\infty(X,\tilde{\mathsf{P}}M)$ is acted upon by the monoid of piecewise smooth reparametrizations of I (still monotone and endpoints-preserving). Observe that the definition of Chen's map C extends to smooth maps $X\to\tilde{\mathsf{P}}M$ and still yields smooth differential forms on X. Furthermore the reparametrization invariance continues to hold in the piecewise smooth setting.

3.2 Igusa's map

In this paragraph we construct a map

$$\mathsf{S} : \Omega(\mathsf{P}M) \to \mathsf{s}C(M)$$

from the differential forms on $\mathsf{P}M$ to the suspension of the singular cochains on M. We will need a sequence of maps from the cubes to the simplices, which relate their cellular structures. These maps were originally considered by Adams ([1]) and – later on – by Chen ([8]). We will use a construction due to Igusa ([17]).

Remark 3.14. As mentioned before, we use the following definition of the k-simplex Δ_k:

$$\Delta_k := \{(t_1, \ldots, t_k) \in \mathbb{R}^k : 1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0\} \subset \mathbb{R}^k.$$

In this convention, the face- and degeneracy-maps that equip $\{\Delta_k\}$ with the structure of a cosimplicial set are given by

$$\partial_i : \Delta_k \to \Delta_{k+1}, \quad (t_1, \ldots, t_k) \mapsto \begin{cases} (1, t_1, \ldots, t_k) & \text{for } i = 0, \\ (t_1, \ldots, t_{i-1}, t_i, t_i, t_{i+1}, \ldots, t_k) & \text{for } 0 < i < k+1, \\ (t_1, \ldots, t_k, 0) & \text{for } i = k+1. \end{cases} \quad \text{and}$$

$$\epsilon_i : \Delta_k \to \Delta_{k-1}, \quad (t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_{i-1}, \widehat{t_i}, t_{i+1}, \ldots, t_k), \quad \text{respectively.}$$

The i-th vertex v_i of Δ_k is the point

$$(\underbrace{1, \ldots, 1}_{i \text{ times}}, \underbrace{0, \ldots, 0}_{k-i \text{ times}}) \in \Delta_k.$$

The simplicial set of (smooth) singular chains $\mathrm{Sing}(M)$ of M is given by

$$\mathrm{Sing}_k(M) := \mathcal{C}^\infty(\Delta_k, M).$$

The simplicial structure maps $\{d_i\}$ and $\{s_i\}$ are defined via $d_i := \partial_i^*$ and $s_i := \epsilon_i^*$.

Observe that the maps P_i and Q_i, which send an element $\mathrm{Sing}(M)$ to its back- and front-face, respectively, are equal to the pull backs of

$$U_i : \Delta_i \to \Delta_k, \quad (t_1, \ldots, t_i) \mapsto (1, \ldots, 1, t_1, \ldots, t_i) \quad \text{and}$$
$$V_i : \Delta_i \to \Delta_k, \quad (t_1, \ldots, t_i) \mapsto (t_1, \ldots, t_i, 0, \ldots, 0), \quad \text{respectively.}$$

Definition 3.15. *Let M be a manifold. The dg-algebra of (smooth) singular cochains $(C(M), \delta, \cup)$ consists of:*

- *The graded vector space $C(M)$ of linear functionals on the vector space generated by $\mathrm{Sing}(M)$ over $\mathbb{R}$.*

- *The differential δ defined by*

$$(\delta\phi)(\sigma) := \sum_{i=0}^{k} (-1)^i (d_i^* \phi)(\sigma) := \sum_{i=0}^{k} (-1)^i \phi(\partial_i^* \sigma).$$

- *The cup product $\cup$ defined by*

$$(\phi \cup \psi)(\sigma) := \phi(V_i^* \sigma)\psi(U_j^* \sigma).$$

Definition 3.16. *For each $k \geq 0$, the map*

$$\Theta_{(k)} : I^{k-1} \to \mathsf{P}\Delta_k,$$

is defined to be the composition

$$I^{k-1} \xrightarrow{\lambda_{(k)}} \mathsf{P}I^k \xrightarrow{\mathsf{P}\pi_k} \mathsf{P}\Delta_k.$$

Here $\pi_k : I^k \to \Delta_k$ is given by $\pi_k(x_1,\ldots,x_k) := (t_1,\ldots,t_k)$, with components

$$t_i := \max\{x_i,\ldots,x_k\}.$$

The map $\lambda_{(k)} : I^{k-1} \to \mathsf{P}I^k$ is defined by sending a point $(x_1,\ldots,x_{k-1})$ to the path which goes backwards through the following $k+1$ points:

$$0 \leftarrow x_1 e_1 \leftarrow (x_1 e_1 + x_2 e_2) \leftarrow \cdots \leftarrow (x_1 e_1 + \cdots + x_{k-1} e_{k-1}) \leftarrow (x_1 e_1 + \cdots + x_{k-1} e_{k-1} + e_k),$$

where $(e_1, \cdots, e_n)$ denotes the standard basis of $\mathbb{R}^n$. In other words, for $j = 0,\ldots,k$ we set:

$$\lambda_{(k)}(x_1,\ldots,x_{k-1})(\frac{k-j}{k}) = x_1 e_1 + \cdots + x_j e_j,$$

where $x_k = 1$, and interpolate linearly.
   By convention, $\Theta_{(0)}$ is the map from a point to a point.
   We will denote the map adjoint to $\Theta_{(k)}$ by $\Theta_k : I^k \to \Delta_k$.

Remark 3.17.

Let us mention some properties of the maps $\Theta_{(k)}$:

1. The maps $\Theta_{(k)}$ are piecewise linear but not smooth. However, one can smooth them via reparametrizations. The reparametrization invariance of elements in the image of Chen's map then guarantees that for all our purposes the maps $\Theta_{(k)}$ behave as if they were smooth.

2. One can check that the degree of the map $\Theta_k : I^k \to \Delta_k$ is $(-1)^k$, i.e.

$$\int_{I^k} \Theta_k^* \alpha = (-1)^k \int_{\Delta_k} \alpha$$

   for every differential form $\alpha \in \Omega(\Delta_k)$.

3. The image of Θ_k lies in the subset $\mathsf{P}(\Delta_k, v_k, v_0)$ of paths starting at the last vertex v_k and ending at the zeroth vertex v_0.

The key properties of the maps $\Theta_{(k)}$ are described in the following lemma from [17]:

Lemma 3.18. *The sequence of maps*

$$\Theta_{(k)} : I^{k-1} \to \mathsf{P}\Delta_k,$$

satisfies the following properties:

16

1. For $1 \leq i \leq k-1$ the diagram

$$
\begin{array}{ccc}
I^{k-2} \xrightarrow{\ \partial_i^-\ } I^{k-1} \xrightarrow{\ \Theta_{(k)}\ } \mathsf{P}(\Delta_k, v_k, v_0) \\
\Theta_{(k-1)} \downarrow \qquad\qquad\qquad\qquad \uparrow \mathsf{P}\partial_i \\
\mathsf{P}(\Delta_{k-1}, v_{k-1}, v_0) \xrightarrow{\ \hat{\phi}_i\ } \mathsf{P}(\Delta_{k-1}, v_{k-1}, v_0)
\end{array}
$$

commutes, where the maps are as follows:

- $\partial_i^- : I^{k-2} \to I^{k-1}$ is $(x_1, \cdots, x_{k-2}) \mapsto (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{k-2})$,
- $\hat{\phi}_i$ is the map induced by the following piecewise smooth reparametrization

$$
\phi_i(t) := \begin{cases} \frac{kt}{k-1} & \text{for } 0 \leq t \leq \frac{k-i-1}{k}, \\ \frac{k-i-1}{k-1} & \text{for } \frac{k-i-1}{k} \leq t \leq \frac{k-i}{k}, \\ \frac{kt-1}{k-1} & \text{for } \frac{k-i}{k} \leq t \leq 1. \end{cases}
$$

  Hence, $\hat{\phi}_i \circ \Theta_{(k-1)} = \phi_i \bullet \Theta_{(k-1)}$.
- $\mathsf{P}\partial_i$ is the smooth map induced by the the i'th face map $\partial_i : \Delta_{k-1} \to \Delta_k$.

2. For $1 \leq i \leq k-1$ the diagram

$$
\begin{array}{ccc}
I^{k-2} \xrightarrow{\ \partial_i^+\ } I^{k-1} \xrightarrow{\ \Theta_{(k)}\ } \mathsf{P}(\Delta_k, v_k, v_0) \\
\cong \downarrow \qquad\qquad\qquad\qquad\qquad \uparrow \mu_i \\
I^{i-1} \times I^{k-i-1} \xrightarrow{\ \Theta_{(i)} \times \Theta_{(k-i)}\ } \mathsf{P}(\Delta_i, v_i, v_0) \times \mathsf{P}(\Delta_{k-i}, v_{k-i}, v_0),
\end{array}
$$

commutes. Here μ_i is the concatenation map

$$
\mu_i(\alpha, \beta)(t) := \begin{cases} U_{k-i}\left(\beta\left(\frac{kt}{k-i}\right)\right) & \text{for } 0 \leq t \leq \frac{k-i}{k}, \\ V_i\left(\alpha\left(\frac{k}{i}\left(t - \frac{k-i}{k}\right)\right)\right) & \text{for } \frac{k-i}{k} \leq t \leq 1, \end{cases}
$$

and $\partial_i^+ : I^{k-2} \to I^{k-1}$ is given by

$$
(x_1, \cdots, x_{k-2}) \mapsto (x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_{k-2}).
$$

Lemma 3.19. *Let μ_i be the i-th concatenation map from Lemma 3.18 and suppose $a_1, \ldots, a_n$ are differential forms on Δ_k. Also, let $f : X \to \mathsf{P}(\Delta_i, v_i, v_0)$ and $g : Y \to \mathsf{P}(\Delta_{k-i}, v_{k-i}, v_0)$ be smooth maps. Then the following factorization property holds:*

$$
\int_{X \times Y} (f \times g)^* (\mu_i)^* \mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)
$$

$$
= \sum_{l=0}^{n} \left(\int_X f^* \mathsf{C}(\mathsf{s}V_i^* a_1 \otimes \cdots \otimes \mathsf{s}V_i^* a_l) \right) \times \left(\int_Y g^* \mathsf{C}(\mathsf{s}U_{k-i}^* a_{l+1} \otimes \cdots \otimes \mathsf{s}U_{k-i}^* a_n) \right).
$$

Here we extended the map C to the augmented bar complex $\mathbb{R} \oplus \mathsf{B}(\mathsf{s}\Omega(M))$ by setting

$$
\mathsf{C}(1) := 1.
$$

17

Proof. The main observation is that the diagram

$$
\begin{array}{ccc}
\Delta_l \times \Delta_{n-l} \times X \times Y & \xrightarrow{\ \cong\ \tau\ } & (\Delta_l \times X) \times (\Delta_{n-l} \times Y) \\[4pt]
\big\downarrow{\scriptstyle \gamma_l \times (\mathrm{id}_{X\times Y})} & & \big\downarrow{\scriptstyle (f^{\mathrm{ev}}) \times (g^{\mathrm{ev}})} \\[4pt]
\Delta_n \times X \times Y & & (\Delta_i)^l \times (\Delta_{k-i})^{n-l} \\
\end{array}
$$

with maps $(\mu_i \circ (f \times g))^{\mathrm{ev}}$ and $(V_i)^l \times (U_{k-i})^{n-l}$ into $(\Delta_k)^n$

is commutative, where γ_l is

$$
\begin{aligned}
\gamma_l : \Delta_l \times \Delta_{n-l} & \to \Delta_n \\
(\vec{u}, \vec{v}) & \mapsto \left(\frac{i}{k}\vec{u} + \frac{k-i}{k}\vec{1}, \frac{k-i}{k}\vec{v} \right).
\end{aligned}
$$

It is easy to check that γ_l is injective in the interior of $\Delta_l \times \Delta_{n-l}$ and maps onto

$$
\Delta_n(l) := \{ 1 \geq t_1 \cdots \geq t_l \geq \frac{k-i}{k} \geq t_{l+1} \geq \cdots \geq t_n \geq 0 \}.
$$

Observe that the interiors of $\Delta_n(l)$ are disjoint for different values of l and that the union over all l differs from Δ_n by a set of measure zero only. Moreover, γ_l is orientation preserving.

Now, we compute

$$
\int_{X \times Y} (\pi_{X \times Y})_* ((\mu_i \circ (f \times g)^{\mathrm{ev}})_{(n)})^* (p_1^* a_1 \wedge \cdots \wedge p_n^* a_n)
$$

$$
= \int_{\Delta_n \times X \times Y} ((\mu_i \circ (f \times g)^{\mathrm{ev}})_{(n)})^* (p_1^* a_1 \wedge \cdots \wedge p_n^* a_n)
$$

$$
= \sum_{l=0}^{n} \int_{\Delta_n(l) \times X \times Y} ((\mu_i \circ (f \times g)^{\mathrm{ev}})_{(n)})^* (p_1^* a_1 \wedge \cdots \wedge p_n^* a_n)
$$

$$
= \sum_{l=0}^{n} \int_{\Delta_l \times \Delta_{n-l} \times X \times Y} ((\gamma_l \times \mathrm{id}_{X \times Y}))^* ((\mu_i \circ (f \times g)^{\mathrm{ev}})_{(n)})^* (p_1^* a_1 \wedge \cdots \wedge p_n^* a_n)
$$

$$
= \sum_{l=0}^{n} (-1)^{(n-l)\dim X} \int_{(\Delta_l \times X) \times (\Delta_{n-l} \times Y)} ((f^{\mathrm{ev}})_{(l)} \times (g^{\mathrm{ev}})_{(n-l)})^* \Big(
$$

$$
((V_i)^l \times (U_{k-i})^{n-l})^* (p_1^* a_1 \wedge \cdots \wedge p_n^* a_n) \Big)
$$

$$
= \sum_{l=0}^{n} (-1)^{(n-l)\dim X} \left(\int_{\Delta_l \times X} (f_{(l)}^{\mathrm{ev}})^* (p_1^* V_i^* a_1 \wedge \cdots \wedge p_l^* V_i^* a_l) \right) \times
$$

$$
\left(\int_{\Delta_{n-l} \times Y} (g_{(n-l)}^{\mathrm{ev}})^* (p_1^* U_{k-i}^* a_{l+1} \wedge \cdots \wedge p_{n-l}^* U_{k-i}^* a_n) \right)
$$

$$
= \sum_{l=0}^{n} (-1)^{(n-l)\dim X} \left(\int_X (\pi_X)_* (f_{(l)}^{\mathrm{ev}})^* (p_1^* V_i^* a_1 \wedge \cdots \wedge p_l^* V_i^* a_l) \right) \times
$$

$$
\left(\int_Y (\pi_Y)_* (g_{(n-l)}^{\mathrm{ev}})^* (p_1^* U_{k-i}^* a_{l+1} \wedge \cdots \wedge p_{n-l}^* U_{k-i}^* a_n) \right).
$$

Taking the additional signs in the definition of Chen's map C into account yields the claimed factorization identity. $\qquad\square$

Definition 3.20. *The map* $\mathsf{S} : \Omega(\mathsf{P}M) \to \mathsf{s}C(M)$ *is the composition of*

$$\Omega(\mathsf{P}M) \quad \to \quad C(M),$$
$$\alpha \quad \mapsto \quad \left(\sigma \mapsto \int_{I^{k-1}} (\Theta_{(k)})^* \mathsf{P}\sigma^* \alpha \right)$$

and $\mathsf{s} : C(M) \to \mathsf{s}C(M)$.

Remark 3.21. Let us evaluate the image of $\mathsf{C}(\mathsf{s}a)$ under S on a k-simplex $\sigma : \Delta_k \to M$:

$$
\begin{aligned}
\mathsf{S}(\mathsf{C}(\mathsf{s}a))(\sigma) &= \int_{I^{k-1}} (\Theta_{(k)})^* \mathsf{P}\sigma^* \mathsf{C}(\mathsf{s}a) = \int_{I^{k-1}} (\Theta_{(k)})^* \mathsf{C}(\mathsf{s}\sigma^* a) \\
&= \int_{I^k} (\Theta_k)^* (\sigma^* a) = (-1)^k \int_{\Delta_k} \sigma^* a.
\end{aligned}
$$

Observe that this is valid only for $k > 0$, since $\mathsf{C}(\mathsf{s}f) = 0$ for f a smooth function.

3.3 An A_∞ version of de Rham's theorem

The aim of this paragraph is to show that the composition

$$B(\mathsf{s}\Omega(M)) \xrightarrow{\quad\mathsf{C}\quad} \Omega(\mathsf{P}M) \xrightarrow{\quad\mathsf{S}\quad} \mathsf{s}C(M)$$

yields an A_∞ quasi-isomorphism between the differential graded algebras $(\Omega(M), -d, \wedge)$ and $(C(M), \delta, \cup)$, see also [15].

Proposition 3.22. *Let* $a_1, \dots, a_n$ *be differential forms on* M. *Then the following equation holds:*

$$
\begin{aligned}
\mathsf{S}(d\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;=\;\; & b_1'(\mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n))) + \\
& \sum_{l=1}^{n-1} b_2' \left(\mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \mathsf{s}a_l)) \otimes \mathsf{S}(\mathsf{C}(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)) \right).
\end{aligned}
$$

Here b_1' and b_2' are the maps corresponding – at the level of the suspension – to the differential and the multiplication of the dg-algebra $C(M)$ of simplicial cochains.

Proof. Let α be an arbitrary differential form on the path space $\mathsf{P}M$ and $\sigma : \Delta_k \to M$ a simplex. We want to compute

$$\int_{I^{k-1}} d(\Theta_{(k)})^* (\mathsf{P}\sigma)^* \alpha = \int_{\partial I^{k-1}} \iota^* (\Theta_{(k)})^* (\mathsf{P}\sigma)^* \alpha.$$

Let $\partial_i^\pm$ be the standard embedings of I^{k-2} into I^{k-1} as top and bottom faces. Then the expression above is equal to:

$$\sum_{i=1}^{k-1} (-1)^i \left(\int_{I^{k-2}} (\partial_i^-)^* (\Theta_{(k)})^* (\mathsf{P}\sigma)^* \alpha \right) - \sum_{i=1}^{k-1} (-1)^i \left(\int_{I^{k-2}} (\partial_i^+)^* (\Theta_{(k)})^* (\mathsf{P}\sigma)^* \alpha \right).$$

19

We now take $\alpha = \mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$. Using the commutativity the first diagram in Lemma 3.18, reparametrization invariance of differential forms in the image of Chen's map C and the naturality of Chen's map, one concludes

$$\int_{I^{k-2}} (\partial_i^-)^*(\Theta_{(k)})^*(\mathsf{P}\sigma)^*\alpha = \int_{I^{k-2}} (\Theta_{(k-1)})^*(\mathsf{P}\partial_i^*\sigma)^*\alpha = \mathsf{S}(\alpha)(\partial_i^*\sigma).$$

Consequently, the strata of I^{k-1} given by the faces $\partial_i^- I^{k-1}$ yield

$$\sum_{i=1}^{k-1} (-1)^i (\mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)))(\partial_i^*\sigma).$$

On the other hand, commutativity of the second diagram in Lemma 3.18 together with Proposition 3.19 imply

$$\int_{I^{k-2}} (\partial_i^+)^*(\Theta_{(k)})^*(\mathsf{P}\sigma)^*\alpha = \sum_{l=0}^{n} \mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l))(V_i^*\sigma)\mathsf{S}(\mathsf{C}(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n))(U_{k-i}^*\sigma).$$

Summing over all the values of i one obtains:

$$\sum_{i=1}^{k-1} (-1)^{i-1} \left(\int_{I^{k-2}} (\partial_i^+)^*(\Theta_{(k)})^*(\mathsf{P}\sigma)^*\alpha \right) =$$

$$= \sum_{l=1}^{n-1} b_2'(\mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l)) \otimes \mathsf{S}(\mathsf{C}(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)))(\sigma)$$

$$+ \mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n))(U_{k-1}^*\sigma) + (-1)^k \mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n))(V_{k-1}^*\sigma).$$

$$= \sum_{l=1}^{n-1} b_2'(\mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l)) \otimes \mathsf{S}(\mathsf{C}(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)))(\sigma)$$

$$+ \mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l))(\partial_0^*\sigma) + (-1)^k \mathsf{S}(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l))(\partial_k^*\sigma).$$

$\square$

Definition 3.23. *Given a smooth manifold M and $n \geq 1$, we define the map*

$$\psi_n : \mathsf{s}\Omega(M)^{\otimes n} \to \mathsf{s}C(M)$$

as follows:

1. For $n = 1$, we set

$$(\psi_1(\mathsf{s}a))\,(\sigma : \Delta_k \to M) := (-1)^k \left(\int_{\Delta^k} \sigma^*a \right).$$

2. For $n > 1$, we set

$$\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n) := (\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n).$$

Remark 3.24. Observe that $\psi_1(\mathsf{s}a)$ coincides with $(\mathsf{S} \circ \mathsf{C})(\mathsf{s}a)$, except for the case when a is of degree 0, i.e. a function. In that case, $(\mathsf{S} \circ \mathsf{C})(\mathsf{s}a) = 0$, while

$$(\psi_1(\mathsf{s}a))\,(\sigma : \{*\} \to M) := a(\sigma(0)).$$

We now come to the main result of this section, originally established by Gugenheim in [15].

Theorem 3.25 (Gugenheim)**.** *The sequence of maps:*

$$\psi_n : (\mathsf{s}\Omega(M))^{\otimes n} \to \mathsf{s}C(M)$$

gives an A_∞ *morphism from* $(\Omega(M), -d, \wedge)$ *to* $(C(M), \delta, \cup)$. *Moreover, this morphism is a quasi-isomorphism and the construction is natural with respect to pull backs along smooth maps.*

Proof. Let $\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n$ be an element of the bar complex of the suspension of $(\Omega(M), -d, \wedge)$. By Theorem 3.10, the following equation holds:

$$
\begin{aligned}
d(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;=\;\; & \mathsf{C}(\overline{D}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;+ \\
& ev_1^*(a_1) \wedge \mathsf{C}(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_n) - (-1)^{[a_1]+\cdots+[a_{n-1}]}\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{n-1}) \wedge ev_0^*(a_n).
\end{aligned}
$$

On the other hand, Proposition 3.22 asserts that

$$
\begin{aligned}
\mathsf{S}(d\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;=\;\; & b_1'((\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;+ \\
& \sum_{l=1}^{n-1} b_2'\left((\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l) \otimes (\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)\right).
\end{aligned}
$$

Together, these two equations yield

$$
\begin{aligned}
(\mathsf{S} \circ \mathsf{C})(\overline{D}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \;=\;\; & b_1'((\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) \\
& + \sum_{l=1}^{n-1} b_2'\left((\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l) \otimes (\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)\right) \\
& - \mathsf{S}\left(ev_1^*(a_1) \wedge \mathsf{C}(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_n)\right) \\
& + (-1)^{[a_1]+\cdots+[a_{n-1}]}\mathsf{S}\left(\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{n-1}) \wedge ev_0^*(a_0)\right).
\end{aligned}
$$

Direct computations lead to

$$\mathsf{S}\left(ev_1^*(a_1) \wedge \mathsf{C}(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_n)\right) = -b_2'\left(\psi_1(\mathsf{s}a_1) \otimes (\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_n)\right)$$

for $|a_1| = 0$ and

$$
\begin{aligned}
\mathsf{S}((\mathsf{C}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{n-1}) \wedge ev_0^*(a_n)) =\;\; & \\
& (-1)^{[a_1]+\cdots+[a_{n-1}]}b_2'\left((\mathsf{S} \circ \mathsf{C})(\mathsf{s}a_2 \otimes \cdots \otimes \mathsf{s}a_{n-1}) \otimes \psi_1(\mathsf{s}a_n)\right)
\end{aligned}
$$

for $|a_n| = 0$, respectively. For $|a_1| > 0$ (respectively, $|a_n| > 0$), the first (second) expression vanishes. Thus, we obtain:

$$
\begin{aligned}
& \sum_{i=1}^{n}(-1)^{[a_1]+\cdots+[a_{i-1}]}\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{i-1} \otimes \mathsf{s}(-da_i) \otimes \mathsf{s}a_{i+1} \otimes \cdots \otimes \mathsf{s}a_n) \;+ \\
& \sum_{i=1}^{n-1}(-1)^{[a_1]+\cdots+[a_i]}\psi_{n-1}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_{i-1} \otimes \mathsf{s}(a_i \wedge a_{i+1}) \otimes \mathsf{s}a_{i+2} \otimes \cdots \otimes \mathsf{s}a_n) \\
& \qquad\qquad = (\mathsf{S} \circ \mathsf{C})(\overline{D}(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) = \\
& b_1'(\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)) + \sum_{l=1}^{n-1} b_2'\left(\psi_l(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_l) \otimes \psi_{n-l}(\mathsf{s}a_{l+1} \otimes \cdots \otimes \mathsf{s}a_n)\right),
\end{aligned}
$$

which is precisely the structure equation for an A_∞ morphism.

We remark that, strictly speaking, the above argument is not valid for the case $n = 2$ with both a_1 and a_2 of degree 0, i.e. smooth functions. However, in this case the defining relation for ψ being an A_∞ morphism is equivalent to the fact that $(a_1 a_2)(x) = a_1(x) a_2(x)$ for every point $x \in M$.

That fact that the linear component

$$\psi_1 : (\mathsf{s}\Omega(M))^{k-1} \quad \to \quad (\mathsf{s}C(M))^{k-1},$$
$$\alpha \quad \mapsto \quad \left(\sigma \mapsto (-1)^k \left(\int_{\Delta^k} \sigma^* \alpha \right) \right)$$

of the A_∞ morphism ψ induces an isomorphism in cohomology is the content of the usual de Rham theorem. Naturality of the construction follows from the naturality of S and C. $\qquad\square$

Proposition 3.26. *The A_∞ quasi-isomorphism ψ has the following properties:*

1. *$\psi_1(\mathsf{s}f) = \mathsf{s}f$ for every function $f \in \Omega^0(M)$.*

2. *For $n > 1$, $\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ vanishes, whenever one of the elements a_i is a function.*

3. *The image of ψ_n lies in the subspace $\hat{C}(M)$ of normalized simplicial cochains, i.e. the space of cochains that vanish on degenerate simplices.*

Proof. The first two claims are direct consequences of the definition of ψ and the properties of Chen's map C. Let us now prove the last claim. We need to show that

$$\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)(\epsilon_l^* \sigma) = 0,$$

where $\epsilon_l : \Delta_k \to \Delta_{k-1}$ is the l-th degeneracy map. Since

$$\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)(\epsilon_l^* \sigma) \quad = \quad \pm \int_{\Delta_n \times I^{k-1}} (\Theta_k)^*_{(n)}(\epsilon_l^{\times n})^*(p_1^* \sigma^*(a_1) \wedge \cdots \wedge p_n^* \sigma^*(a_n)),$$

it is sufficient to show that the differential of

$$\Delta_n \times I^{k-1} \xrightarrow{\ (\Theta_k)_{(n)}\ } (\Delta_k)^{\times n} \xrightarrow{\ \epsilon_l^{\times n}\ } (\Delta_{k-1})^{\times n}$$

is singular almost everywhere.

Fix a point $(t_1, \ldots, t_n, x_1, \ldots, x_{k-1}) \in \Delta_n \times I^{k-1}$. We assume without loss of generality that each of the variables t_m lies in an open interval of the form

$$\left(\frac{k - i_m}{k}, \frac{k - i_m + 1}{k} \right)$$

where i_m is an integer satisfying $1 \le i_m \le k$.

Evaluating $(\Theta_k)_{(n)}$ on $(t_1, \ldots, t_n, x_1, \ldots, x_{k-1})$ yields an element of $(\Delta_k)^n$ whose projection onto the m-th copy is given by

$$\left(\max\{x_1, \ldots, x_{i_m-1}, y_m\}, \ldots, \max\{x_{i_m-1}, y_m\}, y_m, \vec{0} \right),$$

with $y_m := k(t_m - \frac{k-i_m}{k}) x_{i_m}$. We can assume without loss of generality that all the variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are pairwise different.

Now, consider the effect of applying the degeneracy-map $\epsilon_l : \Delta_k \to \Delta_{k-1}$ to all the components of $(\Theta_k)_{(n)}(t_1, \ldots, t_n, x_1, \ldots, x_{k-1})$. The remaining expressions which (potentially) depend on x_l and $(t_m)_{i_m=l}$ also depend on x_{l-1}, i.e. we get either expressions of the form

$$\max\{x_j, \ldots, x_{l-1}, x_l, \ldots\} \quad \text{or} \quad \max\{x_j, \ldots, x_{l-1}, k(t_m - \frac{k-l}{k})x_l, \vec{0}\}$$

with $j \leq l-1$, or expressions independent of x_l and $(t_m)_{i_m=l}$. Hence, for

$$((\epsilon_l)^n \circ (\Theta_k)_{(n)})(t_1, \ldots, t_n, x_1, \cdots, x_n)$$

to depend on $(x_l, (t_m)_{i_m=l})$, we need

$$x_{l-1} < k(t_m - \frac{k-l}{k})x_l \quad \text{for all } m \text{ with } i_m = l, \quad \text{and} \quad x_{l-1} < x_l.$$

Observe that this means that x_{l-1} can only appear through expressions of the form

$$\max\{\ldots, k(t_m - \frac{k-l-1}{k})x_{l-1}, \vec{0}\}$$

where $i_m = l-1$. Therefore the restriction of the differential of $(\epsilon_l)^n \circ (\Theta_k)_{(n)}$ at the point $(t_1, \ldots, t_n, x_1, \ldots, x_{k-1})$ to the subspace spanned by

$$\left\{ \frac{\partial}{\partial x_{l-1}}, \left(\frac{\partial}{\partial t_m}\right)_{i_m=l-1} \right\}$$

is not injective, and neither is the differential on the whole tangent space. $\qquad\square$

4   The integration A_∞ functor

In this section the integration $\int [E]$ of a representation up to homotopy E of a Lie algebroid A is defined. The key concept is the holonomy map

$$\sigma \mapsto \mathsf{Hol}(\sigma, E),$$

which assigns a linear map $\mathsf{Hol}(\sigma, E) \in \mathrm{Hom}^{1-k}(E_{\sigma(v_k)}, E_{\sigma(v_0)})$ to every k-simplex $\sigma : T\Delta_k \to A$.

The main results are:

1. The assignment

$$\sigma \mapsto \mathsf{Hol}(\sigma, E)$$

defines a unital representation up to homotopy of $\Pi_\infty(A)$ (Theorem 4.13).

2. The integration map

$$\int : \mathcal{R}\mathrm{ep}^\infty(A) \quad \to \quad \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A)),$$
$$E \quad \mapsto \quad \mathsf{Hol}(-, E)$$

can be extended naturally to an A_∞ functor of dg-categories (Theorem 4.18).

Before proving these results, we need to establish one more property of the A_∞ quasi-isomorphism

$$\psi : (\Omega(M), -d, \wedge) \to (C(M), \delta, \cup).$$

4.1   Gauge invariance

Let M be a smooth manifold and V a finite dimensional graded vector space. We denote by ψ^V the A_∞ morphism

$$\psi^V := \mathrm{id}_{\mathrm{End}V} \otimes \psi : \mathrm{End}V \otimes (\Omega(M), -d, \wedge) \to \mathrm{End}V \otimes C(M).$$

In this paragraph, we will show that this A_∞ morphism is natural with respect to the gauge action. It turns out that this is a consequence of general arguments regarding A_∞ morphisms between differential graded algebras. This fact will imply that the holonomies $\mathsf{Hol}(\sigma, E)$, associated to elements of $\Pi_\infty(A)$, are independent of the trivialization of the graded vector bundle $\sigma^* E$.

Remark 4.1. Let A and B be differential graded algebras with unit. A Maurer-Cartan element $u \in \mathsf{A}^1$ gives rise to a differential graded algebra A_u with the same multiplication and twisted differential

$$d_u(a) := da + u \wedge a - (-1)^{|a|} a \wedge u.$$

Observe that this is a special instance of the twisting procedure of A_∞-algebras described in Appendix A.2.

Given an A_∞ morphism $\psi : \mathsf{A} \to \mathsf{B}$, we set

$$\psi(u) := \mathsf{s}^{-1}\left(\sum_{k \geq 1} \psi_k(\mathsf{s}u^{\otimes k})\right),$$

provided the sum converges. By Proposition A.13, $\psi(u)$ is a Maurer-Cartan element of B.

As explained in the appendix A.2, there is an A_∞ morphism

$$\psi_u : \mathsf{A}_u \to \mathsf{B}_{\psi(u)},$$

with structure maps $(\psi_u)_n$ given by

$$(\psi_u)_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n) := \sum_{l_0 \geq 0, \cdots, l_n \geq 0} \psi_{n+l_0+\cdots+l_n}((\mathsf{s}u)^{\otimes l_0} \otimes \mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n \otimes (\mathsf{s}u)^{\otimes l_n}).$$

We assume convergence of this sum.

Definition 4.2. *Let $\mathsf{A}^\times$ be the group of invertible elements of A and $MC(\mathsf{A})$ the set of Maurer-Cartan elements of A.*

Remark 4.3. Observe that there is a map

$$\begin{aligned}
\mathsf{A}^\times &\to MC(\mathsf{A}), \\
f &\mapsto u_f := f^{-1} df.
\end{aligned}$$

Conjugation by $f \in \mathsf{A}^\times$

$$\phi_f(a) := f^{-1} a f$$

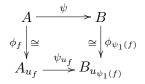yields an isomorphism of differential graded algebras

$$\phi_f : \mathsf{A} \to \mathsf{A}_{u_f}.$$

Proposition 4.4. *Let $\psi : \mathsf{A} \to \mathsf{B}$ be an A_∞ morphism between two differential graded algebras with unit. Assume that ψ satisfies the following properties:*

1. *The map ψ_1 sends the unit to the unit.*

2. *For $n > 1$, $\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n)$ vanishes whenever one of the a_i is invertible.*

Then, for any invertible element $f \in \mathsf{A}$, the following properties are satisfied:

1. *The element $\psi_1(f)$ has an inverse given by $\psi_1(f^{-1})$.*

2. *$\psi(u_f)$ equals $u_{\psi_1(f)}$.*

3. *The diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \psi\ } & B \\[2pt]
\phi_f \downarrow \cong & & \cong \downarrow \phi_{\psi_1(f)} \\[2pt]
A_{u_f} & \xrightarrow{\ \psi_{u_f}\ } & B_{u_{\psi_1(f)}}
\end{array}
$$

   is commutative.

Proof. For the first claim we use the fact that ψ is a morphism to compute:

$$
\begin{aligned}
0 \ &= \ b'_1\psi_2(\mathsf{s}f,\mathsf{s}f^{-1}) + b'_2(\psi_1(f) \otimes \psi_1(f^{-1})) - \psi_2(\mathsf{s}df \otimes \mathsf{s}f^{-1}) + \psi_2(\mathsf{s}f \otimes \mathsf{s}d(f^{-1})) \\
&\quad -\psi_1(b_2(\mathsf{s}f \otimes \mathsf{s}f^{-1})) \\
&= \ b'_2(\psi_1(f) \otimes \psi_1(f^{-1})) - \psi_1(b_2(\mathsf{s}f \otimes \mathsf{s}f^{-1})) = b'_2(\psi_1(f) \otimes \psi_1(f^{-1})) + \mathsf{s}1.
\end{aligned}
$$

In order to establish the second claim, we first have to show that

$$
u_{\psi_1(f)} = \psi_1(f)^{-1}d(\psi_1(f)) \qquad \text{and} \qquad \psi(u_f) = \sum_{k\geq 1}\psi_k((\mathsf{s}f^{-1}df)^{\otimes k})
$$

are equal. To this end, one observes that evaluating the structure equations for ψ being an A_∞ morphism on the element

$$
\mathsf{s}f \otimes (\mathsf{s}f^{-1}df)^{\otimes k}
$$

yields

$$
b'_2(\psi_1(\mathsf{s}f) \otimes \psi_k((\mathsf{s}f^{-1}df)^{\otimes k})) = \psi_{k+1}(\mathsf{s}df \otimes (\mathsf{s}f^{-1}df)^{\otimes k}) - \psi_k(\mathsf{s}df \otimes (\mathsf{s}f^{-1}df)^{\otimes k-1})
$$

and hence

$$
\sum_{k\geq 1} b'_2(\psi_1(\mathsf{s}f) \otimes \psi_k((\mathsf{s}f^{-1}df)^{\otimes k})) = -\psi_1(\mathsf{s}df),
$$

which is equivalent to $u_{\psi_1(f)} = \psi(u_f)$.

In a similar manner one proves the following equations:

$$\sum_{l \geq 0} b_2'(\psi_1(\mathsf{s}f) \otimes \psi_{k+l+1}((\mathsf{s}f^{-1}df)^{\otimes l} \otimes \mathsf{s}(f^{-1}af) \otimes D)) \;=\; -\psi_{k+1}(\mathsf{s}(af) \otimes D),$$

$$\sum_{l \geq 0} \psi_{k'+k+l+2}(C \otimes \mathsf{s}(af) \otimes (\mathsf{s}f^{-1}df)^{\otimes l} \otimes \mathsf{s}(f^{-1}bf) \otimes D) \;=\; \psi_{k'+k+2}(C \otimes \mathsf{s}a \otimes \mathsf{s}(bf) \otimes D),$$

$$\sum_{l \geq 0} \psi_{k'+l+1}(C \otimes \mathsf{s}(af) \otimes (\mathsf{s}f^{-1}df)^{\otimes l}) \;=\; (-1)^{[C]+[a]} b_2'(\psi_{k'+1}(C \otimes \mathsf{s}a) \otimes \psi_1(\mathsf{s}f)),$$

where $C \in (\mathsf{s}A)^{k'}$ and $D \in (\mathsf{s}A)^k$ are arbitrary homogeneous elements of the bar complex

$$\mathsf{B}(\mathsf{s}A) = \bigoplus_{k \geq 1} (\mathsf{s}A)^{\otimes k}.$$

Together, these equations imply

$$\sum_{l_0 \geq 0, \ldots, l_n \geq 0} b_2' \left(\psi_1(\mathsf{s}f) \otimes \psi_{l_0 + \cdots + l_n + n} \left((\mathsf{s}f^{-1}df)^{\otimes l_0} \otimes \mathsf{s}(f^{-1}a_1 f) \otimes (\mathsf{s}f^{-1}df)^{\otimes l_1} \otimes \cdots \right. \right.$$

$$\left. \left. \cdots \otimes \mathsf{s}(f^{-1}a_n f) \otimes (\mathsf{s}f^{-1}df)^{\otimes l_n} \right) \right)$$

$$= (-1)^{[a_1] + \cdots + [a_n] + 1} b_2' \left(\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n) \otimes \psi_1(\mathsf{s}f) \right),$$

which is another way to express

$$\psi_{u_f} \circ \phi_f = \phi_{\psi_1(f)} \circ \psi.$$

$\square$

Definition 4.5. *The gauge action of a differential graded algebra* A *is the right action of the group of invertible elements* $\mathsf{A}^\times$ *on the set* A *given by*

$$a \bullet f := f^{-1}af + f^{-1}df.$$

Corollary 4.6. *Let* M *be a smooth manifold and* V *be a finite dimensional graded vector space. We denote by* ψ^V *the* A_∞ *morphism*

$$\psi^V := \mathrm{id}_{\mathrm{End}V} \otimes \psi : \mathrm{End}V \otimes (\Omega(M), -d, \wedge) \to \mathrm{End}V \otimes C(M),$$

given by the tensor product of $\mathrm{id}_{\mathrm{End}V}$ *and the* A_∞ *quasi-isomorphism* ψ, *see Appendix A.2. Then:*

1. *For any* $a \in \mathrm{End}V \otimes \Omega(M)$, *the sequence*

$$\psi^V(a) := \sum_{k \geq 1} \mathsf{s}^{-1}(\psi_k^V((\mathsf{s}a)^{\otimes k}))$$

converges, in the sense that evaluating every term of the above sum on some fixed simplex yields a sum of endomorphisms which converges absolutely.

2. *The map $a \mapsto \psi^E(a)$ is equivariant with respect to the gauge action, namely:*

$$\psi^E(a \bullet f) = \psi^E(a) \bullet f.$$

Here we use the fact that the invertible elements of $\mathrm{End}V \otimes (\Omega(M), -d, \wedge)$ can also be seen as invertible elements of $\mathrm{End}V \otimes C(M)$.

Proof. The first claim follows from a simple bound that one obtains from the fact that the volume of the n-simplex is $\frac{1}{n!}$. The second claim is a formal consequence of Proposition 4.4, since Lemma 3.26 implies that ψ^V satisfies the hypothesis there. $\qquad\square$

4.2 Integration of representations up to homotopy

In this paragraph we show that the A_∞ quasi-isomorphism ψ from the de Rham algebra to the algebra of singular cochains, which was constructed in Section 3, can be used to integrate representations up to homotopy. In particular, we use ψ to define the holonomy

$$\mathsf{Hol}(\sigma, E)$$

of a simplex $\sigma \in \Pi_\infty(A)$ with respect to some representation up to homotopy E of A. Theorem 4.13 asserts that, in fact, the assignment

$$\sigma \mapsto \mathsf{Hol}(\sigma, E)$$

defines a representation up to homotopy of $\Pi_\infty(A)$.

Before that, we have to modify slightly the map ψ, by reversing the orientations in order to be consistent with Igusa's conventions in [17], as well as the conventions of [3][1].

Definition 4.7. *Given a manifold M, we denote by $\overline{C}(M)$ the dg-algebra $(C(M), \overline{\delta}, \overline{\cup})$ with:*

i) *underlying vector space the space of smooth singular cochains $C(M)$,*

ii) *differential*

$$\overline{\delta} : C^k(M) \to C^{k+1}(M), \quad \overline{\delta} := (-1)^k \delta$$

and

iii) *product given by*

$$\alpha \overline{\cup} \beta := (-1)^{|\alpha||\beta|} \alpha \cup \beta.$$

Let $\overline{i}$ be the natural isomorphism of differential graded algebras

$$\overline{i} : C(M) \to \overline{C}(M),$$
$$\alpha \mapsto (-1)^{\frac{|\alpha|(|\alpha|-1)}{2}} \alpha$$

---

[1]The need for this change of orientation comes from the fact that Igusa's maps $\Theta_{(k)}$ are maps from I^{k-1} to the space of paths in Δ_k from v_k to v_0. This is consistent with the conventions for the nerve of a category in [3], where by a sequence of composable arrows $(g_1, \ldots, g_k)$ we mean a sequence such that the target of g_i is equal to the source of g_{i-1}. The discrepancy of these conventions with the usual orientations of simplices and nerves forces us to introduce this change.

and τ the natural isomorphism of differential graded algebras

$$\tau : (\Omega(M), d, \wedge) \to (\Omega, -d, \wedge)$$
$$\eta \mapsto (-1)^{|\eta|}\eta.$$

We define

$$\overline{\psi} := \overline{i} \circ \psi \circ \tau.$$

Clearly, $\overline{\psi} : (\Omega(M), d, \wedge) \to \overline{C}(M)$ is an A_∞ quasi-isomorphism. Moreover, its components $\overline{\psi}_n : \mathsf{s}\Omega(M)^{\otimes n} \to \mathsf{s}\overline{C}(M)$ are given by

$$\overline{\psi}_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n) = (-1)^{\frac{k(k-1)}{2}+k+n-1}\psi_n(\mathsf{s}a_1 \otimes \cdots \otimes \mathsf{s}a_n),$$

where $k = [a_1] + \cdots + [a_n] + 1$.

Remark 4.8. Let M be a manifold and V a finite dimensional graded vector space. We identify the vector space $\mathrm{End}V \otimes C(M)$ with the space of cochains on M with values in $\mathrm{End}V$ as follows. For an element $\phi \otimes \eta \in \mathrm{End}V \otimes C^k(M)$ and a k-simplex σ in M, we set

$$(\phi \otimes \eta)(\sigma) := \phi\eta(\sigma) \in \mathrm{End}V.$$

A simple computation shows that if $\alpha \in \mathrm{End}^l V \otimes C^k(M)$ and $\beta \in \mathrm{End}^{l'} V \otimes C^{k'}(M)$ then:

$$(\alpha \overline{\cup} \beta)(\sigma) = (-1)^{k(l'+k')}\alpha(V_k^*(\sigma)) \circ \beta(U_{k'}^*(\sigma)).$$

Proposition 4.9. *Let M be a smooth manifold and V a finite dimensional graded vector space.*

a) *There is a natural bijective correspondence between*

   1. *Maurer-Cartan elements in the dg-algebra $\mathrm{End}V \otimes \overline{C}(M)$.*
   2. *Representations up to homotopy of the simplicial set $\Pi_\infty(M) := Sing(M)$, such that the graded vector space associated to every point is V.*

   The correspondence is given by

   $$\alpha \mapsto \mathbf{1} + \alpha \in \mathrm{End}V \otimes C(M).$$

   Here $\mathbf{1}$ is the one-cochain with values in $\mathrm{End}V$, which associates the identity to every 1-simplex.

b) *Suppose that E and E' are representations up to homotopy of $\Pi_\infty(M)$ on trivial vector bundles $M \times V$ and $M \times V'$, respectively. Then there is a natural isomorphism of vector spaces:*

   $$\mathrm{Hom}(V, V') \otimes C(M) \cong \underline{\mathrm{Hom}}(E, E'),$$

   where $\underline{\mathrm{Hom}}(E, E')$ denotes the complex of morphisms between E and E'.

Under this identification, the operator $D : \underline{\mathrm{Hom}}(E, E') \to \underline{\mathrm{Hom}}(E, E')$ corresponds to the map

$$\begin{aligned}
\mathrm{Hom}(V, V') \otimes C(M) &\to \mathrm{Hom}(V, V') \otimes C(M) \\
\eta &\mapsto \overline{\delta}\eta + \alpha'\overline{\cup}\eta - (-1)^{|\eta|}\eta\overline{\cup}\alpha,
\end{aligned}$$

where α and α' are the Maurer-Cartan elements corresponding to E and E' and the product is taken in the algebra $\mathrm{End}(V \oplus V') \otimes \overline{C}(M)$. Moreover, under the identification, composition corresponds to the product in the algebra $\mathrm{End}(V \oplus V' \oplus V'') \otimes \overline{C}(M)$.

Proof. We write $F = \mathbf{1} + \alpha$. The fact that α is a Maurer-Cartan element is equivalent to:

$$\overline{\delta}(F - \mathbf{1}) + (F - \mathbf{1})\overline{\cup}(F - \mathbf{1}) = 0.$$

Since $-\mathbf{1}$ is a Maurer-Cartan element of $\mathrm{End}V \otimes \overline{C}(M)$, this equation becomes

$$\overline{\delta}(F) + (F)\overline{\cup}(F) - \mathbf{1}\overline{\cup}F - F\overline{\cup}\mathbf{1} = 0. \tag{3}$$

We know that F can be written as a sum of homogeneous components

$$F = F_0 + F_1 + F_2 + \dots,$$

where $F_k \in \mathrm{End}^{1-k}(V) \otimes \overline{C}^k(M)$. Looking at the homogeneous components, equation (3) is equivalent to the sequence of equations

$$\overline{\delta}(F_{k-1}) + \sum_{i+j=k} F_j\overline{\cup}F_i - \mathbf{1}\overline{\cup}F_{k-1} - F_{k-1}\overline{\cup}\mathbf{1} = 0.$$

Using the fact that $\mathbf{1}\overline{\cup}F_{k-1} = (-1)^{k-1}(\mathrm{id} \otimes d_0^*)(F_{k-1})$ and $F_{k-1}\overline{\cup}\mathbf{1} = -(\mathrm{id} \otimes d_k^*)(F_{k-1})$, we obtain that the equation above is equivalent to

$$\sum_{i=1}^{k-1}(-1)^{k-1+j}(\mathrm{id} \otimes d_i^*)(F_{k-1}) + \sum_{i+j=k} F_j\overline{\cup}F_i = 0.$$

This equations are precisely the defining relations of a representation up to homotopy. This concludes the proof of the first statement.

For the second statement, we take an element

$$\eta = \eta_0 + \eta_1 + \eta_2 + \dots$$

in $(\mathrm{Hom}(V, V') \otimes \overline{C}(M))^n$ and evaluate $\overline{\delta}\eta + \alpha'\overline{\cup}\eta - (-1)^n\eta\overline{\cup}\alpha$ on a k-simplex σ.

$$\begin{aligned}
\left(\overline{\delta}\eta + \alpha'\overline{\cup}\eta - (-1)^n\eta\overline{\cup}\alpha\right)(\sigma) &= \left(\overline{\delta}\eta_{k-1} + \sum_{i+j=k} \alpha'_j\overline{\cup}\eta_i - \sum_{i+j=k}(-1)^n\eta_j\overline{\cup}\alpha_i\right)(\sigma) \\
&= \left(\sum_{i=1}^{k-1}(-1)^{k-1+i}(\mathrm{id} \otimes d_i^*)\eta_{k-1} + \sum_{i+j=k} F'_j\overline{\cup}\eta_i\right)(\sigma) \\
&\quad - \sum_{i+j=k}(-1)^n(\eta_j\overline{\cup}F_i)(\sigma) \\
&= \sum_{j=1}^{k-1}(-1)^{j+n}\eta_{k-1}(d_j(\sigma)) + \sum_{i+j=k}(-1)^{jn}F'_j \cup \eta_i(\sigma) \\
&\quad + \sum_{i+j=k}(-1)^{n+j+1}\eta_j \cup F_i(\sigma).
\end{aligned}$$

The last expression is precisely the formula for the differential on the spaces of morphisms defined in Equation (2). The remaining statements follow from similar computations. $\square$

Definition 4.10. *Let E be a representation up to homotopy of a Lie algebroid A.*
The holonomy $\mathsf{Hol}(\sigma, E)$ of a simplex $\sigma : T\Delta_k \to A$ of $\Pi_\infty(A)$ is the linear map

$$\mathsf{Hol}(\sigma, E) : E_{\sigma(v_k)} \to E_{\sigma(v_0)}$$

of degree $1 - k$ defined as follows:

1. *Pull back the representation up to homotopy along σ and choose a trivialization*

$$h : \sigma^* E \cong \Delta_k \times V$$

   of the graded vector bundle $\sigma^* E \to \Delta_k$.

2. *By Proposition A.5, the representation up to homotopy on $\Delta_k \times V$ corresponds to a Maurer-Cartan element ω of $\mathrm{End}V \otimes \Omega(\Delta_k)$. Apply the A_∞ quasi-isomorphism*

$$\overline{\psi}^V : \mathrm{End}V \otimes \Omega(\Delta_k) \to \mathrm{End}V \otimes \overline{C}(\Delta_k)$$

   to ω. This yields a cochain $\overline{\psi}^V(\omega)$ on Δ_k with values in $\mathrm{End}V$.

3. *Evaluating this cochain on the fundamental cycle $[\Delta_k] := (\mathrm{id} : \Delta_k \to \Delta_k)$ gives an element of $\mathrm{End}V$, which we interpret as a linear map from the fibre of $\Delta_k \times V$ over v_k to the fibre over v_0. Finally, we set*

$$\mathsf{Hol}(\sigma, E) := h_{v_0}^{-1} \circ <\overline{\psi}^V(\omega), [\Delta_k]> \circ h_{v_k}.$$

Lemma 4.11. *The holonomy $\mathsf{Hol}(\sigma, E)$ of a simplex $\sigma \in \Pi_\infty(A)$ is well-defined.*

Proof. We have to prove that the linear map $\mathsf{Hol}(\sigma, E)$ is independent of the chosen trivialization $h : \sigma^* E \cong \Delta_k \times V$. Changing the trivialization can be encoded in an automorphism of graded vector bundles

$$f : \Delta_k \times V \to \Delta_k \times V,$$

which can also be seen as an invertible element of the *dg*-algebra $\mathrm{End}V \otimes \Omega(\Delta_k)$.

If one uses the trivialization $f^{-1} \circ h$, the Maurer-Cartan element ω changes to

$$\omega' = f^{-1}\omega f + f^{-1}df = \omega \bullet f.$$

Using Corollary 4.6, we compute

$$
\begin{aligned}
<\mathbf{1} + \overline{\psi}^V(\omega \bullet f), [\Delta_k]> &= <\mathbf{1} + \overline{\psi}^V(\omega) \bullet f, [\Delta_k]> \\
&= <\mathbf{1} + f^{-1}\overline{\delta}f, [\Delta_k]> + <f^{-1}\overline{\cup}\overline{\psi}^V(\omega)\overline{\cup}f, [\Delta_k]> \\
&= f^{-1}(v_0) \circ <\mathbf{1}, [\Delta_k]> \circ f(v_k) + f^{-1}(v_0) \circ <\overline{\psi}^V(\omega), [\Delta_k]> \circ f(v_k) \\
&= f^{-1}(v_0) \circ \left(<\mathbf{1} + \overline{\psi}^V(\omega), [\Delta_k]>\right) \circ f(v_k).
\end{aligned}
$$

This means that different choices of trivializations lead to linear maps, which are related by conjugation. Consequently

$$\mathsf{Hol}(\sigma, E) := h_{v_o}^{-1} \circ <\overline{\psi}^V(\omega), [\Delta_k]> \circ h_{v_k}$$

is independent of the trivialization. $\square$

Remark 4.12.

1. The definition of the holonomy $\mathsf{Hol}(\sigma, -)$ of a simplex $\sigma : T\Delta_k \to A$ does not make use of all the defining properties of a representation up to homotopy. In fact, it only uses the fact that the structure operators of the pull back $\sigma^* E \cong \Delta_k \times V$ can be naturally assembled into a differential form ω with values in $\mathrm{End}V$. This is possible because the linear operator

$$D : \Omega(A, E) \to \Omega(A, E)[1],$$

which encodes the representation up to homotopy, satisfies the graded derivation rule

$$D(\omega\eta) = d(\omega)\eta + (-1)^{|\omega|}\omega D(\eta).$$

Hence, our definition of holonomy can be extended to $\mathbb{Z}$-graded A-connections, i.e. linear operators D of $\Omega(A, E)$ of degree 1, which satisfy the graded derivation rule, but do not necessarily square to zero.

2. Any connection ∇ on a vector bundle $E \to M$ gives rise to a $\mathbb{Z}$-graded TM-connection

$$d_\nabla : \Omega(A, E) \to \Omega(A, E)$$

via the Chevalley-Eilenberg formula

$$
\begin{aligned}
d\eta(\alpha_1, \ldots, \alpha_{n+1}) \;=\; & \sum_{i<j}(-1)^{i+j}\eta([\alpha_i, \alpha_j], \cdots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}) \\
& + \sum_i (-1)^{i+1}\nabla_{(\alpha_i)}\eta(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}).
\end{aligned}
$$

We claim that the holonomy

$$\mathsf{Hol}(\gamma, E) : E_{\gamma(1)} \to E_{\gamma(0)}$$

associated to a path γ is equal to the parallel transport of ∇ along $t \mapsto \gamma(1 - t)$.

To see this, we pick a trivialization $\gamma^* E \cong I \times V$ and write

$$\gamma^*\nabla = d + a(t)dt.$$

Evaluating the cochain

$$\overline{\psi}_n^V((\mathsf{s}a)^n) = (-1)^n\psi_n^V((\mathsf{s}a)^n) = (-1)^n(\mathsf{S} \circ \mathsf{C})((\mathsf{s}a)^n)$$

on the fundamental cycle $[I] = (\mathrm{id} : I \to I)$ yields

$$\int_{1 \geq t_1 \geq \cdots \geq t_n \geq 0} a(1 - t_1) \circ \cdots \circ a(1 - t_n)dt_1 \cdots dt_n.$$

Hence – up to the identification $\sigma^* E \cong I \times V$ – $\mathsf{Hol}(\gamma, E)$ is given by

$$H := \mathrm{id}_V + \sum_{n \geq 1} \left(\int_{1 \geq t_1 \geq \cdots \geq t_n \geq 0} a(1 - t_1) \circ \cdots \circ a(1 - t_n)dt_1 \cdots dt_n \right).$$

Observe that the one-parameter family of endomorphisms

$$H_t := \mathrm{id}_V + \sum_{n \geq 1} \left(\int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} a(1 - t_1) \circ \cdots \circ a(1 - t_n) dt_1 \cdots dt_n \right)$$

satisfies

$$H_0 = \mathrm{id}_V \quad \text{and} \quad \frac{d}{dt} H_t = a(1 - t) \circ H_t,$$

which is the differential equation defining the parallel transport of ∇ along $t \mapsto \gamma(1 - t)$.

Theorem 4.13. *Let A be a Lie algebroid and E a representation up to homotopy of A. The assignments*

$$
\begin{aligned}
(\Pi_\infty(A))_0 \ni x & \mapsto E_x, \\
\Pi_\infty(A) \ni \sigma & \mapsto F_k(\sigma) := \mathsf{Hol}(\sigma, E)
\end{aligned}
$$

define a unital representation up to homotopy of the ∞-groupoid $\Pi_\infty(A)$ of A.

Proof. We can assume that $\sigma^* E$ is trivial, i.e. $\sigma^* E \cong \Delta_k \times V$. Since ω is a Maurer-Cartan element of $\mathrm{End}V \otimes \Omega(\Delta_k)$, so is $\overline{\psi}^V(\omega)$ for $\mathrm{End}V \otimes \overline{C}(\Delta_k)$. By Proposition 4.9, such a Maurer-Cartan element corresponds uniquely to a representation up to homotopy of $\Pi_\infty(\Delta_k) = \mathrm{Sing}(\Delta_k)$. The defining relations of such a representation up to homotopy yield the defining relations for $\mathsf{Hol}(-, E)$ being a representation up to homotopy, evaluated on the simplex σ.

Unitality follows from part 3 of Lemma 3.26. $\qquad\qquad\square$

Definition 4.14. *Let A be a Lie algebroid. The integration map*

$$\int : \mathcal{R}ep^\infty(A) \to \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A))$$

assigns to any representation up to homotopy E of A the representation up to homotopy $\int[E]$ of $\Pi_\infty(A)$.

Remark 4.15. Every morphism of Lie algebroids $f : A \to B$ yields a *dg*-functor

$$f^* : \mathcal{R}ep^\infty(B) \to \mathcal{R}ep^\infty(A),$$

as well as a morphism of simplicial sets

$$\Pi_\infty f : \Pi_\infty(A) \to \Pi_\infty(B),$$

which, in turn, yields

$$(\Pi_\infty f)^* : \hat{\mathcal{R}}ep^\infty(\Pi_\infty(B)) \to \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A)).$$

The map $\int$ is natural with respect to pull back, i.e. the diagram

$$
\begin{array}{ccc}
\hat{\mathcal{R}}ep^\infty(\Pi_\infty(B)) & \xrightarrow{\;(\Pi_\infty f)^*\;} & \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A)) \\
{\scriptstyle \int}\big\uparrow & & \big\uparrow{\scriptstyle \int} \\
\mathcal{R}ep^\infty(B) & \xrightarrow{\;\;f^*\;\;} & \mathcal{R}ep^\infty(A)
\end{array}
$$

commutes.

4.3 The A_∞ functor

Next, we extend the map

$$\int : \mathcal{R}\mathrm{ep}^\infty(A) \quad \to \quad \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A))$$

$$E \quad \mapsto \quad \int [E] := \mathsf{Hol}(-, E),$$

which was defined in the previous subsection, to an A_∞ functor of *dg*-categories

$$\int : \mathcal{R}\mathrm{ep}^\infty(A) \to \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A)).$$

The structure maps

$$\int_n : \mathsf{s\underline{Hom}}(E_1, E_0) \otimes \cdots \otimes \mathsf{s\underline{Hom}}(E_n, E_{n-1}) \to \mathsf{s\underline{Hom}}(E_n, E_0)$$

of this A_∞ functor are given in terms of the holonomy $\mathsf{Hol}(\sigma, \phi_1, \ldots, \phi_n)$ of a simplex $\sigma : T\Delta_k \to A$ with respect to a chain of composable morphisms

$$E_0 \xleftarrow{\phi_1} E_1 \longleftarrow \cdots \longleftarrow E_{n-1} \xleftarrow{\phi_n} E_n.$$

Definition 4.16. *Let $E_1, \ldots, E_n$ be representations up to homotopy of a Lie algebroid A and $\phi_i : E_i \to E_{i-1}$ morphisms in $\mathcal{R}ep^\infty(A)$.*

The holonomy $\mathsf{Hol}(\sigma, \phi_1, \ldots, \phi_n)$ of a simplex $\sigma : T\Delta_k \to A$ is the linear map

$$\mathsf{Hol}(\sigma, \phi_1, \ldots, \phi_n) : E_n|_{\sigma(v_k)} \to E_0|_{\sigma(v_0)}$$

of degree $[\phi_1] + \cdots + [\phi_n] - k + 1$ defined as follows:

1. *Pull back the representations up to homotopy E_i and all the morphisms ϕ_i along σ and choose a trivialization*

$$h : \sigma^*(E_0 \oplus \cdots \oplus E_n) \cong \Delta_k \times (V_0 \oplus \cdots \oplus V_n) =: \Delta_k \times V.$$

2. *By Proposition A.5, the representations up to homotopy on $\Delta_k \times V_i$ corresponds to a Maurer-Cartan element ω^i of $\mathrm{End}V \otimes \Omega(\Delta_k)$, while the morphisms correspond to elements $\eta^i \in \mathrm{Hom}(V_i, V_{i-1}) \otimes \Omega(\Delta_k)$.*

3. *Apply the A_∞ quasi-isomorphism*

$$\overline{\psi}^V : \mathrm{End}V \otimes \Omega(\Delta_k) \to \mathrm{End}V \otimes \overline{C}(\Delta_k)$$

   to $\Omega := \omega_0 + \cdots + \omega_n + \eta_1 + \cdots + \eta_n$. This yields a cochain $\overline{\psi}^V(\Omega)$ on Δ_k with values in $\mathrm{End}V$.

4. *Evaluating this cochain on the fundamental cycle $[\Delta_k]$ gives an element in $\mathrm{End}V$. We define $\mathsf{Hol}(\sigma, \phi_1, \ldots, \phi_n)$ to be the composition:*

$$E_n|_{\sigma(v_k)} \xrightarrow{h_{v_k}} V \xrightarrow{<\overline{\psi}^V(\Omega), [\Delta_k]>} V \xrightarrow{h_{v_0}^{-1}} E_0|_{\sigma(v_0)}.$$

Lemma 4.17. *The holonomy* $\mathsf{Hol}(\sigma, \phi_1, \ldots, \phi_n)$ *is well-defined.*

Proof. The proof is literally the same as the one of Lemma 4.11. $\qquad\qquad\square$

Theorem 4.18. *Let A be a Lie algebroid. The assignments*

$$\mathcal{R}ep^\infty(A) \ni E \mapsto \int [E] \in \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A))$$

and

$$
\begin{aligned}
\int_n : \mathsf{s\underline{Hom}}(E_1, E_0) \otimes \cdots \otimes \mathsf{s\underline{Hom}}(E_n, E_{n-1}) &\rightarrow \mathsf{s\underline{Hom}}(E_n, E_0), \\
\phi_1 \otimes \cdots \otimes \phi_n &\mapsto \mathsf{s}\left(\mathsf{Hol}(-, \phi_1, \ldots, \phi_n)\right)
\end{aligned}
$$

define an A_∞ functor

$$\int : \mathcal{R}ep^\infty(A) \rightarrow \hat{\mathcal{R}}ep^\infty(\Pi_\infty(A))$$

between the dg-category of representations up to homotopy of A and the dg-category of unital representations up to homotopy of $\Pi_\infty(A)$.

Proof. We need to prove the equations

$$\beta_1' \circ \int_n + \sum_{i+j=n} \beta_2' \circ \left(\int_i \otimes \int_j\right) = \sum_{i+j+1=n} \int_n \circ \left(\mathrm{id}^{\otimes i} \otimes \beta_1 \otimes \mathrm{id}^{\otimes j}\right) + \sum_{i+j+2=n} \int_{n-1} \circ \left(\mathrm{id}^{\otimes i} \otimes \beta_2 \otimes \mathrm{id}^{\otimes j}\right),$$

$$(4)$$

where the operators

$$
\begin{aligned}
\beta_1' &: \mathsf{s\underline{Hom}}(\textstyle\int [E_n], \int [E_0]) \rightarrow \mathsf{s\underline{Hom}}(\int [E_n], \int [E_0]), \\
\beta_2' &: \mathsf{s\underline{Hom}}(\textstyle\int [E_i], \int [E_0]) \otimes \mathsf{s\underline{Hom}}(\int [E_n], \int [E_i]) \rightarrow \mathsf{s\underline{Hom}}(\int [E_n], \int [E_0]), \\
\beta_1 &: \mathsf{s\underline{Hom}}(E_{i+1}, E_i) \rightarrow \mathsf{s\underline{Hom}}(E_{i+1}, E_i) \quad \text{and} \\
\beta_2 &: \mathsf{s\underline{Hom}}(E_{i+1}, E_i) \otimes \mathsf{s\underline{Hom}}(E_{i+2}, E_{i+1}) \rightarrow \mathsf{s\underline{Hom}}(E_{i+2}, E_i),
\end{aligned}
$$

are the maps corresponding to the composition and the differential on the spaces of morphisms at the level of the suspension.

It suffices to show that the right and left hand side of (4) evaluate to the same linear map on all simplices in $\Pi_\infty(A)$. Since the construction is natural with respect to pull-back, we may assume that $A = T\Delta_k$ and that the simplex σ is the fundamental cochain $[\Delta_k] = (\mathrm{id} : \Delta_k \rightarrow \Delta_k)$. Moreover, by Lemma 4.17 we can assume that all the graded vector bundles E_i are trivial, i.e. $E_i = \Delta_k \times V_i$.

In this situation, Proposition A.5 and Proposition 4.9 provide natural identifications of graded vector spaces:

$$\underline{\mathrm{Hom}}(E_i, E_j) \cong \mathrm{Hom}(V_i, V_j) \otimes \Omega(\Delta_k),$$

$$\underline{\mathrm{Hom}}(\textstyle\int [E_i], \int [E_j]) \cong \mathrm{Hom}(V_i, V_j) \otimes C(\Delta_k).$$

Under these identifications, composition of operators corresponds to multiplication in the algebras $\mathrm{End}V \otimes \Omega(\Delta_k)$ and $\mathrm{End}V \otimes \overline{C}(\Delta_k)$, respectively. Also, the differentials

$$\delta : \underline{\mathrm{Hom}}(E_i, E_j) \to \underline{\mathrm{Hom}}(E_i, E_j)$$

and

$$D : \underline{\mathrm{Hom}}(\int [E_i], \int [E_j]) \to \underline{\mathrm{Hom}}(\int [E_i], \int [E_j])$$

correspond to

$$\begin{aligned}
\mathrm{Hom}(V_i, V_j) \otimes \Omega(\Delta_k)s &\to \mathrm{Hom}(V_i, V_j) \otimes \Omega(\Delta_k), \\
\eta &\mapsto d\eta + \omega^j \wedge \eta - (-1)^{|\eta|}\eta \wedge \omega^i
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}(V_i, V_j) \otimes \overline{C}(\Delta_k) &\to \mathrm{Hom}(V_i, V_j) \otimes \overline{C}(\Delta_k), \\
\mu &\mapsto \overline{\delta}\mu + \alpha^j \overline{\cup}\mu - (-1)^{|\mu|}\mu\overline{\cup}\alpha^i,
\end{aligned}$$

where ω^i and α^i are the Maurer-Cartan elements corresponding to the representations up to homotopy E_i and $\int[E_i]$.

We observe that the map $\int_n$ corresponds to

$$\overline{\int}_n : \mathsf{s}(\mathrm{Hom}(V_1, V_0) \otimes \Omega(\Delta_k)) \otimes \cdots \otimes \mathsf{s}(\mathrm{Hom}(V_n, V_{n-1}) \otimes \Omega(\Delta_k)) \to \mathsf{s}(\mathrm{Hom}(V_n, V_0) \otimes \overline{C}(\Delta_k))$$

given by the formula

$$(\mathsf{s}\eta^1 \otimes \cdots \otimes \mathsf{s}\eta^n) \mapsto \sum_{l_0 \geq 0, \ldots, l_n \geq 0} \overline{\psi}^V_{l_0 + \cdots + l_n + n}((\mathsf{s}\omega^0)^{\otimes l_0} \otimes \mathsf{s}\eta^1 \otimes \cdots \otimes \mathsf{s}\eta^n \otimes (\mathsf{s}\omega^n)^{\otimes l_n}).$$

Let us define $\omega = \omega^0 + \cdots + \omega^n \in \mathrm{End}V \otimes \Omega(\Delta_k)$ and $\alpha = \alpha_0 + \cdots + \alpha_n \in \mathrm{End}V \otimes \overline{C}(\Delta_k)$. Clearly, ω and α are Maurer-Cartan elements and are related by $\overline{\psi}^V(\omega) = \alpha$.

Let b_2, b_2' are the operators corresponding to the products on $\mathrm{End}V \otimes \Omega(\Delta_k)$ and $\mathrm{End}V \otimes \overline{C}(\Delta_k)$ at the level of the suspension and b_1, b_1' correspond to the twisted differentials d_ω and $\overline{\delta}_\alpha$. We need to prove

$$b_1' \circ \overline{\int}_n + \sum_{i+j=n} b_2' \circ \left(\overline{\int}_i \otimes \overline{\int}_j\right) = \sum_{i+j+1=n} \overline{\int}_n \circ \left(\mathrm{id}^{\otimes i} \otimes b_1 \otimes \mathrm{id}^{\otimes j}\right) + \sum_{i+j+2=n} \overline{\int}_{n-1} \circ \left(\mathrm{id}^{\otimes i} \otimes b_2 \otimes \mathrm{id}^{\otimes j}\right).$$
(5)

For this we note that this is exactly the component in $\mathrm{Hom}(V_n, V_0)$ of the equations for the twist $\overline{\psi}^V_\omega$ of $\overline{\psi}^V$ by ω to be an A_∞ morphism. □

5 Examples

In the following we describe some examples of representations up to homotopy of Lie algebroids to which the A_∞ functor

$$\int : \mathcal{R}\mathrm{ep}^\infty(A) \to \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A))$$

can be applied.

5.1 Parallel transport for superconnections

The situation where the Lie algebroid is the tangent bundle TM of a manifold M has been studied by Igusa [17] and Block-Smith [5], and was the starting point of our work. In this case, representations up to homotopy are precisely the $\mathbb{Z}$-graded versions of Quillen's flat superconnections ([22]). Since

$$\mathrm{Hom}_{\mathsf{DGCA}}(\Omega(TM), \Omega(\Delta_k)) \cong \mathrm{Hom}_{\mathcal{C}^\infty}(\Delta_k, M),$$

we know that

$$\Pi_\infty(TM) \cong \mathrm{Sing}(M).$$

Thus, integrating a representation up to homotopy of the tangent Lie algebroid TM amounts to assigning holonomies to smooth singular chains on M. As explained in [5], this procedure is a generalization of the Riemann-Hilbert correspondence. Just as flat connections correspond, via holonomy, to representations of the fundamental groupoid of M, flat superconnections correspond to representations up to homotopy of the infinity groupoid of M.

5.2 Ordinary representations

Let A be an integrable Lie algebroid with integrating source simply connected Lie groupoid G. By Lie's second theorem for Lie algebroids, every one-simplex

$$\sigma \in \mathrm{Hom}_{\mathsf{Lie-alg}}(TI, A) = \mathrm{Hom}_{\mathsf{DGCA}}(\Omega(A), \Omega(I))$$

integrates to a morphism of Lie groupoids

$$\tilde{\sigma} : I \times I \to G$$

from the pair groupoid of I to G. We set the target (source) map to be the projection onto the first (second) factor, respectively.

Let $\tau(\sigma)$ be the value of $\tilde{\sigma}$ on $(1, 0) \in I \times I$, i.e.

$$\tau(\sigma) := \tilde{\sigma}(1, 0).$$

One obtains a morphism of simplicial sets

$$\pi : \Pi_\infty(A) \to \mathsf{N}G$$

from the ∞-groupoid $\Pi_\infty(A)$ of A to the nerve $\mathsf{N}G$ of the Lie groupoid G by setting

$$\pi(\sigma) := (\tau(\sigma_1), \ldots, \tau(\sigma_k)).$$

Here, σ is a k-simplex in $\Pi_\infty(A)$ and σ_i is the one simplex given by the edge joining v_i and v_{i-1}; that is: the one simplex given by pre-composing σ with (the tangent map of)

$$t \mapsto (\underbrace{1, \ldots, 1}_{i-1 \text{ times}}, 1 - t, 0, \ldots, 0).$$

The morphism $\pi : \Pi_\infty(A) \to \mathsf{N}G$ induces a functor of dg-categories

$$\pi^* : \hat{\mathcal{R}}\mathrm{ep}^\infty(G) \hookrightarrow \hat{\mathcal{R}}\mathrm{ep}^\infty(\mathsf{N}G) \to \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A)).$$

The inclusion $\hat{\mathcal{R}}\mathrm{ep}^\infty(G) \hookrightarrow \hat{\mathcal{R}}\mathrm{ep}^\infty(\mathsf{N}G)$ reflects the fact that we require the graded vector spaces $(E_x)_{x \in M}$ underlying a representation up to homotopy of a *Lie* groupoid G over M to fit into a graded *vector bundle* E.

Proposition 5.1. *Let E be an ordinary representation of the Lie algebroid A and denote by $\tilde{E}$ the corresponding representation of G.*

Then

$$\pi^* \tilde{E} = \int [E]$$

holds in $\hat{\mathcal{R}}ep^\infty(\Pi_\infty(A))$.

Proof. By degree reasons we know that the only nontrivial structure operator for $\int[E]$ is F_1. Clearly, the same is true for $\pi^*(\tilde{E})$. Let us denote by $\mu : G \to \mathrm{End}E$ the structure operator for the representation $\tilde{E}$. We need to prove that for any one simplex $\sigma : TI \to A$ one has

$$F_1(\sigma) = \mu(\pi(\sigma)).$$

Naturality of the the integration construction – see Remark 4.15 – implies

$$F_1(\sigma) = F_1^E(\sigma \circ \mathrm{id}) = F_1^{\sigma^*(E)}(\mathrm{id}).$$

We denote by $\theta : I \times I \to \mathrm{End}(\sigma^* E)$ the structure map corresponding to the representation that integrates $\sigma^* E$ and observe that since $\theta = \mu \circ \tilde{\sigma}$ we know that

$$\theta(1,0) = \mu(\tilde{\sigma}(1,0)) = \mu(\widetilde{\sigma_1}(1,0)) = \mu(\pi(\sigma))$$

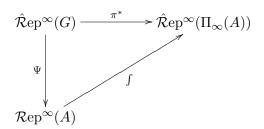holds. Thus, it is sufficient to prove that

$$F_1^{\sigma^*(E)}(\mathrm{id}) = \theta(1,0),$$

i.e. we may assume without loss of generality $A = TI$ and $\sigma = \mathrm{id}$. In this case the statement reduces to the fact that the holonomy of a path is given by parallel transport along the direction-reversed path, see Remark 4.12. $\qquad\qquad\square$

Remark 5.2. The differentiation functor

$$\Psi : \hat{\mathcal{R}}\mathrm{ep}^\infty(G) \to \mathcal{R}\mathrm{ep}^\infty(A)$$

constructed in [4] can be extended naturally to a *dg*-functor. The definition of Ψ involves iterated differentiation and in a sense it is given by the inverse construction of the integration procedure described above. Proposition 5.1 shows that, when restricted to ordinary representations, this is indeed the case. However, we would like to point out that the diagram

$$\begin{array}{ccc}
\hat{\mathcal{R}}\mathrm{ep}^\infty(G) & \xrightarrow{\ \pi^*\ } & \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A)) \\
\Big\downarrow{\scriptstyle \Psi} & \nearrow{\scriptstyle \int} & \\
\mathcal{R}\mathrm{ep}^\infty(A) & &
\end{array}$$

does not commute in general. We expect that it commutes up to homotopy, but this issue will not be discussed here.

5.3 Lie algebras and flat connections

Let us fix a finite dimensional Lie algebra $\mathfrak{g}$ and denote by G the simply connected Lie group integrating it. We think of the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ as the space of functions on the dg-manifold $\mathfrak{g}[1]$ determined by $\mathfrak{g}$.

The space of k-simplices on $\mathfrak{g}$

$$\Pi_\infty(\mathfrak{g})_k = \mathrm{Hom}_{\mathsf{DGCA}}(CE(\mathfrak{g}), \Omega(\Delta_k))$$

can be naturally identified with the space of flat connections on the trivial G bundle over Δ_k. In fact, any such map is in particular a map of graded commutative algebras, and since $CE(\mathfrak{g})$ is free as a graded commutative algebra, one obtains

$$\mathrm{Hom}_{\mathsf{GCA}}(CE(\mathfrak{g}), \Omega(\Delta_k)) \cong \mathrm{Hom}_{\mathsf{Vect}}(\mathfrak{g}^*, \Omega^1(\Delta_k)) \cong \mathfrak{g} \otimes \Omega^1(\Delta_k).$$

One easily checks that the condition of commuting with the differential corresponds to the Maurer-Cartan equation. We conclude that integrating a representation up to homotopy of a Lie algebra amounts to assigning holonomies to the spaces of flat connections on the trivial G-bundles over the simplices.

5.4 The adjoint representation of a Lie algebroid

Given a Lie algebroid A, there is a representation up to homotopy $\mathrm{ad}(A) \in \mathcal{R}\mathrm{ep}^\infty(A)$ – called the adjoint representation of A – which is well defined up to isomorphism. For instance:

- If A is a Lie algebra, $\mathrm{ad}(A)$ coincides with the usual adjoint representation.

- If A is a foliation, $\mathrm{ad}(A)$ is quasi-isomorphic to the Bott connection.

The cohomology associated to the adjoint representation measures the deformations of the Lie algebroid structure, as one would expect from the Lie algebra case – see [2, 4, 12] for more details.

In order to define the adjoint representation, one has to choose a connection ∇ on the vector bundle A. The adjoint representation of A induced by ∇ is the representation up to homotopy of A on the vector bundle $A \oplus TM$, where A is in degree zero and TM in degree one, given by the following structure operator:

$$D = \partial + \nabla^{bas} + K_\nabla.$$

The differential ∂ on the graded vector bundle is the anchor map of A, ∇^{bas} is an A-connection and K_∇ is an endomorphism-valued cochain. The latter two are defined by the formulae

$$
\begin{aligned}
\nabla^{bas}_\alpha(\beta) &:= \nabla_{\rho(\beta)}(\alpha) + [\alpha, \beta], \\
\nabla^{bas}_\alpha(X) &:= \rho(\nabla_X(\alpha)) + [\rho(\alpha), X] \quad, \\
K_\nabla(\alpha, \beta)(X) &:= \nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)] - \nabla_{\nabla^{bas}_\beta X}(\alpha) + \nabla_{\nabla^{bas}_\alpha X}(\beta).
\end{aligned}
$$

This representation up to homotopy is denoted by $\mathrm{ad}_\nabla(A)$. The isomorphism class of this representation, which is independent of the connection ∇, is denoted by $\mathrm{ad}(A)$.

5.5 Poisson manifolds

Let P be a Poisson manifold with Poisson bivector field $\pi \in \Gamma(\wedge^2 TP)$. The cotangent bundle T^*P has the structure of a Lie algebroid with anchor map $TP^* \to TP$ given by contraction with π and bracket determined by the formula

$$[df, dg] = d\{f, g\}.$$

The Poisson sigma model ([18, 24, 7]) is a two dimensional topological field theory associated to a Poisson manifold, whose perturbative quantization gives Kontsevich's formula for a $\star$-product of π. Let us briefly recall the main ingredients of this theory: The fields on an oriented surface Σ are the vector bundle maps $\hat{X} : T\Sigma \to T^*P$. It is convenient to write $\hat{X} = (X, \eta)$, where $X : \Sigma \to P$ is the base map and $\eta \in \Gamma(\operatorname{Hom}(T\Sigma, X^*(T^*P)))$. The action functional is

$$S(X, \eta) := \int_\Sigma \langle \eta, dX \rangle + \frac{1}{2} \langle \pi \circ X, \eta \wedge \eta \rangle.$$

The classical solutions of this theory are those vector bundle maps which are Lie algebroid morphisms from $T\Sigma$ to T^*P.

We conclude that integrating a representation up to homotopy of the Lie algebroid T^*P amounts to assigning holonomies to the classical fields of the Poisson sigma model on the simplex. We hope to make the connection between the integration of representations up to homotopy of T^*P and the Poisson sigma model more precise elsewhere.
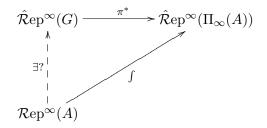
5.6 $\Pi_\infty(-)$ vs $\Pi_1(-)$

Remark 5.3. In Section 4, the integration $\int[E]$ of a representation up to homotopy E of a Lie algebroid A was defined. It is a representation up to homotopy of the ∞-groupoid $\Pi_\infty(A)$ of A.

Let us assume that A can be integrated to a source simply connected Lie groupoid G, which is Hausdorff. In Subsection 5.2, a map of simplicial sets

$$\pi : \Pi_\infty(A) \to \mathsf{N}G$$

was constructed. One might wonder whether the above integrability assumption on A implies that $\int$ factors through π^*, i.e. whether it is possible to complete the diagram

$$
\begin{array}{ccc}
\hat{\mathcal{R}}\mathrm{ep}^\infty(G) & \xrightarrow{\ \pi^*\ } & \hat{\mathcal{R}}\mathrm{ep}^\infty(\Pi_\infty(A)) \\
\big\uparrow{\scriptstyle\exists?} & \nearrow{\scriptstyle\int} & \\
\mathcal{R}\mathrm{ep}^\infty(A) & &
\end{array}
$$

in a commutative way. Intuitively, one should not expect this to be the case, since G only contains information about homotopy classes of A-paths and not about the higher homotopies. In any case, we present an example that shows that this factorization is not possible.

Proposition 5.4. *The integration $\int[E]$ of the representation up to homotopy E of TS^2 associated to the graded vector bundle $E = E^0 \oplus E^1$ given by the trivial vector bundle $S^2 \times \mathbb{R}$ in both degrees, equipped with the structure operator*

$$D = d + \eta,$$

39

where $\eta \in \Omega^2(S^2)$ *is a volume form, viewed as an element of* $\Omega^2(S^2, \text{Hom}(E^1, E^0))$, *is not quasi-isomorphic to any element of* $\pi^* \hat{\mathcal{R}} ep^\infty(G)$.

Remark 5.5. In order to prove the proposition, we need to introduce the cohomology associated to a representation up to homotopy E of a simplicial set $X_\bullet$:

Definition 5.6. *Let E be a representation up to homotopy of $X_\bullet$. The cohomology of $X_\bullet$ with values in E is the graded vector space*

$$H(X_\bullet, E) := H(\underline{\text{Hom}}(\mathbb{R}, E)),$$

where $\mathbb{R}$ denotes the trivial representation up to homotopy of $X_\bullet$ on $\mathbb{R}$.

We can now proceed with the proof of Proposition 5.4.
Proof. Observe that the source simply connected Lie groupoid integrating TS^2 is the pair groupoid $S^2 \times S^2$. Next, suppose that $\mathcal{E}$ is a representation up to homotopy of G, whose pull back $\pi^* \mathcal{E}$ is quasi-isomorphic to $\int [E]$. By definition, this implies that the fiberwise cohomologies are isomorphic, hence

$$\mathcal{H}(\mathcal{E}_x) = \mathcal{H}((\pi^* \mathcal{E})_x) \cong \mathcal{H}(\int [E]_x) = \mathcal{H}(E_x) = E_x = \mathbb{R} \oplus \mathbb{R}[-1].$$

Since any representation up to homotopy can be transfered to its cohomology vector bundle, we may assume without loss of generality that $\mathcal{E}_x = \mathbb{R} \oplus \mathbb{R}[-1]$.

By degree reasons we know that the structure operator of $\mathcal{E}$ is of the form

$$F_1 + F_2.$$

Next, we claim that one can choose an automorphism of the vector bundle $\mathcal{E}$ that conjugates F_1 to the identity. Indeed, choose a point $x \in S^2$ and define the automorphism

$$\phi : \mathcal{E} \to \mathcal{E}$$

by setting
$$\phi(y) = F_1(y, x).$$

Since V is unital, one can easily check that $\text{id} = \phi^{-1} \circ F_1 \circ \phi$. Thus, we can further assume $F_1 = \text{id}$.

The operator $F_2 \in C^2(G)$ is a closed two-cocycle of the pair groupoid $S^2 \times S^2$. Since the nerve of this groupoid is contractible, we know that F_2 is exact and therefore $\mathcal{E}$ is isomorphic to the sum of trivial representations $\mathbb{R} \oplus \mathbb{R}[-1]$. This implies

$$\pi^*(\mathcal{E}) \cong \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}[-1].$$

We can now compute the cohomology associated to this representation:

$$H(\Pi_\infty(TS^2), \pi^* \mathcal{E}) \cong H(\text{Sing}(S^2), \mathbb{R} \oplus \mathbb{R}[-1]) \cong H(S^2) \oplus H(S^2)[-1].$$

In particular, we observe that
$$H^2(\Pi_\infty(TS^2), \pi^* \mathcal{E}) \cong \mathbb{R}.$$

Since quasi-isomorphisms induce isomorphisms in cohomology, it is enough to prove that
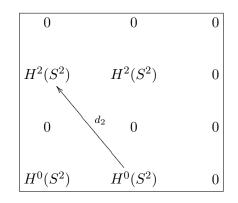
$$H^2(\Pi_\infty(TS^2), \int [E])$$

vanishes, in order to contradict our assumption that $\pi^*\mathcal{E}$ and $\int[E]$ are quasi-isomorphic.

Simple degree considerations imply that the only nontrivial structure operator of $\int[E]$ is $F_2' \in C^2(S^2)$. We claim that $[F_2'] \neq 0 \in H^2(S^2)$. Indeed:

$$F_2'(\sigma) = \pm \int_I \Theta_{(2)}^*(\mathsf{P}\sigma)^*(\mathsf{C}(\mathsf{s}\eta)) = \pm \int_{\Delta_2} \sigma^*\eta.$$

By de Rham's theorem, we conclude that F_2' is a nontrivial cohomology class.

To compute the cohomology of $\mathrm{Sing}(S^2)$ with coefficients in $\int[E]$, we consider the filtration of $C(\mathrm{Sing}(S^2), \int[E])$ given by the cochain degree. It induces a spectral sequence which converges to $H(\mathrm{Sing}(S^2), \int[E])$ and whose second page is

The operator d_2 is given by multiplication with $[F_2']$ and since this class is not zero, the second cohomology

$$H^2(\Pi_\infty(TS^2), \int [E])$$

vanishes. □

Remark 5.7. Mimicking the arguments from above, one can prove that there is no representation up to homotopy $\mathcal{E}$ of G, whose image under the differentiation map

$$\Psi : \hat{\mathcal{R}}\mathrm{ep}^\infty(G) \to \mathcal{R}\mathrm{ep}^\infty(A)$$

from [4] equals E. In fact, if such an $\mathcal{E}$ would exist, the cohomology class of ω would be in the image of the van Est map and would hence vanish, since $H^2(S^2 \times S^2) = 0$.

A   Appendix

A.1   Representations up to homotopy of Lie algebroids

We review the definitions and basic facts regarding representations up to homotopy of Lie algebroids. More details on these constructions, as well as the proofs of the results stated here, can be found in [2, 4]. Throughout the appendix A denotes a Lie algebroid over a manifold M.

Remark A.1. Given a Lie algebroid A, there is a differential graded algebra $\Omega(A) = \Gamma(\wedge A^*)$, with differential defined via the Koszul formula

$$
\begin{aligned}
d\omega(\alpha_1, \ldots, \alpha_{n+1}) &= \sum_{i<j}(-1)^{i+j}\omega([\alpha_i, \alpha_j], \cdots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}) \\
&+ \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}\omega(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}),
\end{aligned}
$$

where ρ denotes the anchor map and $L_X(f) = X(f)$ is the Lie derivative along vector fields. The operator d is a coboundary operator $(d^2 = 0)$ and satisfies the derivation rule

$$
d(\omega\eta) = d(\omega)\eta + (-1)^p\omega d(\eta),
$$

for all $\omega \in \Omega^p(A), \eta \in \Omega^q(A)$.

Given a graded vector bundle $E = \bigoplus_{k\in\mathbb{Z}} E^k$ over M, we denote by $\Omega(A, E)$ the space

$$
\Gamma(E \otimes \wedge A^*),
$$

graded with respect to the total degree. The wedge product gives this space the structure of a graded commutative module over the algebra $\Omega(A)$, i.e. $\Omega(A, E)$ is a bimodule over $\Omega(A)$ and

$$
\omega \wedge \eta = (-1)^{kp}\eta \wedge \omega
$$

holds for $\omega \in \Omega^k(A)$ and $\eta \in \Omega(A, E)^p$. For a detailed explanation of the sign conventions used in the definition in the wedge product, see the appendix of [2]. In order to simplify the notation we will sometimes omit the wedge symbol.

Definition A.2. *A representation up to homotopy of A consists of a graded vector bundle E over M and a linear operator*

$$
D : \Omega(A, E) \to \Omega(A, E)
$$

which increases the total degree by one and satisfies $D^2 = 0$, as well as the graded derivation rule

$$
D(\omega\eta) = d(\omega)\eta + (-1)^k\omega D(\eta)
$$

for all $\omega \in \Omega^k(A), \eta \in \Omega(A, E)$.

Remark A.3. The representations up to homotopy of A can be naturally organized into a *dg*-category $\mathcal{R}ep^\infty(A)$: A morphism $\phi : E \to F$ between two representations up to homotopy of A is a map

$$
\phi : \Omega(A, E) \to \Omega(A, F)
$$

which is $\Omega(A)$-linear in the graded sense. We denote the space of all morphisms from E to F by $\underline{\mathrm{Hom}}(E, F)$. This is a graded vector space and we denote by $\underline{\mathrm{Hom}}^k(E, F)$ the subspace of homogeneous elements of degree k. There is a differential

$$
\delta : \underline{\mathrm{Hom}}^k(E, F) \to \underline{\mathrm{Hom}}^{k+1}(E, F)
$$

given by the formula

$$
\delta(\phi) = D_F \circ \phi - (-1)^k \phi \circ D_E.
$$

Let $\mathrm{Hom}(E, F)$ be the space of morphisms of degree zero which commute with the differentials. Observe that

$$
\mathrm{Hom}(E, F) = Z^0(\underline{\mathrm{Hom}}(E, F)).
$$

Remark A.4. Let E be a graded vector space. The space $\Omega(A, E) = E \otimes \Omega(A)$ is a differential left module over the dg-algebra $\mathrm{End}E \otimes \Omega(A)$. Given $\omega \in \mathrm{End}E \otimes \Omega(A)$, we denote the corresponding operator on $E \otimes \Omega(A)$ by $\omega\wedge$.

One can describe representations up to homotopy on trivial vector bundles as follows:

Proposition A.5. *Let A be a Lie algebroid over M and V a finite dimensional graded vector space.*

a) *There is a natural bijective correspondence between*

1. *Maurer-Cartan elements of the dg-Lie algebra $\mathrm{End}E \otimes \Omega(A)$.*
2. *Representations up to homotopy of A on the trivial vector bundle $M \times V$.*

The correspondence is given by
$$\omega \mapsto d + \omega \wedge .$$

b) *Suppose that E and E' are representations up to homotopy on trivial vector bundles $M \times V$ and $M \times V'$, respectively. There is a natural isomorphism*

$$\begin{aligned} \mathrm{Hom}(V, V') \otimes \Omega(A) &\cong \underline{\mathrm{Hom}}(E, E'). \\ \eta &\mapsto \eta\wedge \end{aligned}$$

Under this identification, the operator $\delta : \underline{\mathrm{Hom}}(E, E') \to \underline{\mathrm{Hom}}(E, E')$ corresponds to the map

$$\begin{aligned} \mathrm{Hom}(V, V') \otimes \Omega(A) &\to \mathrm{Hom}(V, V') \otimes \Omega(A) \\ \eta &\mapsto d\eta + \omega' \wedge \eta - (-1)^{|\eta|}\eta \wedge \omega, \end{aligned}$$

where ω and ω' are the Maurer-Cartan elements corresponding to E and E' and the wedge product is taken in the algebra $\mathrm{End}(V \oplus V') \otimes \Omega(A)$. Furthermore, composition corresponds to multiplication in the algebra $\mathrm{End}(V \oplus V' \oplus V'') \otimes \Omega(A)$.

A.2 A_∞-algebras and morphisms

The notion of an A_∞-algebra was introduced in the sixties by J. Stasheff [25, 26], and has since proved to be important in several areas of mathematics. In order to fix our conventions, we collect here some definitions and basic facts regarding A_∞-algebras.

Definition A.6. *Let $V = \bigoplus_{k \in \mathbb{Z}} V^k$ be a graded vector space. The suspension $\mathsf{s}V$ of V is the graded vector space given by*
$$(\mathsf{s}V)^k := V^{k+1}.$$

Definition A.7. *An A_∞-algebra is a graded vector space A, together with a sequence of linear maps of degree one*
$$b_n : (\mathsf{s}\mathsf{A})^{\otimes n} \to s\mathsf{A},$$

satisfying, for each $n \geq 1$, the equations:

$$\sum_{i+j+k=n} b_{i+k+1} \circ (\mathrm{id}^{\otimes i} \otimes b_j \otimes \mathrm{id}^{\otimes k}) = 0. \tag{6}$$

Examples A.8. If the graded vector space A is concentrated in degree zero, an A_∞-algebra on A is the same as an associative algebra. A differential graded algebra is the same as an A_∞-algebra where $b_n = 0$ for $n \notin \{1, 2\}$.

Observe that b_1 always defines a coboundary operator on the graded vector space A.

Remark A.9. An alternative definition of A_∞-algebras can be described in terms of maps $\tilde{b}_n : \mathsf{A}^{\otimes n} \to \mathsf{A}$. The advantage of using the suspension is that no signs appear in the structure equations. The definition in terms of the structure maps $\tilde{b}_n$ requires some signs that can be determined by requiring the following diagram to commute:

$$
\begin{array}{ccc}
(\mathsf{s}\mathsf{A})^{\otimes n} & \xrightarrow{\ b_n\ } & \mathsf{s}\mathsf{A} \\
{\scriptstyle (s^{-1})^\otimes}\big\downarrow & & \big\uparrow{\scriptstyle s} \\
\mathsf{A}^{\otimes n} & \xrightarrow{\ \tilde{b}_n\ } & \mathsf{A}.
\end{array}
$$

Here, the map $\mathsf{s} : \mathsf{A} \to \mathsf{s}\mathsf{A}$ is given by $\mathsf{s}(v) = v \in \mathsf{s}\mathsf{A}$.

Definition A.10. *Let A and A' be two A_∞-algebras. A morphism $\psi : \mathsf{A} \to \mathsf{A}'$ is a sequence of degree zero maps*

$$
\psi_n : (\mathsf{s}\mathsf{A})^{\otimes n} \to \mathsf{s}\mathsf{A},
$$

satisfying, for each $n \geq 1$, the equation

$$
\sum_{i+j+k=n} \psi_{i+k+1} \circ (\mathrm{id}^{\otimes i} \otimes b_j \otimes \mathrm{id}^{\otimes k}) = \sum_{l_1 + \cdots + l_r = n} b'_r \circ (\psi_{l_1} \otimes \cdots \otimes \psi_{l_r}). \tag{7}
$$

A morphism $\psi : \mathsf{A} \to \mathsf{A}'$ is an A_∞ quasi-isomorphism if the chain map

$$
\psi_1 : (\mathsf{A}, b_1) \to (\mathsf{A}', b'_1)
$$

induces an isomorphism in cohomology.

If $\psi : \mathsf{A} \to \mathsf{A}'$ and $\psi' : \mathsf{A}' \to \mathsf{A}''$ are to morphisms, their composition $\psi' \circ \psi$ is defined by

$$
(\psi' \circ \psi)_n = \sum_{i_1 + \cdots + i_r = n} \psi'_r \circ (\psi_{i_1} \otimes \cdots \otimes \psi_{i_r}).
$$

The identity morphism $\mathrm{id} : \mathsf{A} \to \mathsf{A}$ is defined by setting $\mathrm{id}_1 = \mathrm{id}$ and $\mathrm{id}_n = 0$ for $n \neq 1$.

Remark A.11. In the case where $\psi : \mathsf{A} \to \mathsf{A}'$ is an A_∞ morphism between *dg*-algebras, the structure equations (7) take the form:

$$
b'_1 \circ \psi_n + \sum_{i+j=n} b'_2 \circ (\psi_i \otimes \psi_j) = \sum_{i+j+1=n} \psi_n \circ (\mathrm{id}^{\otimes i} \otimes b_1 \otimes \mathrm{id}^{\otimes j}) + \sum_{i+j+2=n} \psi_{n-1} (\mathrm{id}^{\otimes i} \otimes b_2 \otimes \mathrm{id}^{\otimes j}) \tag{8}
$$

Definition A.12. *Let A be an A_∞-algebra such that $b_n = 0$, except for finitely many values of n. A Maurer-Cartan element of A is an element $x \in (\mathsf{s}\mathsf{A})^0$ satisfying*

$$
\sum_{n \geq 1} b_n(x^{\otimes n}) = 0.
$$

We will sometimes abuse our conventions and say that $y \in \mathsf{A}^1$ is a Maurer-Cartan element if $\mathsf{s}y \in (\mathsf{s}\mathsf{A})^0$ is a Maurer-Cartan element.

Proposition A.13. *Let $\psi : A \to A'$ be an A_∞ morphism between A_∞-algebras such that b_n, b'_n and ψ_n are nonzero only for finitely many values of n. If $x \in (sA)^0$ is a Maurer-Cartan element of A, then*

$$\psi(x) := \sum_{n \geq 1} \psi_n(x^{\otimes n})$$

is a Maurer-Cartan element of A'.

Remark A.14. The hypothesis that only finitely many of the structure operators are nonzero guarantees the convergence of the infinite sum above. The statement remains true for arbitrary A_∞-algebras and morphisms, provided that the infinite sums converge appropriately. Similar issues arise in the definition of twisting by a Maurer-Cartan element given below.

Definition A.15. *Let A be an A_∞-algebra and $x \in (sA)^0$ a Maurer-Cartan element. The A_∞-algebra A_x is given by the structure maps*

$$(b_x)_n(a_1 \otimes \cdots \otimes a_n) := \sum_{l_0 \geq 0, \cdots, l_n \geq 0} b_{n+l_0+\cdots+l_n}(x^{\otimes l_0} \otimes a_1 \otimes x^{\otimes l_1} \otimes \cdots \otimes x^{\otimes l_{n-1}} \otimes a_n \otimes x^{\otimes l_n}).$$

Here, $\{b_n\}$ denotes the family of structure maps of the original A_∞-algebra A. We say that A_x is obtained from A by twisting by x.

Definition A.16. *Suppose $\psi : A \to A'$ be a morphism of A_∞-algebras and let $x \in (sA)^0$ be a Maurer-Cartan element. There is a morphism of A_∞-algebras*

$$\psi_x : A_x \to A'_{\psi(x)}$$

between the twisted A_∞-structures, given by

$$(\psi_x)_n(a_1 \otimes \cdots \otimes a_n) := \sum_{l_0 \geq 0, \cdots l_n \geq 0} \psi_{n+l_0+\cdots+l_n}(x^{\otimes l_0} \otimes a_1 \otimes x^{\otimes l_1} \otimes \cdots \otimes x^{\otimes l_{n-1}} \otimes a_n \otimes x^{\otimes l_n}).$$

We say that ψ_x is obtained from ψ by twisting with x.

Constructing tensor products of A_∞-algebras and morphisms is a complicated issue in general. For our purposes we will only need the following special case:

Proposition A.17. *Let A and A' be differential graded algebras and $\psi : A \to A'$ an A_∞ morphism between them. Then, for any graded algebra E, we define a sequence ϕ of maps $\phi_1, \phi_2, \ldots$, where*

$$\phi_n : s(E \otimes A)^{\otimes n} \to s(E \otimes A')$$

is defined by the formula

$$\phi_n((e_1 \otimes sa_1) \otimes \cdots \otimes (e_n \otimes \ sa_n)) := (-1)^{\sum_{i=1}^n [a_i](|e_{i+1}|+\cdots+|e_n|)}(e_1 \ldots e_n) \otimes \psi_n(sa_1 \otimes \cdots \otimes sa_n).$$

Here, we see $e \otimes sa$ as an element of $s(E \otimes A)$ via

$$s(E \otimes A) \cong E \otimes sA, \quad s(e \otimes a) \mapsto (-1)^{|e|}e \otimes sa.$$

The family ϕ defines an A_∞ morphism from $E \otimes A$ to $E \otimes A'$, which we denote by $\mathrm{id}_E \otimes \psi$.

Remark A.18. The notion of an A_∞ algebra generalizes naturally to that of an A_∞-category in such a way that an A_∞-algebra is an A_∞-category with only one object. For the purpose of the present paper we will only need to consider A_∞ functors between *dg*-categories:

Definition A.19. *Let* C *and* C' *be differential graded categories. An* A_∞ *functor* $\psi : C \to C'$ *consists of the following data:*

1. *A function* $f : Ob(C) \to Ob(C')$.

2. *For* $n \geq 1$ *and any sequence of objects* $v_0, \ldots v_n$, *a degree zero map*

$$\psi_n : \mathsf{s}\underline{\mathrm{Hom}}(v_1, v_0) \otimes \cdots \otimes \mathsf{s}\underline{\mathrm{Hom}}(v_n, v_{n-1}) \to \underline{\mathrm{Hom}}(f(v_n), f(v_0)),$$

satisfying equations (8), *as well as* $\psi_1(\mathsf{s}(\mathrm{id})) = \mathsf{s}(\mathrm{id})$ *and* $\psi_n(\cdots \otimes \mathsf{s}(\mathrm{id}) \otimes \cdots) = 0$ *for* $n > 1$.

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