Polyhedra inscribed in a quadric and anti-de Sitter geometry

Jean-Marc Schlenker

(joint work with Jeffrey Danciger and Sara Maloni)

According to a celebrated result of Steinitz (see e.g. [13, Chapter 4]), a graph Γ is the 1–skeleton of a convex polyhedron in $\mathbb{R}^3$ if and only if Γ is planar and 3–connected. Steinitz [12] also discovered, however, that there exists a 3–connected planar graph which is not realized as the 1–skeleton of any polyhedron inscribed in the unit sphere S, answering a question asked by Steiner [11] in 1832. An understanding of which polyhedral types can or can not be inscribed in the sphere remained elusive until Hodgson, Rivin, and Smith [7] gave a computable but non-explict characterization in 1992, see below. Our first result is on realizability by polyhedra inscribed in other quadric surfaces in $\mathbb{R}^3$. Up to projective transformations, there are two such surfaces: the hyperboloid H, defined by $x_1^2+x_2^2-x_3^2=1$, and the cylinder C, defined by $x_1^2+x_2^2=1$ (with x_3 free).

Definition. A convex polyhedron P is *inscribed* in the hyperboloid H (resp. in the cylinder C) if $P \cap H$ (resp. $P \cap C$) is exactly the set of vertices of P.

Theorem A [3]. *Let Γ be a planar graph. Then the following conditions are equivalent:*

(C): Γ *is the* 1*–skeleton of some convex polyhedron inscribed in the cylinder.*
(H): Γ *is the* 1*–skeleton of some convex polyhedron inscribed in the hyperboloid.*
(S): Γ *is the* 1*–skeleton of some convex polyhedron inscribed in the sphere and* Γ *admits a Hamiltonian cycle.*

The ball $x_1^2+x_2^2+x_3^2<1$ gives the projective model for hyperbolic space $\mathbb{H}^3$, with the sphere S describing the ideal boundary $\partial_\infty\mathbb{H}^3$. In this model, projective lines and planes intersecting the ball correspond to totally geodesic lines and planes in $\mathbb{H}^3$. Therefore a convex polyhedron inscribed in the sphere is naturally associated to a *convex ideal polyhedron* in the hyperbolic space $\mathbb{H}^3$.

Following the pioneering work of Andreev [1, 2], Rivin [8] gave a parameterization of the deformation space of such ideal polyhedra in terms of dihedral angles, as follows.

Theorem B (Andreev '70, Rivin '92). *The possible exterior dihedral angles of ideal hyperbolic polyhedra are the functions $w : E(\Gamma) \to (0,\pi)$ such that*

- *for each vertex v, the sum of the values of w on the edges adjacent to v is equal to 2π,*
- *for each other closed path c in the dual graph Γ^*, $\sum_{e\in c} w(e) > 2\pi$.*

Each such function gives the angles of a unique ideal polyhedron in $\mathbb{H}^3$.

As a corollary, Hodgson, Rivin and Smith [7] showed that deciding whether a planar graph Γ may be realized as the 1–skeleton of a polyhedron inscribed in the sphere amounts to solving a linear programming problem on Γ. To prove Theorem A, we show that, given a Hamiltonian cycle in Γ, there is a similar linear programming problem whose solutions determine polyhedra inscribed in either the cylinder or the hyperboloid.

The solid hyperboloid $x_1^2 + x_2^2 - x_3^2 < 1$ in $\mathbb{R}^3$ gives a picture of the projective model for *anti-de Sitter* (AdS) geometry. Therefore a convex polyhedron inscribed in the hyperboloid is naturally associated to a convex ideal polyhedron in the anti-de Sitter space $\mathbb{AdS}^3$, which is a Lorentzian analogue of hyperbolic space. Similarly, the solid cylinder $x_1^2 + x_2^2 < 1$ (with x_3 free) in an affine chart $\mathbb{R}^3$ of $\mathbb{RP}^3$ gives the projective model for *half-pipe* (HP) geometry. Therefore a convex polyhedron inscribed in the cylinder is naturally associated to a convex ideal polyhedron in the half-pipe space $\mathbb{HP}^3$. Half-pipe geometry, introduced by Danciger [4, 5, 6], is a transitional geometry which, in a natural sense, is a limit of both hyperbolic and anti-de Sitter geometry.

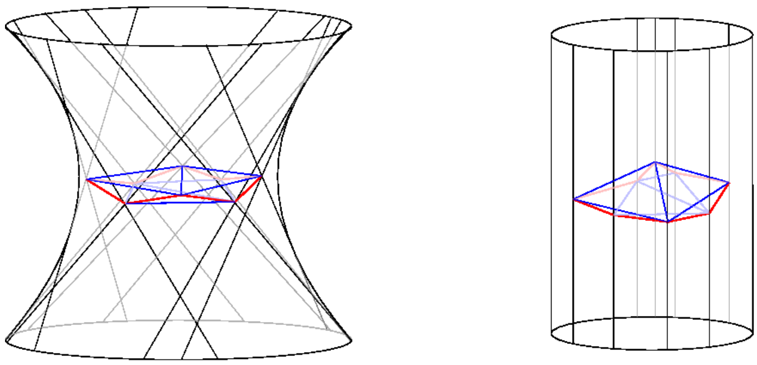

Figure 1. A polyhedron inscribed in the hyperboloid (left) and a combinatorial equivalent polyhedron inscribed in the cylinder (right). The 1–skeleton of any such polyhedron admits a Hamiltonian cycle which we call the *equator* (red).

Theorem C [3]. *The possible exterior dihedral angles of ideal AdS polyhedra are the functions* $w : E(\Gamma) \to \mathbb{R}_{\neq 0}$ *such that*

- $w < 0$ *on a Hamiltonian cycle* γ*,* $w > 0$ *elsewhere,*
- *for each vertex* v*, the sum of the values of* w *on the edges adjacent to* v *is equal to* 0*,*
- *for each "other" closed path* c *in* Γ^**, crossing* γ *exactly twice, the sum of the values of* w *on the edges of* c *is strictly positive.*

Each such function gives the dihedral angles of a unique ideal polyhedron in $\mathbb{AdS}_3$*.*

The equivalence between conditions (H) and (S) in Theorem A follows from a direct argument comparing the conditions occuring in Theorem A and in Theorem C. For condition (C), one has to use instead of Theorem C its analog n half-pipe geometry.

Related results determine the possible induced metrics on ideal polyhedra. In the hyperbolic setting the following result is known.

Theorem D (Rivin '93). *Let h be a hyperbolic metric of finite area on S^2 with at least 3 cusps. There exists a unique ideal polyhedron $\mathbb{H}^3$ with induced metric h on its boundary.*

We have a similar result in the anti-de Sitter setting.

Theorem E [3]. *Let h be a hyperbolic metric of finite area on S^2 with at least 3 cusps, and let γ be a Hamiltonian cycle through the cusps. There exists a unique ideal polyhedron in $\mathbb{A}\mathrm{d}\mathbb{S}_3$ with induced metric h on its boundary and and equator γ.*

In spite of the close analogy between the hyperbolic and AdS statements, the proofs in the AdS case must be done along very different lines. The reason is that the hyperbolic proofs are largely based on the concavity of the volume of ideal hyperbolic polyhedra (in particular ideal simplices), while this property doesn't hold for ideal AdS polyhedra. Other arguments must therefore be developed.

There are a number of open questions stemming from those results on ideal AdS polyhedra. For instance, do the description of their dihedral angles and induced metrics extend to *hyperideal* AdS polyhedra, that is, polyhedra with all vertices *outside* $\mathbb{A}\mathrm{d}\mathbb{S}_3$ but all edges intersecting it? This basically happens in the hyperbolic setting [9, 10].

References

[1] E. M. Andreev, *Convex polyhedra in Lobačevskiĭ spaces*, Mat. Sb. (N.S.) **81 (123)** (1970), 445–478. MR 0259734 (41 #4367)

[2] ———, *Convex polyhedra of finite volume in Lobačevskiĭ space*, Mat. Sb. (N.S.) **83 (125)** (1970), 256–260. MR 0273510 (42 #8388)

[3] J. Danciger, S. Maloni, and J.-M. Schlenker, *Polyhedra inscribed in a quadric*, ArXiv e-prints (2014).

[4] Jeffrey Danciger, *Geometric transitions: From hyperbolic to AdS geometry.*, Ph.D. thesis, Stanford University, June 2011.

[5] ———, *A geometric transition from hyperbolic to anti-de Sitter geometry*, Geom. Topol. **17** (2013), no. 5, 3077–3134. MR 3190306

[6] ———, *Ideal triangulations and geometric transitions*, J. Topol., to appear (2014).

[7] Craig D. Hodgson, Igor Rivin, and Warren D. Smith, *A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 246–251. MR 1149872 (93a:52009)

[8] Igor Rivin, *A characterization of ideal polyhedra in hyperbolic 3-space*, Ann. of Math. (2) **143** (1996), no. 1, 51–70. MR 1370757 (96i:52008)

[9] Jean-Marc Schlenker, *Métriques sur les polyèdres hyperboliques convexes*, J. Differential Geom. **48** (1998), no. 2, 323–405. MR 1630178 (2000a:52018)

[10] ———, *Convex polyhedra in Lorentzian space-forms*, Asian J. Math. **5** (2001), no. 2, 327–363. MR 1868937 (2003f:53127)

[11] Jakob Steiner, *Systematische entwickelung der abhängigkeit geometrischer gestalten von einander*, G. Fincke, 1832.

[12] Ernst Steinitz, *Über isoperimetrische probleme bei konvexen polyedern*, Journal für die reine und angewandte Mathematik **159** (1928), 133–143.

[13] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 1311028 (96a:52011)