FORMALITY THEOREM FOR QUANTIZATIONS OF LIE BIALGEBRAS

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Abstract. Using the theory of props we prove a formality theorem associated with universal quantizations of Lie bialgebras.

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Key words. Hopf algebras, Lie bialgebras, deformation quantization, props.

1. Introduction

1.1. **Two complexes related by a formality map.** Let V be a $\mathbb{Z}$-graded vector space over a field $\mathbb{K}$ and $\mathcal{O}_V := \odot^\bullet V = \oplus_{n\geq 0} \odot^n V$ the graded commutative and cocommutative bialgebra of polynomial functions on V^*. The Gerstenhaber-Schack complex [GS]

$$(1)\qquad \left(\mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V) = \prod_{m,n\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2-m-n],\ d_{\mathfrak{gs}}\right),$$

has a L_∞-algebra structure,

$$\{\mu_n : \odot^n \mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V) \to \mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V)[2-n]\}_{n\geq 1} \quad \text{with} \quad \mu_1 = d_{\mathfrak{gs}},$$

which controls deformations of the standard bialgebra structure on $\mathcal{O}_V$ [MeVa] (an explicit formula for the differential $d_{\mathfrak{gs}}$ is given in §3.4.1 below). This L_∞-algebra depends on the choice of a minimal resolution of the properad of bialgebras, but its isomorphism class is defined canonically. Existence (and non-uniqueness) of such minimal resolutions was proven by Martin Markl in [Ma2]. On the other hand, the completed graded commutative algebra,

$$(2)\qquad \mathfrak{l}_V := \prod_{m,n\geq 1} \odot^m (V[-1]) \otimes \odot^n (V^*[-1]),$$

has a natural degree -2 Poisson structure, $\{\ ,\ \} : \mathfrak{l}_V \otimes \mathfrak{l}_V \to \mathfrak{l}_V[-2]$, which makes $\mathfrak{l}_V$ into a 3-algebra and which is given on generators by

$$\{sv, sw\} = 0, \quad \{s\alpha, s\beta\} = 0, \quad \{s\alpha, sv\} = <\alpha, v>, \quad \forall v, w \in V, \alpha, \beta \in V^*.$$

where $s : V \to V[-1]$ and $s : V^* \to V^*[-1]$ are natural isomorphisms. The Maurer-Cartan elements of $(\mathfrak{l}_V,\ [\ ,\ \})$ are precisely strongly homotopy bialgebra structures in V (see Corollary 5.1 in [Me1]).

It was proven by Etingof and Kazhdan [EK] that there exists a universal quantization of (possibly, infinite-dimensional) Lie bialgebras. Any such a universal quantization morphism is highly non-trivial — it depends on the choice of a Drinfeld associator [D2]. The main result of this paper is a proof of the following

1.2. **Formality Theorem.** *Every universal quantization of Lie bialgebras lifts to a L_∞ quasi-isomorphism,*

$$(3)\qquad F : (\mathfrak{l}_V[2], \{\ ,\ \}) \longrightarrow (\mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V), \mu_\bullet).$$

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As the vector space $\mathfrak{l}_V[2]$ equals the cohomology of the Gerstenhaber-Schack complex $(\mathfrak{gs}^\bullet(\mathcal{O}_V), d_{\mathfrak{gs}})$, Theorem 1.2 implies formality of the L_∞ algebra $(\mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V), \mu_\bullet)$. Our proof of Theorem 1.2 shows that a lifting of a universal Etingof-Kazhdan quantization morphism to a formality map F can be constructed *inductively*, i.e. the lifting itself does *not* involve transcendental methods which seem to be unavoidable in the construction of Drinfeld associators and hence of the universal quantizations of Lie bialgebras; our proof is based, therefore, on a "trivial" procedure provided one reformulates the problem of constructing F in terms of differential graded props, and this reformulation occupies most of the present paper.

1.3. **Corollary.** *Every strongly homotopy Lie bialgebra structure, γ, on a graded vector space V can be deformation quantized, i.e. there exists a strongly homotopy bialgebra structure, $\Gamma(\gamma, \hbar)$, on $\odot^\bullet V[[\hbar]]$ depending on a formal parameter $\hbar$ such that $\Gamma|_{\hbar=0}$ coincides with the standard graded (co)commutative bialgebra structure in $\odot^\bullet V$ while $\frac{d\Gamma}{d\hbar}|_{\hbar=0}$ induces in V the original homotopy Lie bialgebra structure γ.*

1.4. **An outline of the proof of the Formality Theorem.** Our main technical tool in proving Theorem 1.2 is the theory of differential graded (dg, for short) props which allows us to make the idea of *universality* of quantizations of Lie bialgebras rigorous (a very different approach to prop interpretation of deformation quantizations was developed in [EE], while a very similar approach was used by the author in [Me2] to describe deformation quantization of wheeled Poisson structures). We refer to [EE, Ma4, Va] for an introduction into the theory of props and properads, their minimal resolutions and representations.

There is a prop AssB whose representations, $\mathsf{AssB} \to \mathsf{End}_W$, in a dg vector space W are precisely bialgebra structures in W. As $\mathcal{O}_V = \odot^\bullet V$ has a canonical bialgebra structure, there is associated a canonical morphism of props,

$$\rho_0 : \mathsf{AssB} \longrightarrow \mathsf{End}_{\mathcal{O}_V}.$$

The standard machinery developed in [MeVa] gives us a deformation complex of the map ρ_0,

$$\mathsf{Def}(\mathsf{AssB}_\infty \xrightarrow{\rho_0} \mathsf{End}_{\mathcal{O}_V}) \simeq \prod_{\substack{m,n\geq 1\\ m+n\geq 3}} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2-m-n],$$

which comes equipped with a filtered L_∞ algebra structure (depending on the choice of a minimal resolution AssB_∞ of AssB) whose Maurer-Cartan elements γ are in one-to-one correspondence with morphisms of dg props $\rho_0 + \gamma : \mathsf{AssB} \longrightarrow \mathsf{End}_{\mathcal{O}_V}$, that is, with strongly homotopy bialgebra structures in $\mathcal{O}_V$. The differential μ_1 of this $\mathcal{L}_\infty$ structure is determined by the map ρ_0 and was proven in [MeVa] to coincide precisely with the differential $d_{\mathfrak{gs}}$ introduced by Gerstenhaber and Schack in [GS]. More details can be found in §3.

Similarly, there is a prop(erad) LieB of Lie bialgebras which is Koszul and hence admits a simple minimal resolution LieB_∞, see §2 for details. Deformation complex of the zero morphism

$$\mathsf{Def}(\mathsf{LieB}_\infty \xrightarrow{0} \mathsf{End}_V) \simeq \prod_{\substack{m,n\geq 1\\ m+n\geq 3}} \odot^m (V[-1]) \otimes \odot^n (V^*[-1])$$

comes equipped, in accordance with [MeVa], with a graded Lie algebra structure whose Maurer-Cartan elements ν are in one-to-one correspondence with morphisms of dg props $\nu : \mathsf{AssB} \longrightarrow \mathsf{End}_V$, that is, with strongly homotopy bialgebra structures in V. The Lie brackets are precisely the ones $\{\ ,\ \}$ described in §1.1.

The idea of the proof is to relate somehow the dg props AssB_∞ and LieB_∞ to each other rather than to work with their representations. For this we have to resolve a number of problems.

1.4.1. The complex $\mathsf{Def}(\mathsf{AssB}_\infty \xrightarrow{\rho_0} \mathsf{End}_{\mathcal{O}_V})$ does not coincide with the Gerstenhaber-Schack complex (1); more unpleasantly, its cohomology with respect to $d_{\mathfrak{gs}}$ does not equal the space $\mathsf{Def}(\mathsf{LieB}_\infty \xrightarrow{0} \mathsf{End}_V)$. This problem can, however, be ressolved easily: there is an functor, $F^+ : \mathcal{P} \to \mathcal{P}^+$ in the category of dg props introduced in §4.1 below: for any dg prop $\mathcal{P}$, the associated prop $\mathcal{P}^+$ can be uniquely characterized in terms of $\mathcal{P}$ as follows — there is a 1-1 correspondence between representations of $\mathcal{P}^+$ in a graded vector space V equipped with the zero differential, and representations of $\mathcal{P}$ in the same vector space V but equipped with an *arbitrary* (i.e. not fixed a priori) differential. Then we recover the desired complexes using the standard deformation theory of morphisms of props,

$$\mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}) = \mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V) \quad \text{and} \quad \mathsf{Def}(\mathsf{LieB}_\infty^+ \stackrel{0}{\to} \mathsf{End}_V) = \mathfrak{l}_V.$$

1.4.2. The props AssB_∞^+ and LieB_∞^+ can not be related directly as the first one has natural representations in $\mathcal{O}_V$ while the second one in V. We solve this problem in two steps:

- first we notice that there is a sub-prop, $\mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}$, of the endomorphism prop $\mathsf{End}_{\mathcal{O}_V}$ spanned by so-called *polydifferential* operators $\Phi : \mathcal{O}_V^{\otimes m} \to \mathcal{O}_V^{\otimes n}$; the image of the standard representation $\rho_0 : \mathsf{AssB}^+ \to \mathsf{End}_{\mathcal{O}_V}$ lands in $\mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}$. It was shown in [Me3] that the associated L_∞ (sub)algebra

  $$\mathfrak{poly}(\mathcal{O}_V, \mathcal{O}_V) := \mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}) \subset \mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V)$$

  is quasi-isomorphic to $\mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V)$. Thus to prove the Main Theorem it is enough to show existence of a $\mathcal{L}_\infty$ quasi-isomorphism

  $$F : \mathfrak{l}_V[2] \longrightarrow \mathfrak{poly}(\mathcal{O}_V, \mathcal{O}_V). \tag{4}$$

- There is a dg free prop DefQ^+ uniquely characterized by the following property: there is a one-to-one correspondence between polydifferential representations of AssB_∞^+ in $\mathcal{O}_V$ and ordinary representations of DefQ^+ in the vector space V. Its generators and the differential are described in §4.

1.4.3. We construct a highly non-trivial continuous morphism (in fact, a quasi-isomorphism) of topological dg props

$$\mathcal{F}^+ : \mathsf{DefQ}^+ \longrightarrow \widehat{\mathsf{LieB}}_\infty^+ \tag{5}$$

where $\widehat{\mathsf{LieB}}_\infty^+$ is the vertex + genus completion[1] of the prop LieB_∞^+, as follows. It was shown in [EK] that, for any choice of the Drinfeld associator, there is a universal quantization of Lie bialgebras. A differential graded version of this result was settled in the Appendix §8 of [GH]. In our context, existence of a universal quantization of an arbitrary (possibly, infinite-dimensional) dg Lie bialgebra is equivalent to existence of a morphism of dg props

$$\mathcal{EK}^+ : \mathsf{DefQ}^+ \longrightarrow \widehat{\mathsf{LieB}}^+$$

satisfying a certain non-triviality condition (see §5 for details.). Then, using the fact that the natural projection $\widehat{\mathsf{LieB}}_\infty^+ \to \widehat{\mathsf{LieB}}^+$ is a quasi-isomorphism, the required morphism (5) is inductively constructed as a lifting of $\mathcal{EK}^+$ making the diagram

$$\begin{array}{ccc} & & \widehat{\mathsf{LieB}}_\infty^+ \\ & \stackrel{\mathcal{F}^+}{\nearrow} & \downarrow {\scriptstyle qis} \\ \mathsf{DefQ}^+ & \xrightarrow[\mathcal{EK}^+]{} & \widehat{\mathsf{LieB}}^+ \end{array}$$

commutative. Full details are given in §5.4.

1.4.4. The final step is to use the morphism (5) in order to show a proof of the Formality Theorem 2.1. This is done in two different ways in §5.

[1] The fact that the morphism $\mathcal{F}^+$ takes values in the *completed* prop is quite expected. Indeed, all the homogeneous components $F_k : \wedge^k(\mathfrak{l}_V[2]) \to \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$ of (4) are, in general, non-trivial, and every such a component F_k corresponds precisely to the k-vertex summands in the values of $\mathcal{F}^+$ on the generators of DefQ^+.

1.5. **Some notation.** For an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ the associated free prop is denoted by $\Gamma\langle E\rangle$. The endomorphism prop of a graded vector space V is denoted by End_V. The one-dimensional sign representation of the permutation group $\mathbb{S}_n$ is denoted by sgn_n while the trivial representation by $\mathbf{1}_n$. All our differentials have degree $+1$. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$; for $a \in V^i \subset V$ we write $|a| = i$. All our graphs are directed with flow implicitly assumed to go from bottom to top. We work throughout in the category of $\mathbb{Z}$-*graded* vector spaces over a field $\mathbb{K}$ of characteristic 0 with its standard Koszul sign conventions; for example, the value of an element $f_1 \otimes f_2 \in V^* \otimes V^*$ on an element $a \otimes b \in V \otimes V$ is equal to $(-1)^{|f_2||a|} f_1(a) f_2(b)$, where $V^* := \mathrm{Hom}(V, \mathbb{K})$.

2. Prop of strongly homotopy Lie bialgebras

2.1. **Lie bialgebras.** A *Lie bialgebra* is, by definition [D1], a graded vector space V together with two linear maps,

$$\begin{array}{cccc} \vartriangle: & V & \longrightarrow & \wedge^2 V \\ & a & \longrightarrow & \sum a_1 \wedge a_2 \end{array} \quad , \qquad \begin{array}{cccc} [\,,\,]: & \wedge^2 V & \longrightarrow & V \\ & a\otimes b & \longrightarrow & [a,b] \end{array}$$

satisfying,

(i) the Jacobi identity: $[a,[b,c] = [[a,b],c]] + (-1)^{|b||a|}[b,[a,c]]$;

(ii) the co-Jacobi identity: $(\vartriangle \otimes \mathrm{Id}) \vartriangle a + \tau(\vartriangle \otimes \mathrm{Id}) \vartriangle a + \tau^2(\vartriangle \otimes \mathrm{Id}) \vartriangle a = 0$, where τ is the cyclic permutation (123) represented naturally in $V \otimes V \otimes V$;

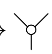
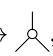
(iii) the Drinfeld compatibility condition: $\vartriangle [a,b] = \sum a_1 \wedge [a_2, b] - (-1)^{|a_1||a_2|} a_2 \wedge [a_1, b] + [a, b_1] \wedge b_2 - (-1)^{|b_1||b_2|}[a, b_2] \wedge b_1$.

for any $a, b, c \in V$.

2.2. **Prop of Lie bialgebras.** It is easy to construct a prop, LieB, whose representations,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}\langle V\rangle,$$

in a graded vector space V are in one-to-one correspondence with Lie bialgebra structures in V. With an association in mind,

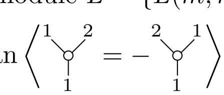
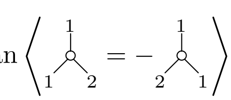
$\vartriangle \leftrightarrow$  , $[\,,\,] \leftrightarrow$ ,

one can define it as a quotient,

$$\mathsf{LieB} := \Gamma\langle L\rangle/(R)$$

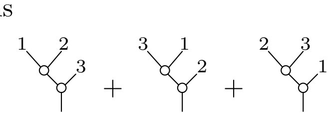
of the free prop, $\Gamma\langle L\rangle$, generated by the $\mathbb{S}$-bimodule $L = \{L(m,n)\}$,

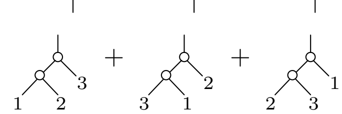
$L(m,n) :=$

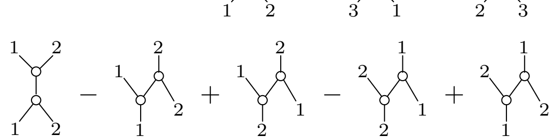
- $sgn_2 \otimes \mathbf{1}_1 \equiv \mathrm{span}$  if $m = 2, n = 1$,
- $\mathbf{1}_1 \otimes sgn_2 \equiv \mathrm{span}$  if $m = 1, n = 2$,
- 0 otherwise

modulo the ideal generated by relations

(6) $R:$

 $\in \Gamma\langle E\rangle(3,1)$

 $\in \Gamma\langle E\rangle(1,3)$

 $\in \Gamma\langle E\rangle(2,2)$

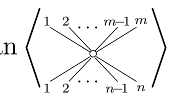
As the ideal is generated by two-vertex graphs, the properad behind LieB is quadratic.

2.3. **Minimal resolution of** LieB**.** It was shown in [Ga] that LieB is Koszul at the dioperadic level. The extension of this result to the level of props turned out to be highly non-trivial [Ko2, MaVo]. The minimal prop resolution, LieB_∞, of LieB is a dg free prop,

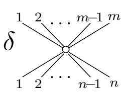
$$\mathsf{LieB}_\infty = \Gamma\langle \mathsf{L}\rangle,$$

generated by the $\mathbb{S}$–bimodule $\mathsf{L} = \{\mathsf{L}(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

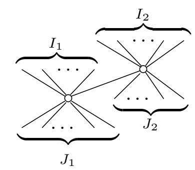
(7) $$\mathsf{L}(m,n) := sgn_m \otimes sgn_n[m+n-3] = \mathrm{span}\left\langle \right\rangle,$$



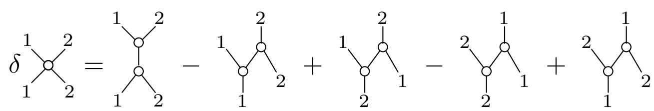
and with the differential of the form



$$\delta \; = \sum_{\substack{[1,\dots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \; \sum_{\substack{[1,\dots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \pm$$



where $\sigma(I_1\sqcup I_2)$ and $\sigma(J_1 \sqcup J_2)$ are the signs of the shuffles $[1,\dots,m] = I_1 \sqcup I_2$ and, respectively, $[1,\dots,n] = J_1\sqcup J_2$. For example,



2.3.1. **Fact.** Representations,

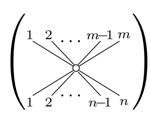
$$\rho : \mathsf{LieB}_\infty \longrightarrow \mathsf{End}\,\langle V\rangle\,,$$

of the dg prop $(\mathsf{LieB}_\infty, \delta)$ in a dg space (V,d) are in one-to-one correspondence with degree 3 elements, γ, in the Poisson algebra $(\mathfrak{l}_V, \{\ ,\ \})$ satisfying the equation, $\{\gamma,\gamma\} = 0$ (see Corollary 5.1 in [Me1]).
Indeed, using natural degree $m+n$ isomorphisms,

$$s^n_m : \mathrm{Hom}\,(\wedge^n V, \wedge^m V) \longrightarrow \odot^n(V[1]) \otimes \odot^m(V^*[1]),$$

we define a degree 3 element,

$$\gamma := s^1_1(d) + \sum_{\substack{m,n\geq 1\\ m+n\geq 3}} s^m_n \circ \rho \left(\right) \in \mathcal{O}_{\mathcal{V}},$$



and then check that the equation $d\circ\rho = \rho\circ\delta$ translates precisely into the Maurer-Cartan equation $\{\gamma,\gamma\} = 0$. The Lie bialgebra structures on V get identified with homogeneous polynomials of order 3,

$$\gamma_3 \in \odot^2(V[-1])\otimes V^*[-1] \quad \oplus \quad V[-1]\otimes\odot^2(V^*[-1]),$$

satisfying the equation $\{\gamma_3,\gamma_3\} = 0$. Indeed, every such a polynomial is equivalent to a pair $(\vartriangle \in \mathrm{Hom}(V, \wedge^2 V),\ [\ ,\ ] \in \mathrm{Hom}(\wedge^2 V, V))$, in terms of which the single equation $\{\gamma_3,\gamma_3\} = 0$ decomposes into relations (i)-(iii) of §2.1.

3. Prop of strongly homotopy bialgebras

3.1. **Associative bialgebras.** A *bialgebra* is, by definition, a graded vector space V equipped with two degree zero linear maps,

$$\begin{array}{rccc} \mu: & V\otimes V & \longrightarrow & V \\ & a\otimes b & \longrightarrow & ab \end{array}\quad , \qquad \begin{array}{rccc} \Delta: & V & \longrightarrow & V\otimes V \\ & a & \longrightarrow & \sum a_1\otimes a_2 \end{array}$$

satisfying,

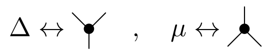
(i) the associativity identity: $(ab)c = a(bc)$;
(ii) the coassociativity identity: $(\Delta\otimes \mathrm{Id})\Delta a = (\mathrm{Id}\otimes\Delta)\Delta a$;
(iii) the compatibility identity: Δ is a morphism of algebras, i.e. $\Delta(ab) = \sum (-1)^{a_2 b_1} a_1 b_1 \otimes a_2 b_2$,

for any $a,b,c \in V$. We often abbreviate “associative bialgebra” to simply “bialgebra”.

3.2. **Prop of bialgebras.** There exists a prop, AssB, whose representations,

$$\rho : \mathsf{AssB} \longrightarrow \mathsf{End}\langle V\rangle,$$

in a graded vector space V are in one-to-one correspondence with the bialgebra structures in V [EE]. With an association in mind,

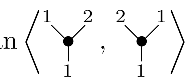


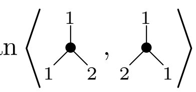
one can define it as a quotient,

$$\mathsf{AssB} := \Gamma\langle E\rangle/(R)$$

of the free prop, $\Gamma\langle E\rangle$, generated by the $\mathbb{S}$-bimodule $E = \{E(m,n)\}$,

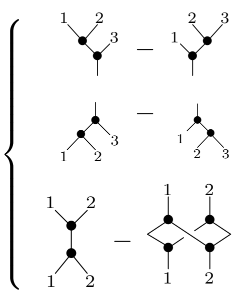
$E(m,n) :=$

$\mathbb{K}[\mathbb{S}_2]\otimes \mathbf{1}_1 \equiv \text{span}$  if $m=2, n=1$,

$\mathbf{1}_1 \otimes \mathbb{K}[\mathbb{S}_2] \equiv \text{span}$  if $m=1, n=2$,

0 otherwise

modulo the ideal generated by relations

$R :$ 

$\in \Gamma\langle E\rangle(3,1)$

$\in \Gamma\langle E\rangle(1,3)$

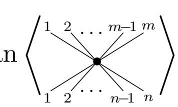
$\in \Gamma\langle E\rangle(2,2)$

which are not quadratic in the properadic sense [Va].

3.3. **A minimal resolution of AssB.** Existence of a minimal resolution of the prop of bialgebras,

$$(\mathsf{AssB}_\infty = \Gamma\langle \mathsf{E}\rangle, \delta) \xrightarrow{\pi} (\mathsf{AssB}, 0)$$

was proven by M. Markl [Ma2] who showed that AssB_∞ is freely generated by a relatively small $\mathbb{S}$-bimodule $\mathsf{E} = \{\mathsf{E}(m,n)\}_{m,n\geq 1, m+n\geq 3}$, with

$$\mathsf{E}(m,n) := \mathbb{K}[\mathbb{S}_m]\otimes\mathbb{K}[\mathbb{S}_n][m+n-3] = \text{span}$$ 

The differential δ is not quadratic, and its explicit value on the generic (m,n)-corolla is not known at present. Rather surprisingly, just existence of $(\mathsf{AssB}_\infty, \delta)$ and a theorem on Gerstenhaber-Schack differential from [MeVa] (recalled in § 3.4.1 below) are enough for our purposes.

3.4. **Deformation theory of dg morphisms.** Let $(\Gamma\langle \mathsf{A}\rangle, \delta)$ be a dg free prop generated by an $\mathbb{S}$-bimodule A, and (P, d) an arbitrary dg prop. It was shown in [MeVa] that the deformation complex of the zero morphism,

$$\mathfrak{g} := \mathsf{Def}(\Gamma\langle\mathsf{A}\rangle \xrightarrow{0} \mathsf{P}) \simeq \mathrm{Hom}_{\mathbb{S}}(\mathsf{A}, \mathsf{P})[-1],$$

has an induced (from the differentials δ and d) filtered L_∞-structure,

$$\{\mu_n : \odot^n(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]\}_{n\geq 1},$$

whose Maurer-Cartan elements, that is, degree 1 elements γ in $\mathfrak{g}$ satisfying a well-defined equation

$$\mu_1(\gamma) + \frac{1}{2!}\mu_2(\gamma,\gamma) + \frac{1}{3!}\mu_2(\gamma,\gamma,\gamma) + \ldots = 0,$$

are in one-to-one correspondence with morphisms, $\gamma : (\Gamma\langle\mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$, of dg props.

If $\gamma : (\Gamma\langle \mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$ is any particular morphism of dg props, then $\mathfrak{g}$ has a canonical γ-*twisted* L_∞-structure,

$$\{\mu_n^\gamma : \odot^n(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]\}_{n\geq 1},$$

which controls deformation theory of the morphism γ. The associated Maurer-Cartan elements, Γ,

$$\mu_1^\gamma(\Gamma) + \frac{1}{2!}\mu_2^\gamma(\Gamma,\Gamma) + \frac{1}{3!}\mu_2^\gamma(\Gamma,\Gamma,\Gamma) + \ldots = 0,$$

are in one-to-one correspondence with those morphisms of dg props, $(\Gamma\langle \mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$ whose values on generators are given by $\gamma + \Gamma$.

3.4.1. **Deformation theory of bialgebras**. If $\gamma : \mathsf{AssB} \to \mathsf{End}_W$ is a bialgebra structure, then the differential μ_1^γ of the induced L_∞ structure in

$$\mathfrak{gs}(W,W) := \mathsf{Def}(\mathsf{AssB} \xrightarrow{\gamma} \mathsf{End}_V) \simeq \mathrm{Hom}_{\mathbb{S}}(\mathsf{E}, \mathsf{End}(V))[-1] = \bigoplus_{\substack{m,n\geq 1\\ m+n\geq 3}} \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[m+n-2],$$

is precisely [MeVa] the Gerstenhaber-Schack differential [GS],

$$d_{\mathfrak{gs}} = d_1 \oplus d_2 : \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \longrightarrow \mathrm{Hom}(V^{\otimes n+1}, V^{\otimes m}) \oplus \mathrm{Hom}(V^{\otimes n}, V^{\otimes m+1}),$$

with d_1 given on an arbitrary $f \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ by

$$\begin{aligned}(d_1 f)(v_0, v_1, \ldots, v_n) \;\; := \;\; & \Delta^{m-1}(v_0)\cdot f(v_1, v_2, \ldots, v_n) - \sum_{i=0}^{n-1}(-1)^i f(v_1, \ldots, v_i v_{i+1}, \ldots, v_n) \\ & +(-1)^{n+1} f(v_1, v_2, \ldots, v_n)\cdot\Delta^{m-1}(v_n) \forall \;\; v_0, v_1, \ldots, v_n \in V,\end{aligned}$$

where multiplication in V is denoted by juxtaposition, the induced multiplication in the algebra $V^{\otimes m}$ by $\cdot$, the comultiplication in V by Δ, and

$$\Delta^{m-1} : (\Delta\otimes \mathrm{Id}^{\otimes m-2})\circ(\Delta\otimes\mathrm{Id}^{\otimes m-3})\circ\ldots\circ\Delta : V \to V^{\otimes m},$$

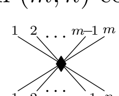
for $m \geq 2$ while $\Delta^0 := \mathrm{Id}$. The expression for d_2 is an obvious "dual" analogue of the one for d_1.

4. The Gerstenhaber-Schack complex of polydifferential operators and a dg prop DefQ^+

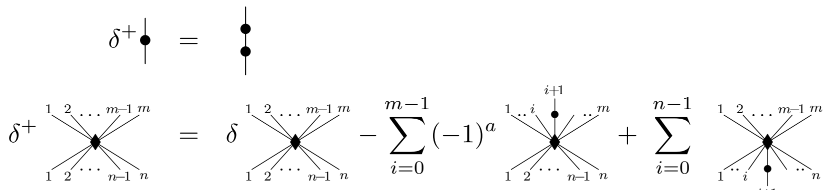
4.1. **An endofunctor in the category of prop(erad)s.** Consider an endofunctor, $^+ : \mathsf{P} \to \mathsf{P}^+$, on the (sub)category of dg *free* props, $\mathsf{P} = (\Gamma\langle E\rangle, \delta)$, $E = \{E(m,n)\}$ being an $\mathbb{S}$-bimodule and δ a differential. Define a new $\mathbb{S}$-module,

$$E^+(m,n) := \begin{cases} E(1)\oplus\mathbb{K}[-1] & \text{if } m=n=1,\\ E(m,n) & \text{otherwise.}\end{cases}$$

If we denote pictorially a generator of the summand $\mathbb{K}[-1] \subset E^+(1,1)$ by ╿•, and a generator (of, say, homological degree a) of $E(m,n)$ by an (m,n)-corolla



then P^+ is defined as the free prop $\Gamma\langle E^+\rangle$ equipped with the following differential,



Next, let $\mathsf{P} = (\Gamma\langle E\rangle/I, \delta)$ be a dg prop with generators E and relations described by an ideal[2] I. Then we set $\mathsf{P}^+ = (\Gamma\langle E^+\rangle/I, \delta^+)$, the point is that the ideal I is respected automatically by the differential δ^+ if it is respected by δ. There is obviously a 1-1 correspondence between representations,

$$\rho^+ : \mathsf{P}^+ \longrightarrow \mathsf{End}_V,$$

in a dg vector space (V, d), and representations

$$\rho : \mathsf{P} \longrightarrow \mathsf{End}_V,$$

in a dg vector space V equipped with the deformed differential $d - \rho^+(\bullet)$. Note that there is a natural forgetful map, $\mathsf{P}^+ \to \mathsf{P}$, of dg props so that every representation of P is automatically a representation of P^+.

As an example, we have an identification of Lie algebras, $\mathfrak{l}_V = \mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty^+ \stackrel{0}{\to} \mathcal{E}nd_V)$, where $\mathfrak{l}_V$ was introduced in §1. The moral is that the functor $^+$ takes care about deformations of the differential in a representation space V.

However the functor $^+$ kills cohomology (at least in the case of vertex completed props). If $\mathsf{P} = (\Gamma\langle E\rangle, \delta)$ is a dg (free) prop and $\widehat{\mathsf{P}}$ is its vertex completion, then the associated dg prop $\widehat{\mathsf{P}}^+$ is always acyclic. Indeed, call the generator(=decorated corolla) $\bullet$ of $\widehat{\mathsf{P}}^+$ *special*, all other generators *non-special*, and consider a filtration of $\widehat{\mathsf{P}}^+$ by the number of non-special corollas. The differential in the associated graded is δ_+ which acts non-trivially only the special corollas. Call a special $(1,1)$-vertex of a graph Γ from P^+ *very special* if it belongs to the path connecting the labelled by 1 input leg of Γ to its first non-special vertex (if there are any). Consider next the filtration of $\widehat{\mathsf{P}}^+$ by the number of vertices which are *not* very special. The associated graded is then isomorphic to $A \otimes C$, where A is the trivial complex and the complex $(C = \oplus_{k\geq 0} C^k, d)$ is defined as follows: C^k is the one-dimensional vector space spanned by the following graph with k vertices ,

$$-\!\bullet\!-\!\bullet\!- \cdots -\!\bullet\!-\!\bullet\!-$$

and the differential d is given by the canonical isomorphism $d : C^k \to C^{k+1}$ for any $k = 0, 2, 4, \ldots$, and by zero map for $k = 1, 3, \ldots$, e.g. $d(\text{---}) = -\!\bullet\!-$. It is clear that $H^\bullet(C) = 0$ making the cohomology of $\widehat{\mathsf{P}}^+$ trivial (as all the spectral sequences involved in the argument converge due to the completeness of the filtration by the number of vertices).

4.2. **Polydifferential operators.** For an arbitrary vector space V we consider a subspace,

$$\mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes\bullet}, \mathcal{O}_V^{\otimes\bullet}) \subset \mathrm{Hom}(\mathcal{O}_V^{\otimes\bullet}, \mathcal{O}_V^{\otimes\bullet}),$$

spanned by polydifferential operators,

$$\begin{array}{rccc} \Phi : & \mathcal{O}_V^{\otimes m} & \longrightarrow & \mathcal{O}_V^{\otimes n} \\ & f_1 \otimes \ldots \otimes f_m & \longrightarrow & \Gamma(f_1, \ldots, f_m), \end{array}$$

of the form,

$$\Phi(f_1, \ldots, f_m) = x^{J_1} \otimes \ldots \otimes x^{J_n} \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|} f_1}{\partial x^{I_1}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_m|} f_m}{\partial x^{I_m}}\right). \tag{8}$$

where x^i are some linear coordinates in V, I and J stand for multi-indices, say, $i_1 i_2 \ldots i_p$, and $j_1 j_2 \ldots j_q$,

$$x^J := x^{j_1} x^{j_2} \ldots x^{j_q}, \qquad \frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^p}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_p}}.$$

and Δ is the standard graded cocommutative coproduct in $\mathcal{O}_V$.

---

[2]We tacitly assume in this paper that the ideal I always satisfies the condition that I belongs to the subspace of the free prop $\Gamma\langle E\rangle$ spanned by graphs with at most k vertices for some $k \in \mathbb{N}$; in this case it makes sense to talk about the vertex completion of P which is denoted by $\widehat{\mathsf{P}}$. This assumption covers all the particular props we consider in this paper

As $\mathcal{O}_V$ is naturally a bialgebra, there is an associated Gerstenhaber-Schack complex [GS],

$$\left(\bigoplus_{m,n\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2], d_{\mathfrak{gs}}\right), \tag{9}$$

with the differential, $d_{\mathfrak{gs}}$, given as in § 3.4.1.

4.2.1. **Lemma.** *The subspace,*

$$\bigoplus_{m,n\geq 1} \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2] \subset \bigoplus_{m,n\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2], \tag{10}$$

is a subcomplex of the Gerstenhaber-Schack complex of the bialgebra $\mathcal{O}_V$, with the Gerstenhaber-Schack differential given explicitly on polydifferential operators (8) by

$$\begin{aligned} d_{\mathfrak{gs}}\Phi \;=\; & \sum_{i=1}^{n}(-1)^{i+1} x^{J_1}\otimes\ldots\otimes\bar{\Delta}(x^{J_i})\otimes\ldots\otimes x^{J_n}\cdot \Delta^n\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^n\left(\frac{\partial^{|I_m|}}{\partial x^{I_m}}\right) \\ & + \sum_{i=1}^{m}(-1)^{i+1}\sum_{I_i=I_i'\sqcup I_i''} \\ & \quad x^{J_1}\otimes\ldots\otimes x^{J_m}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_i'|}}{\partial x^{I_i'}}\right)\cdot\Delta^{n-1}\left(\frac{\partial^{|I_i''|}}{\partial x^{I_i''}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_m|}}{\partial x^{I_m}}\right), \end{aligned}$$

where $\bar{\Delta}:\mathcal{O}_V\rightarrow\mathcal{O}_V\otimes\mathcal{O}_V$ is the reduced diagonal, and the second summation goes over all possible splittings, $I_i = I_i'\sqcup I_i''$, into non-empty disjoint subsets.

Moreover, the inclusion (10) is a quasi-isomorphism.

Proof is a straightforward calculation (full details are shown in [Me3]).

4.2.1. **Normalized polydifferential deformation complex.** Note that polydifferential operators (8) are allowed to have sets of multi-indices I_i and J_i with zero cardinalities $|I_i| = 0$ and/or $|J_i| = 0$ (so that the standard multiplication and comultiplication in $\mathcal{O}_V$ belong to the class of polydifferential operators and it makes sense to consider a L_∞ algebra $\mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}})$). However, the complex in the Lemma above is quasi-isomorphic (cf. [Lo]) to its *normalized* version, i.e. to the one which is spanned by operators which vanish if at least one input function is constant, and never take constant value in each tensor factor. Therefore we can assume from now on that the L_∞-algebra

$$\mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V) := \mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}})$$

is spanned by the *normalized* polydifferential operators (8), i.e. the ones which have a $|I_i|\geq 1$ and $|J_j|\geq 1$ for all $i\in[m]$ and $j\in[n]$. Let us denote by

$$\mu_k : \wedge^k \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V) \longrightarrow \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V)[2-k]$$

the L_∞ structure in $\mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}})$ induced by the differential in AssB_∞^+ and the standard representation $\rho_0 : \mathsf{AssB} \rightarrow \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}$. The set of the associated Maurer-Cartan elements $\Gamma \in \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V)$,

$$\mu_1(\Gamma) + \frac{1}{2!}\mu_2(\Gamma,\Gamma) + \frac{1}{3!}\mu_3(\Gamma,\Gamma,\Gamma) + \ldots = 0, \quad |\Gamma| = 1,$$

is in one-to-one correspondence with polydifferential representations of the form $\rho_0 + \Gamma : \mathsf{AssB} \rightarrow \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}$, where Γ is given by the *normalized* operators; we call such a representation $\rho_0+\Gamma$ a *normalized polydifferential* representation of AssB_∞^+ in $\mathcal{O}_V$.

4.3. **Prop of quantum strongly homotopy bialgebra structures.** In his section we construct a dg free prop, $(\mathsf{DefQ}^+, \mathsf{d})$, whose representations, $\mathsf{DefQ}^+ \longrightarrow \mathsf{End}_V$, in a dg vector space (V, d) are in one-to-one correspondence with normalized polydifferential representations of AssB_∞^+ in $\mathcal{O}_V$, or, equivalently with Maurer-Cartan elements, Γ, of the L_∞-algebra $(\mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V), \mu_\bullet)$.

An arbitrary degree 1 element Γ in $\mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$ has the form,

$$\Gamma = \sum_{m,n\geq 1} \sum_{\substack{I_1,\dots I_n \\ J_1,\dots J_m}} \Gamma^{I_1\dots I_m}_{J_1\dots J_n} x^{J_1} \otimes \dots \otimes x^{J_n} \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right), \tag{11}$$

where $\Gamma^{I_1\dots I_m}_{J_1\dots J_n}$ are the coordinate components of some linear map,

$$V^{\odot|J_1|} \otimes \dots \otimes V^{\odot|J_n|} \longrightarrow V^{\odot|I_1|} \otimes \dots \otimes V^{\odot|I_m|},$$

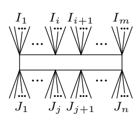
of degree $3 - m - n$. Thus there is a one-to-one correspondence between degree 1 elements $\Gamma \in \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$ and representations in End_V (not in $\mathsf{End}_{\mathcal{O}_V}$!) of the free prop, $\mathsf{DefQ}^+ := \Gamma\langle\mathsf{AssB}\rangle$, generated by the $\mathbb{S}$-bimodule,

$$\mathsf{C} = \{\mathsf{AssB}(p,q) := \mathsf{C}(m) \otimes \mathsf{C}(n)[p+q-3]\}_{p,q\geq 1},$$

where

$$\mathsf{AssB}(p) := \bigoplus_{m\geq 1} \bigoplus_{\substack{[p]=I_1\sqcup\dots\sqcup I_m \\ |I_1|,\dots,|I_k|\geq 1}} \mathrm{Ind}^{\mathbb{S}_m}_{\mathbb{S}_{|I_1|}\times\dots\times\mathbb{S}_{|I_m|}} \mathbf{1}_{|I_1|} \otimes \dots \otimes \mathbf{1}_{|I_m|}.$$

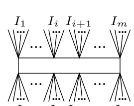
Thus the generators of DefQ can be pictorially represented by degree $3 - m - n$ directed planar corollas of the form,

(12)  $\simeq \quad \Gamma^{I_1,\dots,I_m}_{J_1,\dots,J_n}$

where

- the input legs are labeled by the set $[q] = [1, 2, \dots, q]$ partitioned into n disjoint non-empty subsets,
$$[q] = J_1 \sqcup \dots \sqcup J_j \sqcup J_{j+1} \sqcup \dots \sqcup J_n,$$
and legs in each J_j-bunch are symmetric (so that it does not matter how labels from the set J_j are distributed over legs in j-th bunch);
- the output legs are labeled by the set $[p]$ partitioned into m disjoint non-empty subsets,
$$[p] = I_1 \sqcup \dots \sqcup I_i \sqcup I_{i+1} \sqcup \dots \sqcup I_k,$$
and legs in each I_i-bunch are symmetric.

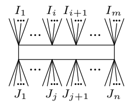
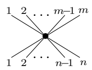
Our next step is to encode of the L_∞-structure $\{\mu_\bullet\}$ in $\mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$ into a differential d in DefQ^+. The idea is simple [Me2]: we replace coefficients $\Gamma^{I_1,\dots,I_m}_{J_1,\dots,J_n}$ in the formal series (11) by the associated generating corollas (12) getting thereby a well-defined element

$\bar\Gamma := \sum_{m,n\geq 1} \sum_{\substack{I_1,\dots I_n \\ J_1,\dots J_m}}$  $\otimes x^{J_1} \otimes \dots \otimes x^{J_n} \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right),$

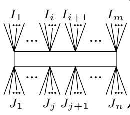
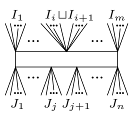
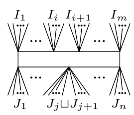
in the tensor product $\mathsf{DefQ}^+ \otimes \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$. We extend L_∞-operations $\{\mu_\bullet\}$ in $\mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)$ to the tensor product $\mathsf{DefQ}^+ \otimes \mathfrak{gs}^\bullet(\mathcal{O}_V)$ by replacing contractions of the coefficients $\Gamma^{I_1\dots I_m}_{J_1\dots J_n}$ by the associated prop compositions in DefQ^+ and then define a derivation, d, in DefQ^+ by comparing coefficients of the monomials, $x^{J_1} \otimes \dots \otimes x^{J_n} \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right)$, in the equality,

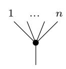
$$\mathsf{d}\bar\Gamma = \sum_{n=1}^{\infty} \frac{1}{n!} \mu_n(\bar\Gamma, \bar\Gamma, \dots, \bar\Gamma) \tag{13}$$

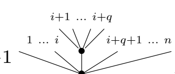
The quadratic L_∞ equations which hold for $\{\mu_\bullet\}$ imply then $\mathsf{d}^2 = 0$ [3]. This completes the construction of the dg prop $(\mathsf{DefQ}^+, \mathsf{d})$ we call from now on the dg *prop of quantum strongly homotopy bialgebra structures*.

4.4. **Examples.** The generators  are kind of "thickenings" of the corresponding generators  of AssB_∞^+ (mimicking thickening of V into $\mathcal{O}_V$), and the differential d in DefQ is also a kind of "thickening" (and twisting by ρ_0) of the differential δ^+ in AssB_∞^+. Once an explicit formula for δ^+ is known it is very straightforward to compute the associated expression for d using (13). Below we show such expressions for those components of δ^+ which we know explicitly.

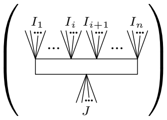
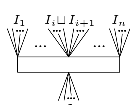
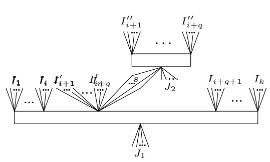
4.4.1. As μ_1 is given explicitly by Lemma 4.2.1, we immediately get from (13) the following expression for d modulo terms with number of vertices ≥ 2,

(14) $\mathsf{d}\Big($  $\Big) = \sum_{i=1}^{m-1}(-1)^i$  $+ \sum_{j=1}^{n-1}(-1)^j$  $+ O(2).$

4.4.2. On the generators  , $n \geq 1$, of AssB_∞^+ the differential is given by

δ^+  $= \sum_{i=0}^{n-1}\sum_{q=1}^{n-i}(-1)^{i+l(n-i-q)+1}$  .

Using (13) get after some tedious calculation (cf. [Me2]) the following expression for the value of d on the associated "thickened" generators of DefQ^+,

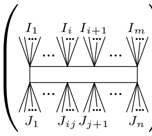
$\mathsf{d}\Big($  $\Big) = \sum_{i=1}^{n-1}(-1)^i$  $+ \sum_{\substack{p+q=n+1\\ 0\leq i\leq p-1\\ 0\leq s}} \sum_{\substack{I_{i+1}=I'_{i+1}\sqcup I''_{i+1}\\ \cdots\cdots\cdots\\ I_{i+q}=I'_{i+q}\sqcup I''_{i+q}\\ J=J_1\sqcup J_2}} \frac{(-1)^\varepsilon}{s!}$ 

where $\varepsilon = i + l(n-i-q) + 1$.

4.5. **Important remark.** The latter example shows that the free prop DefQ^+ makes sense as a *differential* prop only if it is considered as genus completed. This immediately raises a question: what is a representation of the completed prop DefQ^+ in an arbitrary vector space V? A morphism of dg props

$$\rho : \mathsf{DefQ}^+ \longrightarrow \mathsf{End}_V$$

is uniquely specified by its values on the generators,

$\rho\Big($  $\Big) \subset \mathrm{Hom}(\otimes^n \mathcal{O}_V, \otimes^m \mathcal{O}_V) \subset \mathsf{End}_V$

which we assume from now on *to vanish for sufficiently large values of the cardinalities* $|I_i|$ *and* $|J_j|$, $i \in [m]$, $j \in [n]$, $m, n \geq 1$. Under this assumption the map ρ is well-defined, and when we consider the deformation complex $\mathsf{Def}(\mathsf{DefQ}^+ \to \mathsf{End}_V)$ we always tacitly assume that we work

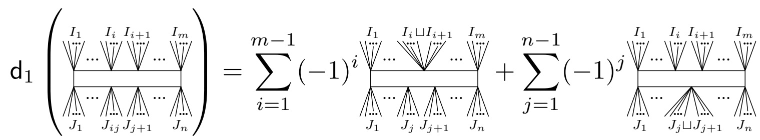
[3] See also §2.5 and §2.5.1 in [Me2] for a detailed explanation of this implication in the case when $\mu_\bullet$ is a dg Lie algebra; below in § 4.4 we illustrate definition (13) with some explicit computations.

within the class of the maps defined just above. Only under these assumptions on representations of DefQ^+ we have an isomorphism of complexes,

$$\mathsf{Def}(\mathsf{DefQ}^+ \to \mathsf{End}_V) \simeq \mathsf{Def}(\mathsf{AssB}_\infty^+ \to \mathsf{End}_{\mathcal{O}_V})$$

Indeed, without the above assumption on representations ρ we shall get an element in $\mathsf{Def}(\mathsf{AssB}_\infty^+ \to \mathsf{End}_{\mathcal{O}_V})$ which is an *infinite* sum of polydifferential operators of the form (8) (with order tending to infinity); such an infinite sum can not be an element of $\mathsf{Def}(\mathsf{AssB}_\infty^+ \to \mathsf{End}_{\mathcal{O}_V})$ (and it has no sense to apply L_∞ operation $\mu_\bullet$ to such an infinite sum).

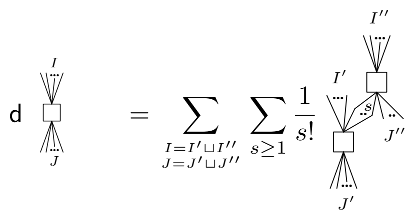
4.6. DefQ^+ **versus** LieB_∞^+**.** It is easy to check that a derivation, d_1, of DefQ^+, given on generators by

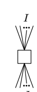
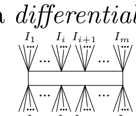
(15) 

is a differential. By (14), the differential d_1 is a part part of the full differential d. As a first approximation to the formality interrelation between $(\mathsf{DefQ}^+, \mathsf{d})$ and $(\mathsf{LieB}_\infty^+, \delta^+)$, we notice the following result which was proven in Proposition 3.5.1 of [Me3].

4.6.1. **Theorem.** *The cohomology prop* $H^\bullet(\mathsf{DefQ}^+, \mathsf{d}_1)$ *is isomorphic to the (non-differential) prop* LieB_∞^+.

4.7. **A quotient of** DefQ^+**.** The formula for the differential in § 4.4.2 implies



which in turn implies that the ideal I in DefQ^+ generated by corollas  is differential. Hence the quotient, $\mathsf{DefQ} := \mathsf{DefQ}^+/\mathsf{I}$, is a *differential* prop. The induced differential we denote by d. It is a free prop generated by corollas  with $m + n \geq 3$. It is clear that there is a one-to-one correspondence between normalized polydifferential representations of the dg prop AssB_∞ in $\mathcal{O}_V$ and ordinary (see §4.5) representations of DefQ in a graded vector space V. The prop DefQ is non-positively graded so that any morphism from DefQ^+ into a non-positively graded prop P factors through the natural projection to DefQ, $\mathsf{DefQ}^+ \to \mathsf{DefQ} \to \mathsf{P}$.

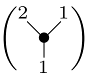
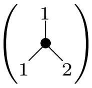
5. Formality morphism of L_∞-algebras via morphism of props

5.1. **On representations of completed props and formal parameters.** The Etingof-Kazhdan quantization morphism $\mathcal{EK}$ (which is discussed in the next section, see (16) below) is a morphism from the dg prop DefQ into a *completed* prop of Lie bialgebras. The value of $\mathcal{EK}$ on a generator of DefQ is a well-defined — probably, infinite! — linear combination of graphs from $\widehat{\mathsf{LieB}}$, but then one has to be careful on how to define a representation of the completed prop $\widehat{\mathsf{LieB}}$ in a vector space V as such an infinite linear combinations of graphs is mapped in general into a divergent infinite sum of linear operators. This the point where in the prop approach to deformation quantization the formal "Planck constant" has to enter the story.

A bialgebra structure on V is a morphism of props,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}_V,$$

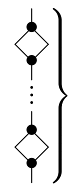
with

$$\rho\Big($$  $$\Big) =: \triangle \in \mathrm{Hom}(V, V\otimes V), \quad \rho\Big($$  $$\Big) =: [\ ,\ ] \in \mathrm{Hom}(V\otimes V, V)$$

being co-Lie and, respectively, Lie brackets in V. However, the morphism ρ can *not* be automatically extended to a representation,

$$\hat{\rho} : \widehat{\mathsf{LieB}} \longrightarrow \mathsf{End}\langle V\rangle,$$

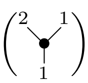
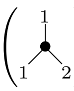
of the completed prop as, for example, a well-defined infinite sum of graphs in $\widehat{\mathsf{LieB}}$,

$$\sum_{n=1}^{\infty}$$  n times

gets mapped into an infinite sum of elements of $\mathrm{Hom}(V, V)$ which is divergent in general.
Thus we interpret instead a bialgebra structure in V as a morphism of props,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}_V[[\hbar]],$$

by setting

$$\rho\Big($$  $$\Big) =: \hbar\, \triangle \in \mathrm{Hom}(V, V\otimes V)[[\hbar]], \quad \rho\Big($$  $$\Big) =: \hbar[\ ,\ ] \in \mathrm{Hom}(V\otimes V, V)[[\hbar]].$$

Here $\hbar$ is a formal parameter and, for a vector space W the symbol $W[[\hbar]]$ stands for the vector space of formal power series in $\hbar$ with coefficients in W. This morphism extends to a *continuous* morphism of *topological* props

$$\hat{\rho} : \widehat{\mathsf{LieB}} \longrightarrow \mathsf{End}_V[[\hbar]],$$

with no divergency problems.

5.2. **Etingof-Kazhdan quantizations.** There are two universal quantizations of Lie bialgebras which were constructed in [EK]: the first one involves universal formulae with traces and hence is applicable only to finite-dimensional Lie bialgebras, while the second one avoids traces and hence is applicable to infinite-dimensional Lie bialgebras as well. We can reinterpret these results in terms of props as follows. Denote by $\mathrm{Mor}(\mathsf{P} \to \mathsf{Q})$ the set of continuous morphisms from a topological prop P to a topological prop Q. Then a Lie bialgebra structure in $V[[\hbar]]$ is the same as an element in $\mathrm{Mor}(\widehat{\mathsf{LieB}} \to \mathsf{End}_V[[\hbar]])$ while a (not necessary commutative/ cocommutative) bialgebra structure in $\mathcal{O}_V[[\hbar]]$ can be understood as an element of $\mathrm{Mor}(\mathsf{AssB}_\infty \to \mathsf{End}_{\mathcal{O}_V}[[\hbar]])$. The Etingof-Kazhdan theorem says that, for any graded vector space V, there exists a non-trivial map of sets

$$\mathrm{Mor}(\widehat{\mathsf{LieB}} \to \mathsf{End}_V[[\hbar]]) \longrightarrow \mathrm{Mor}(\mathsf{AssB}_\infty \to \mathsf{End}_{\mathcal{O}_V}[[\hbar]]).$$

By its very definition, the prop DefQ has the following property

$$\mathrm{Mor}(\mathsf{AssB}_\infty \to \mathsf{End}_{\mathcal{O}_V}[[\hbar]]) = \mathrm{Mor}(\mathsf{DefQ} \to \mathsf{End}_V[[\hbar]])$$

so that we can reinterpret the Etingof-Kazhdan theorem as an existence of a non-trivial map

$$\mathcal{EK}_V : \mathrm{Mor}(\widehat{\mathsf{LieB}} \to \mathsf{End}_V[[\hbar]]) \longrightarrow \mathrm{Mor}(\mathsf{DefQ} \to \mathsf{End}_V[[\hbar]])$$

which in fact does not depend on the choice of a particular vector space V nor on a particular Lie bialgebra structure in V, i.e. on a particular element in the set $\mathrm{Mor}(\widehat{\mathsf{LieB}} \to \mathsf{End}_V[[\hbar]])$. Put another way, the Etingof-Kazhdan theorem can be reformulated as existence of a non-trivial (in the sense made rigorous in item (iii) of the Theorem 5.2 below) morphism of props

$$\mathcal{EK} : (\mathsf{DefQ}, \mathsf{d}) \longrightarrow (\widehat{\mathsf{LieB}}, 0), \tag{16}$$

which induces the above mentioned morphism of sets. This observation is not new — it was first made in [EE] in a closely related (but not identical) form. Using a version [GH] of the Etingof-Kazhdan theorem which holds true for *dg* Lie bialgebras one can also talk about existence of a non-trivial morphism of dg props

$$\mathcal{EK}^+ : (\mathsf{DefQ}^+, \mathsf{d}^+) \longrightarrow (\widehat{\mathsf{LieB}}^+, \delta^+),$$

Similarly, existence of universal quantizations of *finite*-dimensional Lie bialgebras can be reformulated as existence of continuous morphism of topological dg props,

$$\mathcal{EK}^{\circlearrowright} : (\mathsf{DefQ}, \mathsf{d}) \longrightarrow (\widehat{\mathsf{LieB}}^{\circlearrowright}, 0),$$

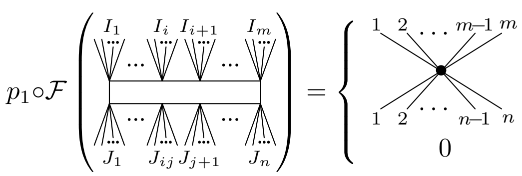
where $\widehat{\mathsf{LieB}}^{\circlearrowright}$ is the vertex (and hence genus) completion of the *wheeled* prop $\mathsf{LieB}^{\circlearrowright}$ of Lie bialgebras (see, e.g., [Me2] for its precise definition). It is existence of the map $\mathcal{EK}$ (or, equivalently, of $\mathcal{EK}^+$, see §5.4) which allows us to prove the formality theorem via a more less elementary lifting procedure; this iterative lifting argument does *not* work for the map $\mathcal{EK}^{\circlearrowright}$.

5.3. **Theorem.** *There exists a morphism of dg props, $\mathcal{F}$, making the diagram*

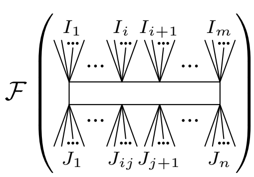
$$\begin{array}{ccc} & & (\widehat{\mathsf{LieB}}_\infty, \delta) \\ & \overset{\mathcal{F}}{\nearrow} & \downarrow{\scriptstyle qis} \\ (\mathsf{DefQ}, \delta) & \xrightarrow[\mathcal{EK}]{} & (\widehat{\mathsf{LieB}}, 0) \end{array}$$

commutative. Moreover, such an $\mathcal{F}$ satisfies the following conditions:

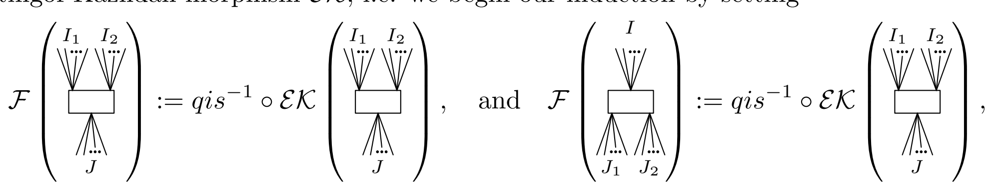
(i) *for any generating corolla* e *in* DefQ*, the composition* $p_k \circ \mathcal{F}(\mathsf{e})$ *is a* finite *linear combination of graphs from* LieB_∞*, where* p_k *is the projection from* $\widehat{\mathsf{LieB}}_\infty$ *to its subspace spanned by decorated graphs with precisely k vertices.*

(ii)  *for* $|I_1| = \ldots = |I_m| = |J_1| = \ldots = |J_n| = 1$; 0 *otherwise.*

Proof. As DefQ is a free prop, a morphism $\mathcal{F}$ is completely determined by its values,

 $\in \widehat{\mathsf{LieB}}_\infty, \quad m + n \geq 3,$

on the generating corollas. We shall construct $\mathcal{F}$ by induction[4] on the "weight", $\mathfrak{w} := m + n - 3$, associated to such corollas. For $\mathfrak{w} = 0$ we set $\mathcal{F}$ to be an arbitrary lift along the surjection qis of the Etingof-Kazhdan morphism $\mathcal{EK}$, i.e. we begin our induction by setting



where qis^{-1} is an arbitrary section of the quasi-isomorphism qis, i.e. an arbitrary lifting of cohomology classes into cycles.

Assume we constructed values of $\mathcal{F}$ on all corollas of weight $\mathfrak{w} \leq N$. Let e be a generating corolla of DefQ with non-zero weight $\mathfrak{w} = N + 1$. Note that de is a linear combination of graphs whose

[4] this induction is an almost literal analogue of the Whitehead lifting trick in the theory of CW-complexes in algebraic topology.

vertices are decorated by corollas of weight $\leq N$ (as $\mathfrak{w}$ is the precisely minus of the degree of e, and d increases degree by $+1$). By induction, $\mathcal{F}(\mathsf{de})$ is a well-defined element in $\widehat{\mathsf{LieB}}_\infty$. As $\mathcal{EK}(\mathsf{e}) = 0$, the element,

$$\mathcal{F}(\mathsf{de})$$

is a closed element in $\widehat{\mathsf{LieB}}_\infty$ which projects under qis to zero. Since the surjection qis is a quasi-isomorphism, this element must be exact. Thus there exists $\mathfrak{e} \in \widehat{\mathsf{LieB}}_\infty$ such that

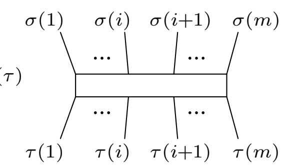
$$\delta\mathfrak{e} = \mathcal{F}(\mathsf{de}).$$

We set $\mathcal{F}(\mathsf{e}) := \mathfrak{e}$ completing thereby inductive construction of $\mathcal{F}$.

Next, if e is a generating corolla in Defq of degree $3-m-n$ then $p_k \circ \mathcal{F}(\mathsf{e})$ is a linear combination of graphs built from k generating corollas in LieB_∞ with total number of half edges attached to these k vertices being equal $3(k-1)+m+n$. There is only a *finite* number of graphs in LieB_∞ satisfying these conditions. This proves Claim (i).

Claim (ii) is obvious in the part *otherwise*. Let

$$e_n^m := \sum_{\sigma\in\mathbb{S}_m}\sum_{\tau\in\mathbb{S}_n}(-1)^{sgn(\sigma)+sgn(\tau)}$$



be the skewsymmetrization of the generating corolla in DefQ with $|I_1| = \ldots = |I_m| = |J_1| = \ldots = |J_n| = 1$. To prove Claim (ii) it is enough to show that $p_1 \circ \mathcal{F}(e_n^m) \neq 0$ for all $m, n \geq 1$, $m+n \geq 3$. We shall show this by induction on the weight $\mathfrak{w} = m+n-3$.

Denote the composition $p_k \circ \mathcal{F}$ by $\mathcal{F}_k$.

If $m+n = 3$, then $\mathcal{F}_1(e_n^m) \neq 0$ by the construction of $\mathcal{F}$.

Assume that $\mathcal{F}_1(e_p^q) \neq 0$ for all e_p^q with weight $\mathfrak{w} \leq N$ and consider e_n^m with non-zero weight $\mathfrak{w} = N+1$. Then

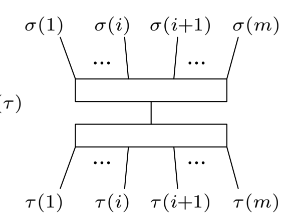
$$\mathcal{F}_1(e_n^m) \neq 0 \quad \Leftrightarrow \delta\left(\mathcal{F}_1(e_n^m)\right) \neq 0.$$

By our construction of $\mathcal{F}$, we have

$$\delta(\mathcal{F}_1(e_n^m)) = \mathcal{F}_2(\mathsf{d}e_n^m) = \mathcal{F}_2(\mathsf{d}_1 e_n^m) + \mathcal{F}_1 \boxtimes \mathcal{F}_1(\mathsf{d}_2 e_n^m),$$

where d_1 is the linear in number of vertices part of the differential d in DefQ and is given by (15), d_2 stands for the quadratic (i.e. spanned by two-vertex graphs) part of d, and $\mathcal{F}_1 \boxtimes \mathcal{F}_1$ means the morphism $\mathcal{F}_1$ applied to decoration of each of the two vertices in every graph summand of $\mathsf{d}_2 e_m^n$. Now $\mathsf{d}_1 e_n^m = 0$, while $\mathsf{d}_2 e_m^n$ contains, for example, the following linear combination of graphs,

$$\sum_{\sigma\in\mathbb{S}_m}\sum_{\tau\in\mathbb{S}_n}(-1)^{sgn(\sigma)+sgn(\tau)}$$



,

as irreducible summands coming from the genus zero part of the differential δ in AssB_∞ (see [Ma2]). Then, by the induction assumption, $\mathcal{F}_1 \boxtimes \mathcal{F}_1(\mathsf{d}_2 e_n^m)$ can not be zero implying $\mathcal{F}_1(e_n^m) \neq 0$. This completes the proof of Claim (ii) and hence of the Theorem. □

5.4. **First Proof of Formality Theorem 1.2.** We are going to construct a sequence of linear maps,

$$F_k : \wedge^k(\mathfrak{l}_V[2]) \longrightarrow \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V), \quad k \geq 1,$$

of degree $1-k$ satisfying quadratic relations of an L_∞-morphism,

$$(17) \qquad \sum_{\sigma\in Sh(2)} F_{k-1}\left(\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}, \ldots, f_{\sigma(k)}\right) =$$

$$= \sum_{i=1}^{k} \sum_{k=k_1+\ldots+k_i} \sum_{\sigma \in Sh(k_1,\ldots,k_r)} \pm \mu_r(F_{k_1}\left(f_{\sigma(1)},\ldots,f_{\sigma(k_1)}),\ldots,F_{k_r}(f_{\sigma(k-k_r+1)},\ldots,f_{\sigma(k)}\right),$$

where $Sh(k_1,\ldots,k_r)$ stands for the subgroup of $(k_1,\ldots,k_r)$-shuffles in $\mathbb{S}_k$, and $f_1,\ldots,f_k$ are arbitrary elements in $\mathfrak{l}_V[2]$. In a linear coordinate system $\{x^j\}$ on V (and the dual coordinate system $\{p_i\}$ on V^*) such an element f is a formal power series,

$$f = \sum_{m,n\geq 1} f^{i_1\ldots i_m}_{j_1\ldots j_n} x^{j_1} \wedge \ldots \wedge x^{j_n} \wedge p_{j_1} \wedge \ldots \wedge p_{j_m}.$$

We define a degree 0 linear map

$$\begin{array}{rccc} F_1: & \mathfrak{l}_V[2] & \longrightarrow & \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V) \\ & f & \longrightarrow & F_1(f) \end{array}$$

by setting

$$F_1(f) := \sum_{m,n\geq 1} f^{i_1\ldots i_m}_{j_1\ldots j_n} x^{j_1} \otimes \ldots \otimes x^{j_n} \cdot \Delta^{n-1}\left(\frac{\partial}{\partial x^{i_1}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial}{\partial x^{i_m}}\right).$$

Clearly, $d_{\mathfrak{gs}} \circ F_1 = 0$ so that this map sends $\mathfrak{l}_V[2]$ into cycles in $\mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V)$.
Next we shall read off the maps F_k for $k \geq 2$ from the components, $\mathcal{F}_k$, of the morphism $\mathcal{F}$ we constructed in the proof of Theorem 5.3. As the morphism $\mathcal{F}$ lands in the prop which does not contain the $(1,1)$-corolla $\bullet$, we are forced to set

$$F_k(f_1,\ldots,f_k) = 0, \quad k \geq 2,$$

if at least one input f_i lies in $V[1] \otimes V^*[1]$. Thus F_k for $k \geq 2$ must factor through the projection,

$$\wedge^k(\mathfrak{l}_V[2]) \longrightarrow \wedge^k \mathfrak{g} \xrightarrow{F_k} \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V),$$

where

$$\mathfrak{g} := \bigoplus_{\substack{m,n\geq 1 \\ m+n\geq 3}} \wedge^m V \otimes \wedge^n V^*[2-m-n] \subset \mathfrak{l}_V[2].$$

From now on we identify (see Corollary 5.1 in [Me1]) the latter with the vector space of $\mathbb{S}$-equivariant linear maps,

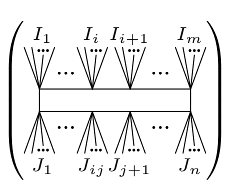
$$\mathfrak{g} \equiv \mathrm{Hom}(\mathsf{L},\mathsf{End}_V)[-1] \simeq \mathsf{Def}(\mathsf{LieB}_\infty \xrightarrow{0} \mathsf{End}_V)$$

where the $\mathbb{S}$-bimodule L is given by (7)).
The maps F_k are defined once the compositions,

$$F^{|I_1|,\ldots,|I_m|}_{k,|J_1|,\ldots,|J_n|} : \wedge^k \mathfrak{g} \xrightarrow{F_k} \mathsf{poly}(\mathcal{O}_V,\mathcal{O}_V) \xrightarrow{proj} \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes n},\mathcal{O}_V^{\otimes m}) \xrightarrow{proj}$$

$$\xrightarrow{proj} \mathrm{Hom}\left(\odot^{|J_1|}V \otimes \ldots \otimes \odot^{|J_n|}V, \odot^{|I_1|} \otimes \ldots \otimes \odot^{|I_m|}V\right).$$

are defined for all $k \geq 2$, $m+n \geq 3$, and $|J_1|,\ldots,|J_n|,|I_1|,\ldots,|I_m| \geq 1$.
We construct the value $F^{|I_1|,\ldots,|I_m|}_{k,|J_1|,\ldots,|J_n|}(f_1,\ldots,f_k)$ on arbitrary $f_1,\ldots,f_k \in \mathrm{Hom}(\mathsf{L},\mathsf{End}\langle V\rangle)[-1]$ in three steps:
Step 1. Consider

$\mathcal{F}_k$  $= \sum_G G\langle l_1,l_2,\ldots,l_k\rangle_{Aut(G)} \in \mathsf{LieB}_\infty$

where the sun runs over a family of graphs with k vertices, $Vert(G) = \{v_1,\ldots,v_k\}$, decorated with with the unique generators $l_i \in \mathsf{L}(|In(v_i)|,|Out(v_i)|$, $i=1,\ldots,k$ from L, where $|In(v_i)|$ (resp.

$|Out(v_i)|$) stands for the number of input (resp. output) half edges attached to the vertex v_i and

$$G\langle l_{v_1}, l_{v_2}, \ldots, l_{v_k}\rangle := \left(\sum_{s:[k]\to Vert(G)} l_{s(1)}\otimes\ldots\otimes l_{s(k)}\right)_{\mathbb{S}_k}$$

stands for the *unordered* tensor product of l_i over the set $Vert(G)$.
Step 2. Define next

$$\mathcal{F}_k(f_1,\ldots,f_k) := \sum_G \sum_{\sigma\in\mathbb{S}_k} (-1)^{Koszul\ sgn} G\langle f_{\sigma(1)}(l_1), f_{\sigma(2)}(l_2), \ldots, f_{\sigma(k)}(l_k)\rangle_{Aut(G)}.$$

This is a sum of graphs whose vertices are decorated by elements of the endomorphism prop End_V (cf. [Ma4, MeVa]). Hence we can apply to $\mathcal{F}_k(f_1,\ldots,f_k)$ the vertical and horizontal End_V-compositions to get a well defined element,

$$comp_{\mathsf{End}_V}\left(\mathcal{F}_k(f_1,\ldots,f_k)\right) \in \mathsf{End}\langle V\rangle,$$

which, in fact, lies by the construction in the subspace

$$\mathrm{Hom}\left(\odot^{|J_1|}V\otimes\ldots\otimes\odot^{|J_n|}V, \odot^{|I_1|}V\otimes\ldots\otimes\odot^{|I_m|}V\right)\subset \mathsf{End}_V.$$

Step 3. Finally set

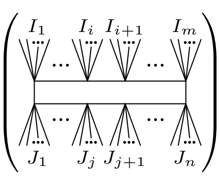
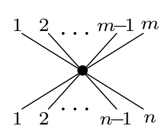
$$F_{k,|J_1|,\ldots,|J_n|}^{|I_1|,\ldots,|I_m|}(f_1,\ldots,f_k) := comp_{\mathsf{End}\langle V\rangle}\left(\mathcal{F}_k(l_1,\ldots,l_k)\right).$$

It is easy to check that the constructed map F_k has degree $1-k$. It is also a straightforward untwisting of the definitions of differentials d in Defq and δ in LieB_∞ to show that equations (17) follow directly from the basic property, $\delta\circ\mathcal{F} = \mathcal{F}\circ\mathsf{d}$, of the morphism $\mathcal{F}$ (cf. again [Ma3, MeVa]).
□

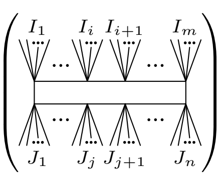
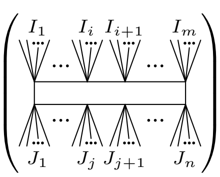
5.5. **Second Proof of Formality Theorem 1.2.** There exists a commutative diagram of dg props,

$$\begin{array}{ccc} & & (\widehat{\mathsf{LieB}}^+_\infty, \delta^+) \\ & \overset{\mathcal{F}^+}{\nearrow} & \downarrow{\scriptstyle qis} \\ (\mathsf{DefQ}^+, \delta^+) & \xrightarrow[\mathcal{EK}^+]{} & (\widehat{\mathsf{LieB}}^+, \delta^+) \end{array}$$

with

$p_1\circ\mathcal{F}^+$ () = {  for $|I_1| = \ldots = |I_m| = |J_1| = \ldots = |J_n| = 1$; 0 otherwise.

Indeed, one can define $\mathcal{F}^+$ and, respectively, $\mathcal{EK}^+$ by their values on the generators,

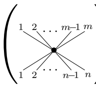
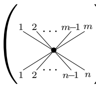
$\mathcal{F}^+$(resp. $\mathcal{EK}^+$) () = {
 for $m=n=1, |I_1| = |J_1| = 1$,
0 for $m=n=1, |I_1|+|J_1|\geq 3$,
$\mathcal{F}$(resp. $\mathcal{EK}$) () for $m+n\geq 3$

It is easy to check that the maps $\mathcal{EK}^+$ and $\mathcal{F}^+$ commute with the differentials. In fact the above observation about $\mathcal{EK}^+$ is equivalent to the extension of the Etingof-Kazhdan quantizations from Lie bialgebras to *differential* Lie bialgebras [GH]. (Of course, one can construct the lifting $\mathcal{F}^+$ from the extended Etingof-Kazhdan quantization map $\mathcal{EK}^+$ by an inductive procedure similar to the above construction of the lifting $\mathcal{F}$ as the prop DefQ^+ is elementally cofibrant — in the sense explained in [Ma1] — over its dg subprop generated by corollas of non-negative homological degrees.)

The prop $\widehat{\mathsf{LieB}}_\infty^+$ is completed and hence has no, in general, meaningful representations in a graded vector space V, but does admit non-trivial continuous representations in the topological vector space $V[[\hbar]] := V \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]]$, where $\hbar$ is a formal variable of homological degree zero. For example, for any representation

$$\gamma : \mathsf{LieB}_\infty \longrightarrow \mathsf{End}_V$$

i.e. for any strongly homotopy Lie bialgebra structure in dg vector space (V, d), there is an associated well-defined continuous representation $\gamma_\hbar$ of the topological dg prop given on the generators by,

$\gamma_\hbar \left(\right.$  $\left. \right) = \hbar^{m+n-2} \gamma \left(\right.$  $\left. \right), \quad m, n \geq 1.$

The morphism (5) induces a continuous L_∞ quasi-morphism,

$$F^\hbar : \mathsf{Def}_{cont}(\widehat{\mathsf{LieB}}_\infty^+ \stackrel{0}{\to} \mathsf{End}_V[[\hbar]]) \longrightarrow \mathsf{Def}_{cont}(\mathsf{DefQ}^+ \stackrel{\rho_0}{\to} \mathsf{End}_V[[\hbar]]) \simeq \mathsf{Def}(\mathsf{AssB}_\infty^+ \stackrel{\rho_0}{\to} \mathsf{End}_{\mathcal{O}_V}^{\mathsf{poly}}[[\hbar]])$$

that is,

$$F^\hbar : \mathfrak{l}_V[[\hbar]] \longrightarrow \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]].$$

There is a morphism of Lie algebras,

$$\begin{array}{lccc} \mathfrak{i}: & \mathfrak{l}_V & \longrightarrow & \mathfrak{l}_V[[\hbar]] \\ & m \in \odot^m(V[-1]) \otimes \odot^n(V[-1]) & \longrightarrow & \hbar^{m+n-2} m \end{array} \quad (18)$$

such that the composition

$$F_k^\hbar \circ \wedge^k \mathfrak{i} : \wedge^k(\mathfrak{l}_V[2]) \longrightarrow \mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$$

takes values in the subspace $\mathsf{poly}(\mathcal{O}_V, \mathcal{O}_V)[\hbar]$. Then the family of maps

$$F = \{F_k := F_k^\hbar|_{\hbar=1} \circ \wedge^k \mathfrak{i}\}$$

gives us finally the required quasi-isomorphism (4), and hence (3). The second proof is completed.

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