BT Quantization on K3 Surfaces

Hector Castejon Diaz
Université du Luxembourg
Mathematics Research Unit

Advisor: Prof. Dr. Martin Schlichenmaier

Berezin-Toeplitz Quantization [3]

Consider a compact *Kähler manifold* (M, g, I, ω).

Definition. *A* quantum line bundle *on M is a holomorphic bundle $L \to M$ together with a hermitian metric h and a connection ∇ compatible with both h and the complex structure such that the* prequantum *condition*

$$\text{curv}_{L,\nabla} = -i\omega$$

is fulfilled. M is called quantizable *if there exists such a line bundle.*

For a general tensor product $L^m := L^{\otimes m}$, one denotes by $h^{(m)}$ and $\nabla^{(m)}$ the induced metric and connection.

Berezin-Toeplitz Operators

Consider $L^2(M, L^m)$ the L^2-completion of the space of smooth sections of L^m and $\Gamma(M, L^m)$ the space of holomorphic sections. Denote by

$$\Pi^{(m)} : L^2(M, L^m) \to \Gamma(M, L^m)$$

the orthogonal projection induced by the hermitian metric.

Definition. *For $f \in C^\infty(M)$, the* Toeplitz operator *$T_f^{(m)}$ of* level m *is defined by*

$$T_f^{(m)} := \Pi^{(m)}(f\cdot) : \Gamma(M, L^m) \to \Gamma(M, L^m).$$

K3 Surfaces

Definition. *A* K3 surface *M is a simply connected complex surface admitting 3 Kähler structures $(g, I, \omega_I, J, \omega_J, K, \omega_K)$ sharing the same metric g such that $IJ = K$.*

I am assuming that all three Kähler structures are quantizable to study some relations between them. The existence of three quantizable Kähler structures implies the existence of a infinitude of quantum structures (see *Twistor Space* at [1]); each of them determining a quantum line bundle. One can easily see that such line bundles are not isomorphic. However:

Lemma. *There exists a constant $c \in \mathbb{Z}$ such that $h^0(M, L) = c$ for any quantum line bundle L of M.*

Note that, while there always exists an isomorphism between two vector spaces of the same (finite) dimension, I am looking for a *canonical* relation.

The *transcendental lattice* is the orthogonal complement $T_I(M)$ of the Neron-Severi group (group of line bundles on a variety) with respect to a fixed complex structure I:

$$T_I(M) := NS_I(M)^\perp \subset H^2(M, \mathbb{Z})$$

Lemma. *If all three Kähler structures are quantizable, then rk$(T_I(M)) = 2$ and one of the generators can be assumed to be ω_J.*

Proposition. *If $T_I(M)$ has a symmetry as a lattice, then there exist $a, b, c \in \mathbb{Q}$ and an I-holomorphic automorphism of M such that*

$$L_K^a = L_J^b \otimes f^* L_J^c.$$

To study the spaces of holomorphic sections, I am using the projection used in the construction of the Toeplitz Operators.

One can also see that $(T_I(M) \cap T_J(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \langle \omega_K \rangle \otimes_{\mathbb{Z}} \mathbb{Q}$. Currently I am studying if, under some symmetries on all three transcendental lattices the intersection is indeed generated by ω_K. In this case, one can find a quantum line bundle L with respect to a fourth Kähler structure I_0, ω_0 such that all other quantum line bundles are of the form

$$f_I^*(L^{\otimes a}) \otimes f_J^*(L^{\otimes b}) \otimes f_K^*(L^{\otimes c})$$

for three automorphisms f_I, f_J and f_K each one holomorphic with respect to a different complex structure.

References

[1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Rovcek, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), no. 4, 535–589. MR 877637 (88g:53048)

[2] Daniel Huybrechts, *Lectures on k3-surfaces*, Visited on January 12, 2015., p. 338.

[3] Martin Schlichenmaier, *Berezin-toeplitz quantization for compact kähler manifolds. A review of results.*, 2010, pp. Art. ID 927280, 38. MR 2608952 (2011g:53201)