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Grothendieck–Teichmüller group and Poisson cohomologies

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Abstract. We study actions of the Grothendieck–Teichmüller group GRT on Poisson cohomologies of Poisson manifolds, and prove some "go" and "no-go" theorems associated with these actions.

Mathematics Subject Classification (2010). 53D55, 16E40, 18G55, 58A50.

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1. Introduction

It is proven in [15, 17] that the Grothendieck–Teichmüller group, GRT_1, acts up to homotopy on the set, $\{\pi\}$, of Poisson structures (depending on a formal parameter $\hbar$) on an arbitrary smooth manifold. Universal formulae for such an action can be represented as sums over Feynman graphs with weights given by integrals over compactified configuration spaces introduced in [11, 13].

Any Poisson structure makes the algebra of polyvector fields, $(\mathcal{T}_{poly}(M)[[\hbar]], \wedge)$ into a *Poisson complex*, more precisely, into a differential graded (dg, for short) associative algebra with the differential $d_\pi = [\pi, \ ]_S$, where $[\ ,\ ]_S$ is the Schouten bracket. The cohomology of this complex is sometimes denoted by $H^\bullet(M,\pi)$ and is called the *Poisson cohomology* of (M,π). The dg algebra $(\mathcal{T}_{poly}(M)[[\hbar]], \wedge, d_\pi)$ is of special type — both operations $\wedge$ and d_π respect the Schouten bracket in the sense of *dg Gerstenhaber algebra*. The main purpose of our paper is to study

- a class of universal $\mathcal{A}ss_\infty$ structures on $\mathcal{T}_{poly}(\mathbb{R}^d)$ which are consistent with the Schouten bracket in the sense of strong homotopy *non-commutative* Gerstenhaber ($nc\mathcal{G}_\infty$, for short) algebras;
- universal actions of the group GRT_1 on this class,

and then use these technical gadgets to give a constructive proof of the following

Main Theorem. *Let π be a Poisson structure on M, γ an arbitrary element of GRT_1, and let $\gamma(\pi)$ be the Poisson structure on M obtained from π by an action of γ. Then there exists a morphism,*

$$F^\gamma : H^\bullet(M,\pi) \longrightarrow H^\bullet(M,\gamma(\pi)), \tag{1.1}$$

of associative algebras.

The morphism (1.1) is, in general, highly non-trivial. In one of the simplest cases, when π is a linear Poisson structure on an affine manifold $M = \mathbb{R}^d$ (which is equivalent to the structure of a Lie algebra on the dual vector space $\mathfrak{g} := (\mathbb{R}^d)^*$), the morphism F^γ becomes an algebra *automorphism* of the Chevalley–Eilenberg cohomology of the $\mathfrak{g}$-module $\odot^\bullet \mathfrak{g} := \oplus_{n\geq 0} \odot^n \mathfrak{g}$,

$$H^\bullet(\mathbb{R}^d, \pi) = H^\bullet(\mathbb{R}^d, \gamma(\pi)) = H^\bullet(\mathfrak{g}, \odot^\bullet \mathfrak{g}),$$

and its restriction to $H^0(\mathfrak{g}, \odot^\bullet \mathfrak{g}) = (\odot^\bullet \mathfrak{g})^{\mathfrak{g}}$ coincides precisely with Kontsevich's generalization of the classical Duflo map (see Theorems 7 and 8 in §4.8 of [8]). Thus in this special case our main theorem extends Kontsevich's action of GRT_1 on $(\odot^\bullet \mathfrak{g})^{\mathfrak{g}}$ to the full cohomology $H^\bullet(\mathfrak{g}, \odot^\bullet \mathfrak{g})$, and also gives explicit formulae for that extension.

The existence of the *algebra* morphism (1.1) is far from obvious. We prove in this paper a kind of "no-go" theorem which says that there does *not* exist a *universal* (i.e. given by formulae applicable to any Poisson structure) $\mathcal{A}ss_\infty$-morphism of dg associative algebras,

$$\left(\mathcal{T}_{poly}(M)[[\hbar]], \wedge, d_\pi\right) \longrightarrow \left(\mathcal{T}_{poly}(M)[[\hbar]], \wedge, d_{\gamma(\pi)}\right).$$

Hence the algebra morphism (1.1) can not be lifted to the level of the associated Poisson complexes in such a way that the wedge multiplication is respected in the strong homotopy sense.

Our main technical tool is the deformation theory of universal $nc\mathcal{G}_\infty$-structures on polyvector fields. This is governed by the mapping cone of a natural morphism of graph complexes introduced and studied by Thomas Willwacher in [17]. Using some of his results we show that there exists an exotic universal GRT_1-deformation of the standard the dg associative algebra $(\mathcal{T}_{poly}(M), \wedge, d_{\gamma(\pi)})$ into an $\mathcal{A}ss_\infty$-algebra,

$$\left(\mathcal{T}_{poly}(M)[[\hbar]],\ \mu^\gamma_\bullet = \{\mu^\gamma_n\}_{n\geq 1}\right),\ \ \gamma \in GRT_1,$$

whose differential μ^γ_1 is independent of γ and equals d_π, while the higher homotopy operations $\mu^\gamma_{n\geq 2}$ are independent of π and are fully determined by $\wedge$ and γ. This universal $\mathcal{A}ss_\infty$- algebra structure on $\mathcal{T}_{poly}(M)$ is homotopy equivalent to $(\mathcal{T}_{poly}(M), \wedge, d_{\gamma(\pi)})$, i.e. there exists a universal continuous $\mathcal{A}ss_\infty$ isomorphism,

$$\mathcal{F}^\gamma : \left(\mathcal{T}_{poly}(M)[[\hbar]], \mu^\gamma_\bullet\right) \longrightarrow \left(\mathcal{T}_{poly}(M)[[\hbar]], \wedge, d_{\gamma(\pi)}\right) \qquad (1.2)$$

which on cohomology induces the map (1.1) and hence proves the main theorem.

The above mentioned deformation, $\left(\mathcal{T}_{poly}(M)[[\hbar]], \mu^\gamma_\bullet\right)$, of the standard dg algebra structure on the space of polyvector fields on a Poisson manifold is of a special type — it respects the Schouten bracket. Omitting reference to a particular Poisson structure π on M, we can say that we study in this paper universal deformations of the standard Gerstenhaber algebra structure on polyvector fields in the class of $nc\mathcal{G}_\infty$-algebras (rather than in the class of $\mathcal{G}_\infty$-algebras). The 2-coloured operad $nc\mathcal{G}_\infty$ is a minimal resolution of the 2-coloured Koszul operad, $nc\mathcal{G}$, of non-commutative Gerstenhaber algebras, and, moreover, it admits a very natural geometric realization via configuration spaces of points in the pair, $\mathbb{R} \subset \mathbb{C}$, consisting of the complex plane $\mathbb{C}$ and a line $\mathbb{R}$ drawn in the plane [1]. We prove that, up to $nc\mathcal{G}_\infty$-isomorphisms there are only two universal $nc\mathcal{G}_\infty$-structures on polyvector fields, the one which comes from the standard Gerstenhaber algebra structure, and the exotic one which was introduced in [1] in terms of a de Rham field theory on a certain operad of compactified configuration spaces.

1.1. Some notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$. The cardinality of a finite set A is denoted by $\#A$. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$ and and s^k for the associated isomorphism $V \to V[k]$; for $v \in V^i$ we set $|v| := i$. For a pair of graded vector spaces V_1 and V_2, the symbol $\mathrm{Hom}_i(V_1, V_2)$ stands for the space of homogeneous linear maps of degree i, and $\mathrm{Hom}(V_1, V_2) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(V_1, V_2)$; for example, $s^k \in \mathrm{Hom}_{-k}(V, V[k])$. For an operad $\mathcal{P}$ we denote by $\mathcal{P}\{k\}$ the unique operad which has the following property: for any graded vector space V there is a one-to-one correspondence between representations of $\mathcal{P}\{k\}$ in V and representations of $\mathcal{P}$ in $V[-k]$; in particular, $\mathcal{E}nd_V\{k\} = \mathcal{E}nd_{V[k]}$.

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2. Compactified configuration spaces of points in the flag $\mathbb{R} \subset \mathbb{C}$

2.1. $\mathcal{L}ie_\infty$-algebras. For a finite set A let $Conf_A(\mathbb{C})$ stand for the set of all injections, $\{A \hookrightarrow \mathbb{C}\}$. For $\#A \geq 2$ the orbit space

$$C_A(\mathbb{C}) := \frac{\{A \hookrightarrow \mathbb{C}\}}{z \to \mathbb{R}^+ z + \mathbb{C}},$$

is naturally a real $(2\#A - 3)$-dimensional manifold (if $A = [n]$, we use the notations $C_n(\mathbb{C})$). Its Fulton–MacPherson compactification, $\overline{C}_A(\mathbb{C})$, can be made into a compact smooth manifold with corners [7] (or into a compact semialgebraic manifold). Moreover, the collection

$$\overline{C}(\mathbb{C}) = \{\overline{C}_A(\mathbb{C})\}_{\#A\geq 2}$$

has a natural structure of a non-unital pseudo-operad in the category of oriented smooth manifolds with corners. The associated operad of chains, $\mathcal{C}hains(\overline{C}(\mathbb{C}))$, contains a suboperad of fundamental chains, $\mathcal{F}\mathcal{C}hains(\overline{C}(\mathbb{C}))$, which is precisely the operad, $\mathcal{L}_\infty\{1\}$, of degree shifted L_∞-algebras (see [12] for a review).

2.2. OCHA versus strong homotopy non-commutative Gerstenhaber algebras. For arbitrary finite sets A and B consider the space of injections,

$$Conf_{A,B}(\mathbb{C}) := \{A \sqcup B \hookrightarrow \mathbb{C}, B \hookrightarrow \mathbb{R} \subset \mathbb{C}\},$$

and, for $2\#A + \#B \geq 2$, consider the quotient space,

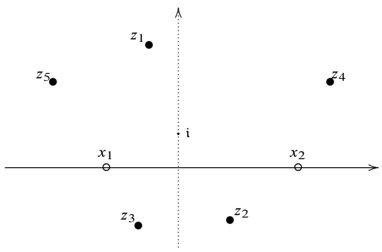
$$C_{A,B}(\mathbb{C}) := \frac{Conf_{A,B}(\mathbb{C})}{z \to \mathbb{R}^+ z + \mathbb{R}},$$

by the affine group $\mathbb{R}^+ \ltimes \mathbb{R}$. As $\mathbb{C} \setminus \mathbb{R} = \mathbb{H} \sqcup \mathbb{H}^-$, where $\mathbb{H}$ (resp. $\mathbb{H}^-$) is the upper (resp., lower) half-plane, we can consider subspaces,

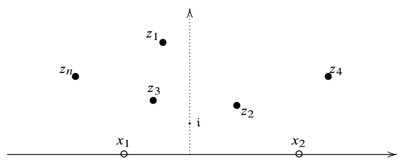
$$Conf_{A,B}(\mathbb{H}) := \{A \hookrightarrow \mathbb{H}, B \hookrightarrow \mathbb{R}\} \subset Conf_{A,B}(\mathbb{C})$$

and

$$C_{A,B}(\mathbb{H}) := \frac{Conf_{A,B}(\mathbb{H})}{z \to \mathbb{R}^+ z + \mathbb{R}} \subset C_{A,B}(\mathbb{C})$$



$C_{5,2}(\mathbb{C})$



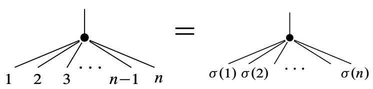
$C_{5,2}(\mathbb{H})$

The Fulton–MacPherson compactification, $\overline{C}_{A,B}(\mathbb{H})$, of $C_{A,B}(\mathbb{H})$ was introduced in [7]. The fundamental chain complex, $\mathcal{F}\mathcal{C}hains(\overline{C}(\mathbb{H}))$, of the disjoint union,

$$\overline{C}(\mathbb{H}) := \overline{C}_\bullet(\mathbb{C}) \bigsqcup \overline{C}_{\bullet,\bullet}(\mathbb{H}),$$

is a dg quasi-free 2-coloured operad [5] generated by

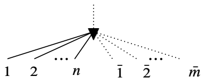
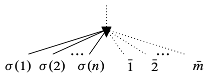
(i) degree $3 - 2n$ corollas,



$$, \quad \forall \sigma \in \mathbb{S}_n,\ n \geq 2 \qquad (2.1)$$

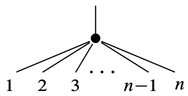
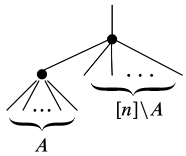
representing $\overline{C}_n(\mathbb{C})$, and

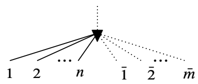
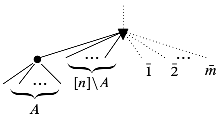
(ii) degree $2 - 2n - m$ corollas,

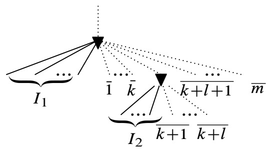
 $=$  $,\quad 2n + m \geq 2, \forall\, \sigma \in \mathbb{S}_n$ (2.2)

representing $\overline{C}_{n,m}(\mathbb{H})$.

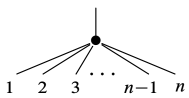
The differential in $\mathcal{F}Chains(\overline{C}(\mathbb{H}))$ is given on the generators by [7, 5]

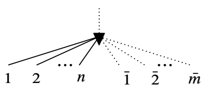
∂  $= \sum_{\substack{A \subsetneq [n] \\ \#A \geq 2}}$  (2.3)

∂  $= - \sum_{\substack{A \subsetneq [n] \\ \#A \geq 2}}$  (2.4)

$+ \sum_{\substack{k,l,[n]=I_1 \sqcup I_2 \\ 2\#I_1 + m \geq l+1 \\ 2\#I_2 + l \geq 2}} (-1)^{k+l(n-k-l)}$ 

Representations of $(\mathcal{F}Chains(\overline{C}(\mathbb{H})), \partial)$ in a pair of dg vector spaces $(A, \mathfrak{g})$ were called in [5] *open-closed homotopy algebras* or OCHAs for short. Such a representation, ρ, is uniquely determined by its values on the generators,

$\nu_n \;:=\; \rho\Big($  $\Big) \in \mathrm{Hom}(\mathfrak{g}^{\odot n}, \mathfrak{g})[3 - 2n], \quad n \geq 2,$

$\mu_{n,m} \;:=\; \rho\Big($  $\Big) \in \mathrm{Hom}(\mathfrak{g}^{\odot n} \otimes A^{\otimes m}, A)[2 - 2n - m],$

$2n + m \geq 2,$

which satisfy quadratic relations given by the above formulae for the differential ∂ and give us, therefore, the following list of algebraic structures in $(A, \mathfrak{g})$:

(i) an $\mathcal{L}_\infty\{1\}$-algebra structure, $\nu_\bullet = \{\nu_n : \odot^n \mathfrak{g} \to \mathfrak{g}[3 - 2n]\}_{n \geq 1}$, in $\mathfrak{g}$;

(ii) an $\mathcal{A}_\infty$-algebra structure, $\mu_\bullet = \{\mu_{0,m} : \otimes^m A \to A[2 - m]\}_{m \geq 1}$, in A; if $[\ ,\ ]_G$ stands for the standard Gerstenhaber bracket on the Hochschild cochains $C(A, A) = \prod_{n \geq 0} \mathrm{Hom}(A^{\otimes n}, A)[1 - n]$),then $\mu_\bullet$ defines a differential on $C(A, A)$, $d_\mu := [\mu_\bullet,\ ]_G$;

(iii) an $\mathcal{L}_\infty$-morphism, F, from the L_∞-algebra $(\mathfrak{g}, \nu)$ to the dg Lie algebra $(C(A, A), [\ ,\ ]_G, d_\mu)$.

If ρ is an arbitrary representation of $(\mathcal{F}\mathcal{C}hains(\overline{C}(\mathbb{H})), \partial)$ and $\gamma \in \mathfrak{g}$ is an arbitrary Maurer–Cartan element[1],

$$\sum_{n\geq 0} \frac{1}{n!}\nu_n(\gamma^{\otimes n}) = 0, \quad |\gamma| = 2,$$

of the associated $\mathcal{L}_\infty$-algebra $(\mathfrak{g}, \nu_\bullet)$, then the maps

$$\begin{array}{rccc} \mu_m: & \otimes^m A & \longrightarrow & A[[\hbar]][2-m], \quad m \geq 0, \\ & x_1 \otimes \ldots x_m & \longrightarrow & \sum_{n\geq 1} \frac{\hbar^n}{n!}\mu_{n,m}(\gamma^{\otimes n} \otimes x_1 \otimes \ldots x_m) \end{array}$$

make the topological (with respect to the adic topology) vector space $A[[\hbar]]$ into a topological, *non-flat* (in general) $\mathcal{A}_\infty$-algebra (here $\hbar$ is a formal parameter, and $A[[\hbar]] := A \otimes \mathbb{K}[[\hbar]]$). Non-flatness originates from the generators (2.2) with $m = 0$, $n \geq 1$, which correspond to the boundary strata in $\overline{C}(\mathbb{H})$ that are given by groups of points in the upper half plane collapsing to a point on the real line. It is clear how to get rid of such strata — one should allow configurations of points everywhere in $\mathbb{C}$, and hence consider the Fulton–MacPherson compactifications [1] of the configuration spaces $C_{A,B}(\mathbb{C})$ rather than $C_{A,B}(\mathbb{H})$. The disjoint union

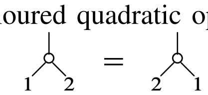
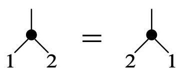
$$\overline{CF}(\mathbb{C}) := \overline{C}_\bullet(\mathbb{C}) \bigsqcup \overline{C}_{\bullet,\bullet}(\mathbb{C}),$$

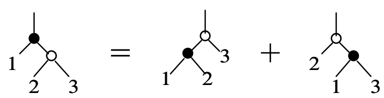
has a natural structure of a dg quasi-free 2-coloured operad in the category of compact manifolds with corners. This operad is free in the category of sets. The suboperad,

$$nc\mathcal{G}_\infty := \mathcal{F}\mathcal{C}hains(\overline{CF}(\mathbb{C})),$$

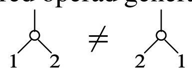
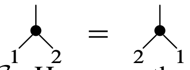
of the associated chain operad $\mathcal{C}hains(\overline{CF}(\mathbb{C}))$ generated by fundamental chains is free in the category of graded vector spaces and is canonically isomorphic as a dg operad to the quotient operad

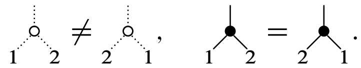
$$nc\mathcal{G}_\infty := \mathcal{F}\mathcal{C}hains(\overline{C}(\mathbb{H}))/I,$$

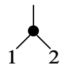
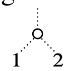
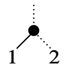
where I is the (differential) ideal generated by corollas (2.2) with $m = 0$, $n \geq 1$. The notation $nc\mathcal{G}_\infty$ stems from the fact [1, 4] that this operad is a minimal resolution of a 2-coloured quadratic operad which governs the type of algebras introduced in [3] under the name of *Leibniz pairs*. Let us compare this quadratic operad with the operad, $\mathcal{G}$, of Gerstenhaber algebras. The latter is a 1-coloured quadratic operad generated by commutative associative product in degree 0,  and Lie bracket of degree -1, , satisfying the compatibility condition

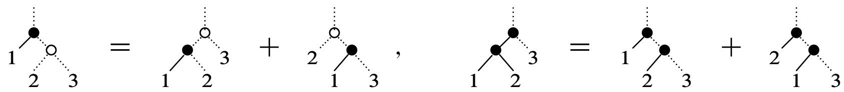
 (2.5)

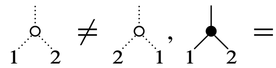
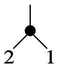
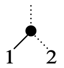
[1] We tacitly assume here that the L_∞-algebra $(X_c, \nu_\bullet)$ is appropriately filtered so that the MC equation makes sense. In our applications below $\nu_{n\geq 3} = 0$ so that one has no problems with convergence of the infinite sum.

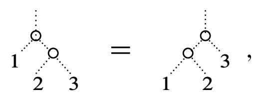
This condition satisfies the distributive law so that the 1-coloured operad $\mathcal{G}$ is Koszul. In fact, this condition makes sense even if we assume that the associative product is *not* commutative so that one might attempt to define an operad of *non-commutative* Gerstenhaber algebras as a 1-coloured operad generated by associative non-commutative product product of degree 0,  , and Lie bracket of degree -1,  , formally satisfying the same relations as the ones in the operad $\mathcal{G}$. However, the compatibility condition (2.5) now fails to obey the distributive law (there are new unwanted relations already for graphs with three vertices, see Remark 1.7 in [10]), and the resulting 1-coloured operad fails to be Koszul. However, this problem with non-Koszulness disappears if we think of the generating operations as living in two different (say, dashed and straight) colours,



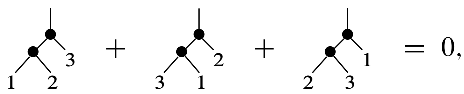
To make sense of the Gerstenhaber compatibility condition (2.5) in two colours, we can notice that the generator  plays a two-fold role in the compatibility conditions of the operad $\mathcal{G}$: it represents a Lie algebra structure, and also a *morphism* from that Lie algebra into the Lie algebra of derivations of the associative algebra represented by  . In the two coloured version we have to assign these two roles to two different actors, that is, we have to introduce a new degree -1 generator,  , for the role of the *morphism*, and then substitute (2.5) with the following two relations,

 (2.6)

The 2-coloured operad generated by binary operations   and  ,

 (2.7)

Jacobi relations for the Lie brackets,

 (2.8)

and the compatibility relations (2.6) was introduced in [3] (with slightly different grading conventions which in two colours are irrelevant) under the name of the *operad of Leibniz pairs*. However algebras over the operad of Leibniz pairs have nothing to do with pairs of Leibniz algebras. We prefer to call this quadratic operad the *2-coloured operad of noncommutative Gerstenhaber algebras* ($nc\mathcal{G}$ for short) as this name specifies its structure non-ambiguously; this is the only natural way to generalize the notion of Gerstenhaber algebras to the case of a non-commutative product while keeping the Koszulness property. Moreover, any Gerstenhaber algebra is automatically an algebra over $nc\mathcal{G}$. In particular, for any smooth manifold M the associated space of polyvector fields, $\mathcal{T}_{poly}(M)$ equipped with the Schouten bracket $[\ ,\ ]_S$ and the wedge product $\wedge$ is an $nc\mathcal{G}$-algebra. It was proven in [15] that $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_S, \wedge)$ is rigid as a $\mathcal{G}_\infty$ algebra. It follows from Willwacher's proof [17] of the Furusho theorem that $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_S, \wedge)$ admits a unique (up to homotopy and rescalings) universal $nc\mathcal{G}_\infty$ deformation whose explicit structure is described in [1] (see also (4.3) below for its explicit graph representation).

2.3. Configuration space model for the 4-coloured operad of morphisms of $nc\mathcal{G}_\infty$-algebras. A geometric model for the 4-coloured operad of morphisms of OCHA algebras was given in [12]. The same ideas work for the operad, $\mathcal{M}or(nc\mathcal{G})_\infty$, of morphisms of $nc\mathcal{G}_\infty$-algebras provided one replaces everywhere in §6 of [12] the upper-plane $\mathbb{H}$ with the full complex plane $\mathbb{C}$.

3. T. Willwacher's theorems

3.1. Universal deformations of the Schouten bracket. The deformation complex of the graded Lie algebra $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_S)$ is the graded Lie algebra,

$$\mathsf{CoDer}\left(\underbrace{\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)[2])}_{\substack{\text{standard coalgebra}\\ \text{structure}}}\right) = \prod_{n\geq 0} \mathrm{Hom}(\odot^n \mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d))[2-2n]$$

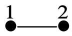
of coderivations of the graded-cocommutative coalgebra $\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)[2])$ equipped with the differential, δ, given by

$$\delta(D) := [\ ,\ ]_S \circ D - (-1)^{|D|} D \circ [\ ,\ ]_S, \quad \forall\ D \in \mathsf{CoDer}\left(\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)[2])\right).$$

Here $\circ$ stands for the composition of coderivations. There is a universal (i.e. independent of the dimension d) version of this deformation complex, GC_2, which was introduced by Kontsevich in [6] and studied in detail in [17]. In this subsection we recall some ideas, results and notations of [17] which we later use to prove our main theorem.

3.2. Operad $\mathcal{G}ra$. To define Kontsevich's dg Lie algebra GC_2 it is easiest to start by defining a certain operad of graphs. For arbitrary integers $n \geq 1$ and $l \geq 0$ let $\mathsf{G}_{n,l}$ stand for the set of graphs $\{\Gamma\}$ with n vertices and l edges such that (i) the vertices of Γ are labelled by elements of $[n] := \{1, \dots, n\}$, (ii) the set of edges, $E(\Gamma)$, is totally ordered up to an even permutation (that is, *oriented*); it has at most two different orientations. For $\Gamma \in \mathsf{G}_{n,l}$ we denote by Γ_{opp} the oppositely oriented graph. Let $\mathbb{K}\langle \mathsf{G}_{n,l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{n,l}$ modulo the relation[2] $\Gamma_{opp} = -\Gamma$, and consider the $\mathbb{Z}$-graded $\mathbb{S}_n$-module,

$$\mathcal{G}ra(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}_{n,l}\rangle[l].$$

For example,  is a degree -1 element in $\mathcal{G}ra(2)$. The $\mathbb{S}$-module, $\mathcal{G}ra := \{\mathcal{G}ra(n)\}_{n\geq 1}$, is naturally an operad with the operadic compositions given by

$$\begin{array}{rccc} \circ_i : & \mathcal{G}ra(n) \otimes \mathcal{G}ra(m) & \longrightarrow & \mathcal{G}ra(m+n-1) \\ & \Gamma_1 \otimes \Gamma_2 & \longrightarrow & \sum_{\Gamma \in \mathsf{G}^i_{\Gamma_1,\Gamma_2}} (-1)^{\sigma_\Gamma} \Gamma \end{array} \tag{3.1}$$

where $\mathsf{G}^i_{\Gamma_1,\Gamma_2}$ is the subset of $\mathsf{G}_{n+m-1,\#E(\Gamma_1)+\#E(\Gamma_2)}$ consisting of graphs, Γ, satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \dots, i+m-1\}$ is isomorphic to Γ_2 and the quotient graph, Γ/Γ_2, obtained by contracting that subgraph to a single vertex, is isomorphic to Γ_1 (see §2 in [17] or §7 in [12] for examples). The sign $(-1)^{\sigma_\Gamma}$ is determined by the equality $\wedge_{e\in E(\Gamma)} e = (-1)^{\sigma_\Gamma} (\wedge_{e'\in E(\Gamma_1)} e') \wedge (\wedge_{e''\in E(\Gamma_2)} e'')$ where the edge products over the sets $E(\Gamma_1)$ and $E(\Gamma_1)$ are taken in accordance with the given orientations. The unique element in $\mathsf{G}_{1,0}$ serves as the unit element in the operad $\mathcal{G}ra$.

3.3. A canonical representation of $\mathcal{G}ra$ in $\mathcal{T}_{poly}(\mathbb{R}^d)$. The operad $\mathcal{G}ra$ has a natural representation in the vector space $\mathcal{T}_{poly}(\mathbb{R}^d)[2]$ for any dimension d,

$$\begin{array}{rccc} \rho : & \mathcal{G}ra(n) & \longrightarrow & \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}(n) = \mathrm{Hom}(\mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes n}, \mathcal{T}_{poly}(\mathbb{R}^d)) \\ & \Gamma & \longrightarrow & \Phi_\Gamma \end{array} \tag{3.2}$$

given by the formula,

$$\Phi_\Gamma(\gamma_1, \dots, \gamma_n) := \mu\left(\prod_{e\in E(\Gamma)} \Delta_e \left(\gamma_1(x_{(1)}, \psi_{(1)}) \otimes \gamma_2(x_{(2)}, \psi_{(2)}) \otimes \dots \otimes \gamma_n(x_{(n)}, \psi_{(n)})\right)\right)$$

[2]Abusing notations we identify from now an equivalence class $[\Gamma]$ with any of its representative Γ.

where, for an edge e connecting vertices labeled by integers i and j,

$$\Delta_e = \sum_{a=1}^{n} \frac{\partial}{\partial x^a_{(i)}} \otimes \frac{\partial}{\partial \psi_{(j)a}} + \frac{\partial}{\psi_{(i)a}} \otimes \frac{\partial}{\partial x^a_{(j)}}$$

and μ is the multiplication map,

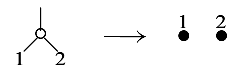
$$\begin{array}{cccc} \mu: & \mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes n} & \longrightarrow & \mathcal{T}_{poly}(\mathbb{R}^d) \\ & \gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_n & \longrightarrow & \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n. \end{array}$$

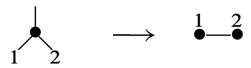
Here we used a coordinate identification, $\mathcal{T}_{poly}(\mathbb{R}^d) = C^\infty(x^1, \ldots, x^d)[\psi_1, \ldots, \psi_d]$, where $C^\infty(x^1, \ldots, x^d)$ is the ring of smooth functions of coordinates $x^1, \ldots, x^d$ on $\mathbb{R}^d$, and ψ_a are formal variables of degree one symbolizing $\partial/\partial x^a$.

3.4. Kontsevich graph complex. There is a morphism of operads [18]

$$\mathcal{G} \longrightarrow \mathcal{G}ra$$

given on the generators of the operad of Gerstenhaber algebras by


(3.3)


(3.4)

The latter map also gives us a canonical morphism of operads

$$i : \mathcal{L}ie\{1\} \longrightarrow \mathcal{G}ra.$$

The *full Kontsevich graph complex* fGC_2 is, by definition, the deformation complex controlling deformations of the morphism i,

$$\mathsf{fGC}_2 := \mathsf{Def}(\mathcal{L}ie\{1\} \to \mathcal{G}ra)$$

There are several explicit constructions of deformation complexes of (pr)operadic morphisms given, for example, in [14]. To construct $\mathsf{Def}(\mathcal{L}ie\{1\} \to \mathcal{G}ra)$ one has to replace $\mathcal{L}ie\{1\}$ by its the minimal resolution, $\mathcal{L}ie\{1\}_\infty$, which is a quasi-free dg operad generated by the $\mathbb{S}$-module

$$E = \{E(n) := \mathbf{1}_n[2n-3]\}.$$

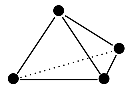
Then, as a $\mathbb{Z}$-graded vector space,

$$\begin{aligned} \mathsf{Def}(\mathcal{L}ie\{1\} \to \mathcal{G}ra) \equiv \mathsf{Def}(\mathcal{L}ie\{1\}_\infty \to \mathcal{G}ra) &:= \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{S}_n}(E(n), \mathcal{G}ra(n))[-1] \\ &= \prod_{n\geq 0} \mathcal{G}ra(n)^{\mathbb{S}_n}[2-2n], \end{aligned}$$

i.e. an element of fGC_2 can be understood as an $\mathbb{S}_n$-symmetrization a of graph from $\mathsf{G}_{n,l}$ to which we assign the degree $2n - l - 2$, for example

$$\overset{1}{\bullet}\!\!-\!\!\overset{2}{\bullet} + \overset{2}{\bullet}\!\!-\!\!\overset{1}{\bullet} =: \bullet\!\!-\!\!\bullet$$

is a degree 1 element in fGC_2. As labelling of vertices of elements from fGC_2 by integers is symmetrized, we often represent such elements as a single graph with vertices *unlabelled*, e.g.

$\bullet\!\!-\!\!\bullet$, 

One should not forget, however, that such a graph is in reality a symmetrization sum of some labelled graph from $\mathsf{G}_{n,l}$.

The Lie algebra structure in $\mathsf{fGC}_2 = \mathsf{Def}(\mathcal{L}ie\{1\}_\infty \to \mathcal{G}ra)$ is completely determined by the differential on $\mathcal{L}ie\{1\}_\infty$ [14]. It is an elementary exercise to see that the Lie brackets in fGC_2 can expressed in terms of operadic composition in $\mathcal{G}ra$ as follows,

$$[\Gamma, \Gamma'] := Sym(\Gamma \circ_1 \Gamma' - (-1)^{|\Gamma||\Gamma'|}\Gamma \circ_1 \Gamma),$$

where Sym stands for the symmetrization of vertex labels. The usefulness of this Lie algebra structure on $\mathsf{fGC}_2 := \mathsf{Def}(\mathcal{L}ie\{1\}_\infty \to \mathcal{G}ra)$ stems from the fact [14] that the set of its Maurer–Cartan elements is in one-to-one correspondence with morphisms of operads $\mathcal{L}ie\{1\}_\infty \to \mathcal{G}ra$. It is easy to check that the element $\bullet\!\!-\!\!\bullet$ is Maurer–Cartan,

$$[\bullet\!\!-\!\!\bullet, \bullet\!\!-\!\!\bullet] = 0.$$

It corresponds precisely to the morphism (3.4). This element makes fGC_2 into a complex with the differential

$$\delta_{\bullet\!-\!\bullet} := [\bullet\!\!-\!\!\bullet,\ ].$$

This dg Lie algebra contains a dg Lie subalgebra, GC_2, spanned by connected graphs with at least trivalent vertices and no tadpoles; this subalgebra is precisely the original (odd) *Kontsevich graph complex* [6, 17]. One of the main theorems of [17] asserts an isomorphism of Lie algebras,

$$H^0(\mathsf{GC}_2, \delta_{\bullet\!-\!\bullet}) \simeq \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ stands for the Grothendieck–Teichmüller Lie algebra and H^0 for cohomology in degree zero.

Note that the canonical representation (3.2) induces a morphism of dg Lie algebras,

$$\rho^{ind} : \mathsf{fGC}_2 = \mathsf{Def}\,(\mathcal{L}ie\{1\}_\infty \to \mathcal{G}ra) \longrightarrow \mathsf{Def}\left(\mathcal{L}ie\{1\}_\infty \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}\right)$$
$$= \mathsf{CoDer}\left(\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)[2])\right).$$

The image of this map consists of coderivations of the coalgebra $\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)[2]$ which are *universal* i.e. make sense in any dimension. In particular, $\rho(\bullet\!-\!\bullet)$ is precisely the Schouten bracket in $\mathcal{T}_{poly}(\mathbb{R}^d)$. Therefore, one can say that the graph complex fGC_2 (or GC_2) describes *universal* deformations of the Schouten bracket. T. Willwacher's theorem gives us universal homotopy actions of the Grothendieck–Teichmüller group $GRT_1 = \exp(\mathfrak{grt}_1)$ on $\mathcal{T}_{poly}(\mathbb{R}^d)$ by $\mathcal{L}ie_\infty$ automorphisms of the Schouten bracket.

3.5. T. Willwacher's twisted operad $f\mathcal{G}raphs^{\circlearrowleft}$. For any operad $\mathcal{P}$ and morphism of operads, $\mathcal{L}ie\{k\}_\infty \to \mathcal{P}$, there is an associated operad $Tw(\mathcal{P})$ whose representations, $\rho^{tw} : Tw(\mathcal{P}) \to \mathcal{E}nd_V$, can be obtained from representations, $\rho : \mathcal{P} \to \mathcal{E}nd_V$, of $\mathcal{P}$ by "twisting" ρ by Maurer–Cartan elements of the associated (via the map $\mathcal{L}ie\{k\}_\infty \to \mathcal{P}$) $\mathcal{L}ie\{k\}_\infty$ structure on V. Omitting general construction (see [17] for its details), we shall describe explicitly the dg operad $f\mathcal{G}raphs^{\circlearrowleft} := Tw(\mathcal{G}ra)$ obtained from $\mathcal{G}ra$ by twisting the morphism (3.4). For arbitrary integers $m \geq 1$, $n \geq 0$ and $l \geq 0$ we denote by $\mathsf{G}_{m,n;l}$ a set of graphs $\{\Gamma\}$ with m white vertices, n black vertices are and l edges such that (i) the white vertices of Γ are labelled by elements of $[m]$, (ii) the black vertices of Γ are at least trivalent and are labelled by elements of $[\bar{n}] = \{\bar{1}, \ldots, \bar{n}\}$, (iii) and the set of edges, $E(\Gamma)$, is totally ordered up to an even permutation. The set of black (respectively, white) vertices of Γ will be denoted by $V_\bullet(\Gamma)$ (resp. $V_\circ(\Gamma)$).

Let $\mathbb{K}\langle\mathsf{G}_{m,n;l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{m,n;l}$ modulo the relation $\Gamma_{opp} = -\Gamma$, and consider the $\mathbb{Z}$-graded $\mathbb{S}_m$-module,

$$f\mathcal{G}raphs^{\circlearrowleft}(m) := \prod_{n=0}^{\infty}\bigoplus_{l=0}^{\infty}\mathbb{K}\langle\mathsf{G}_{m,n;l}\rangle^{\mathbb{S}_n}[l-2n],$$

where invariants are taken with respect to the permutations of $[\bar{n}]$-labellings of black vertices. For example, $\overset{1}{\circ}\!-\!\bullet$ is a degree 1 element in $f\mathcal{G}raphs^{\circlearrowleft}(1)$ and $\overset{1}{\circ}\ \ \overset{2}{\circ}$ is a degree 0 element in $f\mathcal{G}raphs^{\circlearrowleft}(2)$. The operadic composition, $\Gamma \circ_i \Gamma'$, in

$$f\mathcal{G}raphs^{\circlearrowleft} = \left\{f\mathcal{G}raphs^{\circlearrowleft}(m)\right\}$$

is defined by substitution of the graph $\Gamma' \in \mathbb{K}\langle\mathsf{G}_{m',n';l}\rangle^{\mathbb{S}_{n'}}$ into the i-th white vertex v of $\Gamma \in \mathbb{K}\langle\mathsf{G}_{m,n;l}\rangle^{\mathbb{S}_n}$, reconnecting all edges of Γ incident to v in all possible ways to vertices of Γ' (in a full analogy to the case of $\mathcal{G}ra$), and finally symmetrizing over labellings of the $n+n'$ black vertices. Consider linear maps,

$$\delta_{\bullet\!-\!\bullet}\Gamma := -(-1)^{|\Gamma|}Sym\left(\Gamma \circ_{\bar{1}} \bullet\!-\!\bullet\right)$$

and

$$\delta_{\circ\!-\!\bullet}\Gamma := Sym\left(\overset{1}{\circ}\!-\!\bullet \circ_1 \Gamma \ - \ (-1)^{|\Gamma|}\sum_{v\in V(\circ)}\Gamma \circ_v \overset{1}{\circ}\!-\!\bullet\right)$$

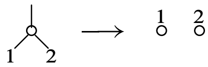
where Sym stands for the symmetrization of black vertex labellings. Note that in this case $\delta^2_{\bullet\!-\!\bullet} \neq 0$ and $\delta^2_{\circ\!-\!\bullet} \neq 0$ in general, but their sum $\delta_{\circ\!-\!\bullet} + \delta_{\bullet\!-\!\bullet}$ makes $f\mathcal{G}raphs^{\circlearrowleft}$ into an operad of *complexes* [17].

The dg suboperad of $f\mathcal{G}raphs^{\circlearrowleft}$ consisting of graphs Γ which have no connected component consisting solely of black vertices is denoted in [17] by $\mathcal{G}raphs^{\circlearrowleft}$. The inclusions of the suboperads of graphs without tadpoles, $f\mathcal{G}raphs$ and $\mathcal{G}raphs$, into $f\mathcal{G}raphs^{\circlearrowleft}$ and, respectively, $\mathcal{G}raphs^{\circlearrowleft}$, are quasi-isomorphisms. [17] Hence we may without loss of generality replace the operads $f\mathcal{G}raphs^{\circlearrowleft}$ and $\mathcal{G}raphs^{\circlearrowleft}$ by these suboperads $f\mathcal{G}raphs$ and $\mathcal{G}raphs$.

There is a morphism of dg operads [19]

$$\mathcal{A}ss_\infty \longrightarrow \mathcal{A}ss \longrightarrow \mathcal{G}raphs$$

where the first arrow is a natural projection and the second map is given on the generators of the operad $\mathcal{A}ss$ by



The standard construction [14] gives us a dg Lie algebra, $\mathsf{Def}(\mathcal{A}ss \to \mathcal{G}raphs)$, whose elements, Γ, are linear combinations of graphs from $\mathbb{K}\langle \mathsf{G}_{m,n;l}\rangle^{\mathbb{S}_n}$, $m, n, l \geq 0$, equipped with a total order on the set of white vertices of Γ (so that in pictures we can depict vertices of such graphs as lying on a line) and with degree $2n + m - l - 1$. The differential on $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$ is a sum,

$$\delta = \delta_{\circ\circ} + \delta_{\circ\!-\!\bullet} + \delta_{\bullet\!-\!\bullet}, \tag{3.5}$$

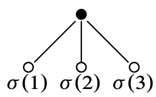
where

$$\delta_{\circ\circ}\Gamma := \left((\overset{1}{\circ}\ \overset{2}{\circ}) \circ_1 \Gamma + (\overset{1}{\circ}\ \overset{2}{\circ}) \circ_2 \Gamma \right) - (-1)^{|\Gamma|} \sum_{v \in V(\circ)} \Gamma \circ_v (\overset{1}{\circ}\ \overset{2}{\circ})$$

The first cohomology group of this deformation complex was computed in appendix E of [17],

$$H^i\left(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)\right) = \begin{cases} \mathfrak{grt}_1 \oplus \mathbb{R}[-1] & \text{for } i = 1, \\ \mathbb{R}[\mathbb{S}_2] & \text{for } i = 0, \\ 0 & \text{for } i \leq -1, \end{cases} \tag{3.6}$$

where the summand $\mathbb{R}[-1]$ in $H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs))$ is generated by the following graph

$$\sum_{\sigma \in \mathbb{S}_3} (-1)^\sigma$$  $$\in \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs). \tag{3.7}$$

and $H^0(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs))$ is generated by $\overset{1}{\circ}\!-\!\overset{2}{\circ}$.

Lemma 3.5.1. $H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) = \mathfrak{grt}_1 \oplus \mathbb{R}[-1]$.

Proof. As a complex $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$ is isomorphic to the tensor product of complexes

$$\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs) \otimes \odot^{\bullet \geq 0}(\mathsf{GC}_2[-2])$$

so that

$$\begin{aligned} H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) &= \sum_{i \in \mathbb{Z}} H^i(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)) \\ &\qquad \otimes H^{-i-1}(\odot^{\bullet \geq 0}\mathsf{GC}_2) \\ &= H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)), \end{aligned}$$

because $H^{\leq -1}(\mathsf{GC}_2) = 0$ and $H^{\leq -1}((\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)) = 0$ according to Thomas Willwacher [17, 19]. □

Note that in general the inclusion map of complexes,

$$\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs) \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs),$$

induces an *injection* on cohomology,

$$H^i(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)) \hookrightarrow H^i(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs))$$

since $\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$ is direct summand of $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$.

3.6. A mapping cone of the Willwacher map. It was proven in [17] that there is a degree 1 morphism of complexes,

$$\mathfrak{W} : \left(\mathsf{GC}_2, \delta_{\bullet\!-\!\bullet}\right) \longrightarrow (\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs), \delta).$$

which induces an *injection* on cohomology [17, 19]

$$[\mathfrak{W}] : H^i(\mathsf{GC}_2) \longrightarrow H^i(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)).$$

The map $\mathfrak{W}$ sends a graph $\gamma \in \mathsf{GC}_2$ (with, say, n black vertices) to a linear combination of graphs with n black vertices and one white vertex,

$$\mathfrak{W}(\gamma) := \frac{1}{\#V(\gamma)!} \sum_{v \in V(\gamma)} \begin{array}{c}\gamma\\ \bullet_v\\ |\\ \circ\\ 1\end{array} =: \begin{array}{c}\gamma\\ \bullet\\ |\\ \circ\\ 1\end{array} \tag{3.8}$$

where $\begin{array}{c}\gamma\\ \bullet_v\\ |\\ \circ\\ 1\end{array}$ stands for the graph obtained by attaching $\begin{array}{c}|\\ \circ\end{array}$ to the vertex v of γ; the set of edges of $\mathfrak{W}(\gamma)$ is ordered by putting the new edge after the edges of γ. Let $\mathsf{MaC}(\mathfrak{W})$ be the mapping cone of the map $\mathfrak{W}$, that is, the direct sum (without the standard degree shift as the map $\mathfrak{W}$ has degree $+1$)

$$\mathsf{MaC}(\mathfrak{W}) = \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2$$

equipped with the differential

$$\begin{array}{rccc} d: & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 \\ & (\Gamma, \gamma) & \longrightarrow & (\delta\Gamma + \mathfrak{W}(\gamma), \delta_{\bullet\!-\!\bullet}\gamma). \end{array}$$

There is a natural representation [17] of the Lie algebra GC_2 on the vector space $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$,

$$\begin{array}{rccc} \circ: & \mathsf{GC}_2 \times \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \\ & (\gamma, \Gamma) & \longrightarrow & \Gamma \cdot \gamma := \sum_{v \in V_{black}(\Gamma)} \Gamma \circ_v \gamma \end{array}$$

given by substitution of the graph γ into black vertices of the graph Γ. This action can be used to make $\mathsf{MaC}(\mathfrak{W})$ into a Lie algebra with the brackets,

$$[(\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)] := \Big([\Gamma_1, \Gamma_2] + \Gamma_1 \cdot \gamma_2 - (-1)^{|\Gamma_2||\gamma_1|} \Gamma_2 \cdot \gamma_1,\ [\gamma_1, \gamma_2] \Big). \quad (3.9)$$

The differential d respects these brackets so that

$$(\mathsf{MaC}(\mathfrak{W}), [\ ,\ ], d) \quad (3.10)$$

is a differential graded Lie algebra. For future reference we need the following

Lemma 3.6.1. $H^1(\mathsf{MaC}(\mathfrak{W}), d) = \mathbb{R}[-1]$.

Proof. There is a short exact sequence of dg Lie algebras,

$$0 \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \stackrel{\alpha}{\longrightarrow} \mathsf{MaC}(\mathfrak{W}) \stackrel{\beta}{\longrightarrow} \mathsf{GC}_2 \longrightarrow 0$$

where

$$\begin{array}{rccc} \alpha: & \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs) & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 \\ & \Gamma & \longrightarrow & (\Gamma, 0) \end{array}$$

and

$$\begin{array}{rccc} \beta: & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 & \longrightarrow & \mathsf{GC}_2 \\ & (\Gamma, \gamma) & \longrightarrow & \gamma \end{array}$$

are the natural maps. We have, therefore, a piece of the associated long exact sequence of cohomology groups,

$$H^i(\mathsf{GC}_2) \stackrel{[\mathfrak{W}]}{\to} H^{i+1}(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) \stackrel{[\alpha]}{\to} H^{i+1}(\mathsf{MaC}(\mathfrak{W}))$$
$$\stackrel{[\beta]}{\to} H^{i+1}(\mathsf{GC}_2) \stackrel{[\mathfrak{W}]}{\to} H^{i+2}(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs))$$

As the map $[\mathfrak{W}]$ is injective, we obtain

$$H^{i+1}(\mathsf{MaC}(\mathfrak{W})) = \frac{H^{i+1}(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs))}{[\mathfrak{W}](H^i(\mathsf{GC}_2))}.$$

Since $H^0(\mathsf{GC}_2) = \mathfrak{grt}_1$ and $H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) = \mathfrak{grt}_1 \oplus \mathbb{R}[-1]$ the claim follows. □

4. Universal $nc\mathcal{G}_\infty$ deformations of the standard Gerstenhaber algebra structure in $\mathcal{T}_{poly}(\mathbb{R}^d)$

4.1. Two-coloured version of $\mathcal{G}ra$. Let $\mathcal{G}ra = \{\mathcal{G}ra(n)\}_{n\geq 1}$ be the operad defined in §3.2; from now one we assume that vertices of graphs from $\mathcal{G}ra$ are coloured in black. For arbitrary integers $m \geq 1$, $n \geq 0$ and $l \geq 0$ we denote by $\mathsf{G}'_{m,n;l}$ the set of tadpoles-free graphs $\{\Gamma\}$ with m white vertices, n black vertices and l edges, such that

- (i) the set of white vertices, $V_\circ(\Gamma)$, of Γ is equipped with a total order (so that in our pictures white vertices will depicted as lying on a line),
- (ii) there is a bijection $V_\circ(\Gamma) \to [m]$ (which does not, in general, respect total orders),
- (iii) there is a bijection from the set, $V_\bullet(\Gamma)$, of black vertices of Γ to the set $[\bar{n}] = \{\bar{1}, \dots, \bar{n}\}$,
- (iv) the black vertices of Γ are at least trivalent,
- (v) the set of edges, $E(\Gamma)$, is equipped with an orientation, i.e. it is totally ordered up to an even permutation.

Note that graphs from $\mathsf{G}'_{m,n;l}$ can have connected components consisting of graphs with solely black vertices. Let $\mathbb{K}\langle \mathsf{G}'_{m,n;l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{m,n;l}$ modulo the relation $\Gamma_{opp} = -\Gamma$, where the graph Γ_{opp} is identical to Γ except that it has the opposite orientation. Consider the following collection of $\mathbb{Z}$-graded $\mathbb{S}$-modules,

$$\mathcal{G}ra^{\circ\bullet} := \left\{ \bigoplus_{N=m+n} \mathrm{Ind}_{\mathbb{S}_N}^{\mathbb{S}_m\times\mathbb{S}_{\bar{n}}} \left\{ \mathcal{G}ra(m,n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}'_{m,n;l}\rangle[l] \right\}_{m\geq 1, n\geq 0}, \{\mathcal{G}ra(n)\}_{n\geq 1} \right\}.$$

It has a structure of a 2-coloured operad with compositions

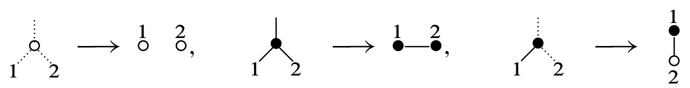
$$\begin{array}{llll} \circ_i : & \mathcal{G}ra(m_1,n_1)\otimes\mathcal{G}ra(m_2,n_2) & \longrightarrow & \mathcal{G}ra(m_1+m_2-1, n_1+n_2), \quad i\in[m_1] \\ \circ_i : & \mathcal{G}ra(m,n_1)\otimes\mathcal{G}ra(n_2) & \longrightarrow & \mathcal{G}ra(m, n_1+n_2-1), \quad i\in[n_1] \\ \circ_i : & \mathcal{G}ra(n_1)\otimes\mathcal{G}ra(n_2) & \longrightarrow & \mathcal{G}ra(n_1+n_2-1), \quad i\in[n_1], \end{array}$$

given by graph substitutions as in the case of $\mathcal{G}ra$.

Proposition 4.1.1. *There is a morphism of operads*

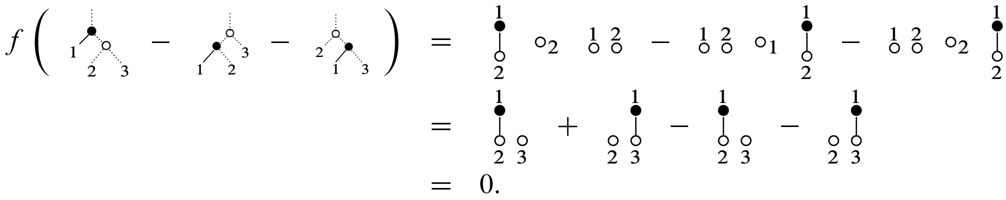
$$f : nc\mathcal{G} \longrightarrow \mathcal{G}ra^{\circ\bullet}$$

given on generators as follows,



(4.1)

Proof. We have to check that the map f respects relations (2.7), (2.8) and (2.6). For example,



Analogously one checks all other relations. □

Theorem 4.1.2. *The deformation complex,* $\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet})$, *of the morphism*

$$f_o : nc\mathcal{G}_\infty \xrightarrow{proj} nc\mathcal{G} \xrightarrow{f} \mathcal{G}ra^{\circ\bullet}$$

is isomorphic as a dg Lie algebra to $\mathsf{MaC}(\mathfrak{W})$.

Proof. As a graded vector space $\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet})$ is identical to the space of homomorphisms, $\mathrm{Hom}_{\mathbb{S}}(E, \mathcal{G}ra^{\circ\bullet})[-1]$, of $\mathbb{S}$-modules, where $E = \{E(N)\}$ is the $\mathbb{S}$-module of generators of $nc\mathcal{G}_\infty$. The latter $\mathbb{S}$-module splits as a direct sum,

$$E(N) = E_1(N) \oplus E_2(N),$$

where $E_1(N)$ is spanned as a vector space by corollas (2.1) and hence is given by

$$E_1(N) = sgn_N[2n-3]$$

where sgn_N is the the one-dimensional sign representation of $\mathbb{S}_N$. The $\mathbb{S}_N$-module $E_2(N)$ is spanned by corollas (2.2) and hence equals

$$E_2(N) = \bigoplus_{\substack{N=m+n \\ m\geq 1, n\geq 0}} \mathrm{Ind}_{\mathbb{S}_N}^{\mathbb{S}_m\times\mathbb{S}_n} \mathbb{K}[\mathbb{S}_m] \otimes sgn_n[2n+m-2].$$

Therefore, we have an isomorphism of graded vector spaces

$$\begin{aligned}
\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet}) &= \prod_N \mathrm{Hom}_{\mathbb{S}}\left(E_2(N), \mathcal{G}ra^{\circ\bullet}(N)\right)[-1] \\
&\qquad \oplus \prod_N \mathrm{Hom}_{\mathbb{S}}\left(E_1(N), \mathcal{G}ra^{\circ\bullet}(N)\right)[-1] \\
&= \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 \\
&= \mathsf{MaC}(\mathfrak{W})
\end{aligned}$$

One reads the Lie algebra structure in $\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet})$ from the differential (2.3) and (2.4) and easily concludes that it is given precisely by the Lie bracket [,] given in (3.9). Next, there is a 1-1 correspondence between Maurer–Cartan elements, Γ,

$$[\Gamma, \Gamma] = 0,$$

and morphisms of operads $nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet}$ (cf. [14]). The morphism f_0 is represented by the following Maurer–Cartan element,

$$\Gamma_0 = \left(\circ\ \circ + \overset{\bullet}{\underset{\circ}{|}}\ ,\ \ \bullet\!\!-\!\!\bullet \right) \tag{4.2}$$

so that the differential in $\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet})$ is given by $[\Gamma_0, \ ]$ and hence coincides precisely with the differential d in $\mathsf{MaC}(\mathfrak{W})$. The theorem is proven. □

4.2. A canonical representations of $\mathcal{G}ra^{\circ\bullet}$ in polyvector fields and an exotic $nc\mathcal{G}_\infty$ structure. There is a representation of the two-coloured operad $\mathcal{G}ra^{\circ\bullet}$ in the two-coloured endomorphism operad, $\mathcal{E}nd_{\mathcal{T}_{poly}(R^d),\mathcal{T}_{poly}(R^d)}$, of two copies of the space $\mathcal{T}_{poly}(R^d)$ given by formulae which are completely analogous to (3.2). Hence there is an induced of morphism of dg Lie algebras

$$\mathsf{MaC}(\mathfrak{W}) = \mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{G}ra^{\circ\bullet}) \longrightarrow \mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{E}nd_{\mathcal{T}_{poly}(R^d),\mathcal{T}_{poly}(R^d)}).$$

The dg Lie algebra $\mathsf{Def}(nc\mathcal{G}_\infty \to \mathcal{E}nv_{\mathcal{T}_{poly}(R^d),\mathcal{T}_{poly}(R^d)})$ describes $nc\mathcal{G}_\infty$ deformation of the standard Gerstenhaber algebra structure on $\mathcal{T}_{poly}(R^d)$. The dg Lie algebra $\mathsf{MaC}(\mathfrak{W})$ controls, therefore, *universal* deformations of this structure, i.e the ones which make sense in any dimension d.

In particular any Maurer–Cartan element,

$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$$

in the dg Lie algebra $\mathsf{MaC}(\mathfrak{W})$ gives us a universal $nc\mathcal{G}_\infty$-structure in $\mathcal{T}_{poly}(R^d)$. Such a structure can be viewed as a deformation of the standard Gerstenhaber algebra structure (corresponding to the graph (4.2)) as the above equation can be rewritten as

$$[\Gamma_0 + \Gamma, \Gamma_0 + \Gamma] = 0.$$

The dg Lie algebra $\mathsf{MaC}(\mathfrak{W})$ is naturally filtered by the number of black and white vertices. We assume from now on that $\mathsf{MaC}(\mathfrak{W})$ is completed with respect to this filtration. Then there is a well-defined action of degree zero elements, g, of $\mathsf{MaC}(\mathfrak{W})$ on the set of Maurer–Cartan elements,

$$\text{Ă} \longrightarrow \Gamma^g := e^{ad_g}\text{Ă} - \frac{e^{ad_g} - 1}{ad_g} dg.$$

The orbits of this action are $nc\mathcal{G}_\infty$-isomorphism classes of universal $nc\mathcal{G}_\infty$ structures on polyvector fields.

Infinitesimal homotopy non-trivial $nc\mathcal{G}_\infty$ deformations of the standard Gerstenhaber algebra structure on polyvector fields are classified by the cohomology group $H^1(\mathsf{MaC}(\mathfrak{W}))$. Lemma 3.6.1 says that there exists at most one homotopy non-trivial universal $nc\mathcal{G}_\infty$ deformation of the standard Gerstenhaber algebra structure on polyvector fields. The associated Maurer–Cartan element in $\mathsf{MaC}(\mathfrak{W})$ was given explicitly in [1] in term of periods over the compactified configuration spaces $\overline{C}_{\bullet,\bullet}(\mathbb{C})$,

$$\Gamma_0 + \Gamma = \left(\sum_{m\geq 1, n\geq 0} \sum_{\Gamma\in \mathsf{G}''_{m,n;2n+m-2}} \int_{\overline{C}_{m,n}(\mathbb{C})} \Omega_\Gamma\, \Gamma, \quad \bullet\!\!-\!\!\bullet \right) \tag{4.3}$$

where

- $\mathsf{G}''_{m,n;2n+m-2}$ is the set of equivalence classes of graphs from $\mathsf{G}'_{m,n;2n+m-2}$ which are linearly independent in the space $\mathbb{K}\langle \mathsf{G}'_{m,n;2n+m-2}\rangle$ and have no tadpoles;
- $\Omega_\Gamma := \bigwedge_{e\in Edges(\Gamma)} \pi_e^*(\omega)$,
- for an edge $e \in Edges(\Gamma)$ beginning at a vertex (of any colour) labelled by i and ending at a vertex (of any colour) labelled by j, p_e is the natural surjection

$$\begin{array}{rccc} \pi_e : & C_{n,m}(\mathbb{C}) & \longrightarrow & C_2(\mathbb{C}) = S^1 \\ & (z_1,\ldots,z_i,\ldots,z_j,\ldots,z_{n+m}) & \longrightarrow & \frac{z_i - z_j}{|z_i - z_j|}. \end{array}$$

- The 1-form $\omega := \frac{1}{2\pi} dArg(z_i - z_j)$ is the standard homogenous volume form on S^1 normalized so that $\int_{S^1}\omega = 1$.

The lowest (in total number of vertices) term in Γ is given by the graph (3.7) whose weight is equal to $1/24$. Hence Lemma 3.6.1 and [1] imply the following theorem.

Theorem 4.2.1. *Up to $nc\mathcal{G}_\infty$ isomorphisms, there are only two different universal $nc\mathcal{G}_\infty$ structures on polyvector fields, the standard Gerstenhaber one corresponding to the Maurer–Cartan element (4.2) and the exotic one given by (4.3).*

5. No-Go Theorem

5.1. A class of universal $\mathcal{A}ss_\infty$ structures on Poisson manifolds. For any degree 2 element $\hbar\pi$ in $\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]$ the operad $f\mathcal{G}raphs$ admits a canonical representation

$$\rho^\pi : f\mathcal{G}raphs \longrightarrow \mathcal{E}nd^{cont}_{\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]}$$

of graded operads, which sends a graph Γ from $\mathcal{G}raphs$ with, say, m white vertices and n black vertices into a continuous (in the $\hbar$-adic topology) operator $\rho(\Gamma) \in \mathrm{Hom}(\otimes^m \mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]$ which is constructed exactly as in the formula (3.2) except that black vertices are decorated by the element $\hbar\pi$. (From now on we take our operad $f\mathcal{G}raphs$ to be completed with respect to the filtration by the number of black vertices; hence we need to use a degree zero formal parameter $\hbar$ to ensure convergence of operators $\rho(\Gamma)$ in the $\hbar$-adic topology.) The representation ρ^π is a representation of dg operads if π is Poisson and we equip the space of polyvector fields with the Poisson-Lichnerowitz differential.

The Lie algebra GC_2 acts (on the right) on the operad $f\mathcal{G}raphs$,

$$\begin{array}{rccc} R: & f\mathcal{G}raphs \times \mathsf{GC}_2 & \longrightarrow & f\mathcal{G}raphs \\ & (\Gamma, \gamma) & \longrightarrow & \Gamma \cdot \gamma \end{array}$$

by operadic derivations, where $\Gamma \cdot \gamma$ is obtained from Γ by inserting γ into black vertices [17]. Let $I'_{\bullet\!-\!\bullet}$ be the operadic ideal in $f\mathcal{G}raphs$ generated by graphs of the form $\Gamma \cdot \bullet\!-\!\bullet$. There is natural projection map of operads,

$$f\mathcal{G}raphs \longrightarrow f\mathcal{G}raphs' := f\mathcal{G}raphs / I'_{\bullet\!-\!\bullet},$$

and, for π being a (graded) Poisson structure on $\mathbb{R}^d$, that is, for π satisfying $[\pi, \pi]_S = 0$, the canonical representation ρ^π factors through this projection,

$$\rho^\pi : f\mathcal{G}raphs \longrightarrow f\mathcal{G}raphs' \longrightarrow \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}.$$

The induced representation $f\mathcal{G}raphs' \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}$ we denote by the same letter ρ^π. It induces in turn a map of Lie algebras,

$$\begin{aligned} &\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs') \\ &\qquad \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}) = \mathsf{CoDer}(\otimes^{\bullet \geq 1}(\mathcal{T}_{poly}(\mathbb{R}^d)[1]) \end{aligned}$$

where

$$\mathsf{CoDer}(\otimes^{\bullet \geq 1}(\mathcal{T}_{poly}(\mathbb{R}^d)[1]) = \prod_{m \geq 1} \mathsf{Hom}(\otimes^m \mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d))[1-m]$$

is the Gerstenhaber Lie algebra of coderivations of the tensor coalgebra $\otimes^{\bullet \geq 1}(\mathcal{T}_{poly}(\mathbb{R}^d)[1])$. Hence any Maurer–Cartan element Γ,

$$[\Gamma, \Gamma] = 0,$$

in the Lie algebra $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$ induces, for any fixed Poisson structure on $\mathbb{R}^d$, a universal $\mathcal{A}ss_\infty$ algebra structure on $\mathcal{T}_{poly}(\mathbb{R}^d)$. Moreover, two such universal $\mathcal{A}ss_\infty$ structures, Γ_1 and Γ_2, are universally $\mathcal{A}ss_\infty$ isomorphic if and only if there exists a degree zero element $h \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$ such that

$$\Gamma_2 = e^{\mathrm{ad}_h} \Gamma_1,$$

where ad_h stands for the adjoint action. Note that, due to the filtrations of the Lie algebra $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$ by the numbers of white and black vertices, there is no convergence problem in taking the exponent of ad_h.

It is easy to see that

$$\Gamma_0 = \circ\circ + \circ\!\!-\!\!\bullet$$

is a Maurer–Cartan element in $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$ (but *not* in $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$!), and the associated $\mathcal{A}ss_\infty$ structure in $(\mathcal{T}_{poly}(\mathbb{R}^d), \pi)$ is the standard structure of a Poisson complex, that is, a wedge product $\wedge$ (corresponding to the graph $\circ\circ$) and the differential $d_\pi = [\hbar\pi, \ ]_S$ (corresponding to the graph $\circ\!\!-\!\!\bullet$). Hence Γ_0 makes $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$ into a complex with the differential $d := [\Gamma_0, \ ]$. It is clear that the natural projection of Lie algebras

$$p : \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$$

is compatible with the differentials.

Let γ be a degree zero cycle in the graph complex GC_2, representing some cohomology class from $\mathfrak{grt}_1$. Then

$$\Gamma_0^\gamma := \Gamma_0 \cdot e^\gamma = \circ\circ + \circ\!\!-\!\!\bullet + \overset{\gamma}{\underset{\circ}{\overset{\bullet}{|}}} + \frac{1}{2!} \overset{\gamma\cdot\gamma}{\underset{\circ}{\overset{\bullet}{|}}} + \dots$$

is again a Maurer–Cartan element in $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$. The associated $\mathcal{A}ss_\infty$ structure in $(\mathcal{T}_{poly}(\mathbb{R}^d), \pi)$ consists of the standard wedge product $\wedge$ (corresponding to the graph $\circ\circ$) and the differential $d_{g(\pi)} = [g(\hbar\pi), \ ]_S$, where $g = \exp(\gamma)$ is the element of the group GRT_1 corresponding to γ. The element in the difference $\Gamma_0^\gamma - \Gamma_0$ with lowest number of vertices is

$$\overset{\gamma}{\underset{\circ}{\overset{\bullet}{|}}}$$

It defines a cycle in both complexes $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$ and $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$. It is shown in [17] that $\overset{\gamma}{\underset{\circ}{\overset{\bullet}{|}}}$ is *not* a coboundary in $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$, it therefore defines a non-trivial cohomology class in $H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs))$.

5.1.1. Lemma. *For any* $[\gamma] \in \mathfrak{grt}_1$ *an associated cycle* $\overset{\gamma}{\underset{\circ}{\overset{\bullet}{|}}}$ *is* not *a coboundary in* $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$*, that is, it defines a non-trivial cohomology class in* $H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs'))$*. In fact the natural map*

$$[p] : H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) \longrightarrow H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs'))$$

is an injection.

Let us first prove the following corollary to this lemma, and then the lemma itself.

5.1.2. No-go theorem. *For any* $[\gamma] \in \mathfrak{grt}_1$, *the Maurer–Cartan elements* Γ_0 *and* Γ_0^γ *are not gauge equivalent in the Lie algebra* $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$. *Equivalently, the universal* $\mathcal{A}ss_\infty$ *structures in* $\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]$ *corresponding to these elements are not universally* $\mathcal{A}ss_\infty$ *isomorphic.*

Proof. Comparing the terms with the same number of black and white vertices in the equation

$$\Gamma_0^\gamma = e^{\mathrm{ad}_h}\Gamma_0,$$

we immediately see that

$$\underset{\circ}{\overset{\gamma}{\bullet}} = [h', \circ\circ + \circ\!\!-\!\!\bullet] = -dh'$$

for some summand h' in h. This contradicts Lemma 5.1.1. □

To prove Lemma 5.1.1 we need the following

5.1.3. Lemma. *For any* $[\gamma] \in \mathfrak{grt}_1$, *an associated cycle* $\underset{\circ}{\overset{\gamma}{\bullet}}$ *in the complex* $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$ *is cohomologous to an element* γ_w *which has no black vertices.*

Proof. Let us represent the total differential δ in $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$ as a sum of two differentials (see (3.5))

$$\delta = \delta_{\circ\circ} + \delta'.$$

The cohomology of the complex $(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs), \delta')$ (which contains elements of the form $\underset{\circ}{\overset{\gamma}{\bullet}}$ and is a *direct* summand of the full complex $(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs), \delta')$ was computed in [9] (see also Proposition 5 in [17]). We need from that computation only the following fact: any δ'-cocycle in $\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$ which contains at least one black vertex is δ'-exact. As

$$\delta\ \underset{\circ}{\overset{\gamma}{\bullet}} = \delta_{\circ\circ}\ \underset{\circ}{\overset{\gamma}{\bullet}} = 0,$$

we conclude that $\delta'\ \underset{\circ}{\overset{\gamma}{\bullet}} = 0$ and hence $\underset{\circ}{\overset{\gamma}{\bullet}} = -\delta'\gamma_\circ$ for some degree zero graph $\gamma_\circ$ in $\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$; in fact, it is easy to see that $\gamma_\circ$ is γ with every black vertex labelled by, say, 1 made white (remember that γ is symmetrized over numerical labellings of vertices so that nothing depends on the choice of a particular label in this construction of $\gamma_\circ$). We can, therefore, write,

$$\underset{\circ}{\overset{\gamma}{\bullet}} = -(\delta_{\circ\circ} + \delta')\gamma_\circ + \delta_{\circ\circ}\gamma_\circ.$$

If $\delta_{\circ\circ}\gamma_\circ$ contains black vertices, then again

$$\delta\,\delta_{\circ\circ}\gamma_\circ = \delta'(\delta_{\circ\circ}\gamma_\circ) = 0 \quad\Rightarrow\quad \delta_{\circ\circ}\gamma_\circ = -\delta'\gamma_{\circ\circ}$$

and hence

$$\overset{\gamma}{\underset{\circ}{\bullet}} = -\delta(\gamma_0 + \gamma_{\circ\circ}) + \delta_{\circ\circ}\gamma_{\circ\circ}\,.$$

Continuing this process we finally obtain an equality

$$\overset{\gamma}{\underset{\circ}{\bullet}} = -\delta(\gamma_0 + \gamma_{\circ\circ} + \ldots + \gamma^{max}_{\circ\ldots\circ}) + \delta_{\circ\circ}\gamma_{\circ\ldots\circ} \tag{5.1}$$

where $\gamma_w := \delta_{\circ\circ}\gamma^{max}_{\circ\ldots\circ}$ has no black vertices. □

Proof of Lemma 5.1.1. Since

$$H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)) = H^1(\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs))$$

and since $\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs')$ is a direct summand of $\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs')$, it is enough to study the natural projection map

$$p : \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs) \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs').$$

Consider the following *direct* summands, C and C', of both complexes of the form

$$\mathsf{Ker}\, d \bigcap \{\text{Subspace of graphs with no black vertices}\}$$

As the ideal used to construct the quotient operad $\mathcal{G}raphs'$ out of $Graphs$ consists of graphs with at least two black vertices, we conclude that the map p sends C isomorphically to C'. Then Lemma 5.1.3 (and its obvious analogue for the graph (3.7)) implies the required result. □

5.2. Quotient mapping cone. Let $I_{\bullet\!-\!\bullet}$ be the ideal in the operad $\mathcal{G}ra$ generated by the graph $\overset{1}{\bullet}\!\!-\!\!\overset{2}{\bullet}$, let $\mathcal{G}ra' := \mathcal{G}ra/I_{\bullet\!-\!\bullet}$, and let

$$\mathsf{GC}'_2 := \mathsf{Def}\left(\mathcal{L}ie\{1\}_\infty \xrightarrow{0} \mathcal{G}ra'\right)$$

be the deformation complex of the zero map (this is just a Lie algebra). There is an induced Willwacher map

$$\mathfrak{W}' : \mathsf{GC}'_2 \longrightarrow \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs')$$

and hence an associated Lie algebra structure on the *quotient* mapping cone,

$$\mathsf{MaC}(\mathfrak{W}') := \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs') \oplus \mathsf{GC}'_2.$$

There is a natural surjection of Lie algebras,

$$S : \mathsf{MaC}(\mathfrak{W}) \longrightarrow \mathsf{MaC}(\mathfrak{W}'). \tag{5.2}$$

For future reference we make an evident observation that our class of universal $\mathcal{A}ss_\infty$ structures on polyvector fields can be identified with a class of Maurer–Cartan elements of the quotient mapping cone $\mathsf{MaC}(\mathfrak{W}')$ which have the form $(\Gamma, 0)$ for some $\Gamma \in \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs')$.

6. Proof of the main theorem

6.1. $nc\mathcal{G}_\infty$ isomorphisms of $nc\mathcal{G}_\infty$ algebras. As the two-coloured operad $\mathcal{G}ra^{\circ\bullet}$ has a canonical representation in the space of polyvector fields $\mathcal{T}_{poly}(\mathbb{R}^d)$, any morphism of operads

$$F : nc\mathcal{G}_\infty \longrightarrow \mathcal{G}ra^{\circ\bullet}$$

induces a universal $nc\mathcal{G}_\infty$ structure in $\mathcal{T}_{poly}(\mathbb{R}^d)$. On the other hand, we proved in the previous section that there is a one-to-one correspondence between such morphisms F and degree 1 elements,

$$\breve{\mathsf{A}} = (\Gamma, \gamma)$$

in the Lie algebra $\mathsf{MaC}(\mathfrak{W}) = \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2$ satisfying the Maurer–Cartan condition

$$[\breve{\mathsf{A}}, \breve{\mathsf{A}}] = ([\Gamma, \Gamma] + 2\Gamma \circ \gamma,\ [\gamma, \gamma]) = 0.$$

Two universal $nc\mathcal{G}_\infty$ structures corresponding to Maurer–Cartan elements $\breve{\mathsf{A}}$ and $\breve{\mathsf{A}}'$ are $nc\mathcal{G}_\infty$-isomorphic if and only if the Maurer–Cartan elements $\breve{\mathsf{A}}$ and $\breve{\mathsf{A}}'$ are gauge equivalent, that is,

$$\breve{\mathsf{A}}' = e^{\mathrm{ad}_{\mathsf{H}}} \breve{\mathsf{A}} \tag{6.1}$$

for some degree zero element $\mathsf{H} = (H, h)$ in $\mathsf{MaC}(\mathfrak{W})$.

6.2. $nc\mathcal{G}_\infty$ structures versus $\mathcal{A}ss_\infty$ structures on (affine) Poisson manifolds. Let us denote by $\mathcal{MC}$ the set of all Maurer–Cartan elements in the Lie algebra $\mathsf{MaC}(\mathfrak{W})$. By Theorem 4.2.1, any element $\Gamma \in \mathcal{MC}$ is gauge equivalent either to (4.2) or to (4.3). Both these Maurer–Cartan elements belong to the subset $\mathcal{MC}_{\mathcal{A}ss} \subset \mathcal{MC}$ consisting of elements of the form $(\Gamma, \bullet\!\!-\!\!\bullet)$ for some $\Gamma \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$. As projection (5.2) sends such and element into a Maurer–Cartan element in $\mathsf{MaC}(\mathfrak{W}')$ of the form $(\Gamma, 0)$, the subset $\mathcal{MC}_{\mathcal{A}ss} \subset \mathcal{MC}$ gives us universal $\mathcal{A}ss_\infty$ structures on polyvector fields. We are interested now in the gauge transformations of the set $\mathcal{MC}$ which preserve the subset $\mathcal{MC}_{\mathcal{A}ss}$, as such transformations can sometimes induce (via the surjection (5.2)) $\mathcal{A}ss_\infty$ isomorphisms of our class of universal $\mathcal{A}ss_\infty$ structures on polyvector fields. It is clear that the gauge transformation (6.1) associated to a degree zero element

$$\mathsf{H} = (H \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs), \quad h \in \mathsf{GC}_2)$$

preserves the subset $\mathcal{MC}_{\mathcal{A}ss} \subset \mathcal{MC}$ if and only if $\delta_{\bullet\!-\!\bullet} h = 0$, i.e. if h is a cycle in the Kontsevich graph complex. In this case one has

$$e^{\mathrm{ad}_{\mathsf{H}}}(\Gamma, \bullet\!-\!\bullet) = \left(e^{\mathrm{ad}_{(H \circ e^h)}}(\Gamma \circ (e^{-h})) + \dots\,,\ \bullet\!-\!\bullet\right)$$

where $e^{\mathrm{ad}_{(a \circ e^h)}}$ is computed with respect to the Lie bracket in $\mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$ and, for an element $A \in \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$ and an element $\gamma \in \mathsf{GC}_2$ we set

$$A \circ (e^\gamma) := \sum_{n=0}^{\infty} \frac{1}{n!}(\dots((A \circ \underbrace{\gamma) \circ \gamma) \dots \circ \gamma)}_{n} \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$$

It is clear from these formulae that gauge transformations of the set $\mathcal{MC}_{\mathcal{A}ss}$ associated with degree zero elements in $\mathsf{MaC}(\mathfrak{W})$ of the form

$$\mathsf{H} = (H, 0)$$

will induce — via the projection(5.2)) — $\mathcal{A}ss_\infty$ isomorphisms of $\mathcal{A}ss_\infty$ structures associated to elements of $\mathcal{MC}_{\mathcal{A}ss}$.

6.3. A naive action of GRT_1 on $\mathcal{MC}_{\mathcal{A}ss}$. For any $[\gamma] \in \mathfrak{grt}_1$ an associated degree zero element $\mathsf{H}_\gamma = (0, \gamma) \in \mathsf{MaC}(\mathfrak{W})$ gives us a gauge transformation of $\mathcal{MC}$ which preserves the subset $\mathcal{MC}_{\mathcal{A}ss}$. For example, in the case of the standard Gerstenhaber algebra structure (4.2) one has

$$\breve{\mathsf{A}}_0{}^\gamma := e^{\mathrm{ad}_{\mathsf{H}_\gamma}} \breve{\mathsf{A}}_0 = \left(\circ\ \circ \ + \ \overset{g}{\underset{\circ}{\bullet}}\ ,\ \bullet\!-\!\bullet \right) \tag{6.2}$$

where $g = \exp(-\gamma) \in GRT_1$. The associated (via the projection (5.2)) $\mathcal{A}ss_\infty$ structure on polyvector fields is precisely the standard differential Gerstenhaber algebra structure in which the differential is twisted by the action of g on the Poisson structure (see Main Theorem in the introduction).

To construct a less naive action of GRT_1 on $\mathcal{MC}_{\mathcal{A}ss}$ we need some technical preparations.

6.4. Splitting of the Lie algebra $\mathsf{MaC}(\mathfrak{W})$. The natural epimorphism of differential Lie algebras,

$$\mathsf{MaC}(\mathfrak{W}) \longrightarrow \mathsf{GC}_2,$$

has a section in the category of *non*-differential Lie algebras given explicitly in the following proposition.

Proposition 6.4.1. *There is a morphism of Lie algebras* $s : \mathsf{GC}_2 \to \mathsf{MaC}(\mathfrak{W})$ *given by*

$$\begin{array}{rccc} s: & \mathsf{GC}_2 & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2 \\ & \gamma & \longrightarrow & (\gamma_\circ, \gamma) \end{array}$$

where

$$\gamma_\circ := \sum_{v \in V(\gamma)} \gamma_{v\to\circ}$$

and $\gamma_{v\to\circ}$ *stands for the graph* γ *whose (black) vertex* v *is made white .*

Proof. Denoting $\varepsilon := |\gamma_1||\gamma_2|$, we have

$$\begin{aligned} s([\gamma_1,\gamma_2]) &= \left(\sum_{v\in V([\gamma_1,\gamma_2])} [\gamma_1,\gamma_2]_{v\to\circ} \ , \ [\gamma_1,\gamma_2] \right) \\ &= \left(\sum_{w\in V(\gamma_1)} \left(\sum_{v\in V(\gamma_2)} \gamma_1 \circ_w (\gamma_2)_{v\to\circ} + \sum_{v\in\{V(\gamma_1)\setminus w\}} (\gamma_1)_{v\to\circ} \circ_w \gamma_2 \right) \right. \\ &\qquad \left. -(-1)^\varepsilon (1 \leftrightarrow 2) \ , \ [\gamma_1,\gamma_2] \right) \\ &= \left(\sum_{\substack{w\in V(\gamma_1)\\ v\in V(\gamma_2)}} \gamma_1 \circ_w (\gamma_2)_{v\to\circ} - (-1)^\varepsilon \sum_{\substack{w\in V(\gamma_2)\\ v\in V(\gamma_1)}} \gamma_2 \circ_w (\gamma_1)_{v\to\circ} + (\gamma_1)_\circ \circ \gamma_2 \right. \\ &\qquad \left. -(-1)^\varepsilon (\gamma_2)_\circ \circ \gamma_1 \ , \ [\gamma_1,\gamma_2] \right) \\ &= ([(\gamma_1)_\circ, (\gamma_2)_\circ] + (\gamma_1)_\circ \circ \gamma_2 - (-1)^\varepsilon (\gamma_2)_\circ \circ \gamma_1 \ , \ [\gamma_1,\gamma_2]) \\ &= [s(\gamma_1), s(\gamma_2)] \,. \end{aligned}$$

□

Corollary 6.4.2. *There is an isomorphism of Lie algebras*

$$\begin{array}{rccc} \mathfrak{s}: & \mathsf{MaC}(\mathfrak{W}) & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs) \oplus \mathsf{GC}_2 \\ & (a,\gamma) & \longrightarrow & (a - \gamma_\circ, \gamma). \end{array}$$

and hence an isomorphism of gauge groups,

$$e^{\mathsf{MaC}(\mathfrak{W})^0} \simeq e^{\mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)^0} \times e^{\mathsf{GC}_2^0}.$$

Consider now an action of GRT_1 on $\Gamma_0 \in \mathcal{MC}_{\mathcal{A}ss}$ via the morphism $\mathfrak{s}$,

$$\begin{aligned}
e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}\breve{\mathsf{A}}_0 &= \left(\circ\circ + \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix} + \left[\gamma_\circ, \circ\circ + \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix}\right] + \gamma_\circ \circ \bullet\!\!-\!\!\bullet - \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix} \circ \gamma + \mathcal{O}(\gamma^2)\ ,\ \bullet\!\!-\!\!\bullet\right)\\
&= \left(\circ\circ + \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix} - \delta_{\circ\circ}\gamma_\circ + \begin{smallmatrix}\gamma\\ \bullet\\ |\\ \circ\end{smallmatrix} - \begin{smallmatrix}\gamma\\ \bullet\\ |\\ \circ\end{smallmatrix} + \mathcal{O}(\gamma^2)\ ,\ \bullet\!\!-\!\!\bullet\right)\\
&= \left(\circ\circ + \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix} - \delta_{\circ\circ}\gamma_\circ + \mathcal{O}(\gamma^2)\ ,\ \bullet\!\!-\!\!\bullet\right)
\end{aligned}$$

As terms of the form $\begin{smallmatrix}\gamma\\ \bullet\\ |\\ \circ\end{smallmatrix}$ cancel out, the $\mathcal{A}ss_\infty$ structure on polyvector fields corresponding to $e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}\breve{\mathsf{A}}_0$ has the differential, $\begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix}$, unchanged by the action of GRT_1 at the price of adding higher homotopies to the standard wedge product. This rather unusual universal $\mathcal{A}ss_\infty$ structure is $\mathcal{A}ss_\infty$ isomorphic to the naive GRT_1 deformation (6.2) since

$$e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}\breve{\mathsf{A}}_0 = e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}e^{-\mathrm{ad}_{\mathsf{H}_\gamma}}\breve{\mathsf{A}}_0^\gamma$$

and $e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}e^{-\mathrm{ad}_{\mathsf{H}_\gamma}}$ is of the form $e^{\mathrm{ad}_{\mathsf{H}}}$ for some $\mathsf{H} = (H \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs), 0)$. However this fact does not prove our Main Theorem as the multiplication operation in the $\mathcal{A}ss_\infty$ algebra corresponding to $e^{\mathrm{ad}_{\mathfrak{s}(\gamma)}}\breve{\mathsf{A}}_0$ is given by the graph

$$\circ\circ - \delta_{\circ\circ}\gamma_\circ + \mathcal{O}(\gamma^2)$$

and hence is *not* equal to the standard wedge product. However it is now clear how to achieve a GRT_1 deformation of the standard dg algebra structure on polyvector fields in such a way that the differential and the wedge product stay unchanged. In the notations of Lemma 5.1.3, consider a degree zero map, given by

$$\begin{array}{rccl}\hat{\mathfrak{s}}: & \mathsf{GC}_2 & \longrightarrow & \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs) \oplus \mathsf{GC}_2\\ & \gamma & \longrightarrow & (\gamma_\circ + \gamma_{\circ\circ} + \ldots + \gamma^{max}_{\circ\ldots\circ}, \gamma).\end{array}$$

Then, for γ a cycle in GC_2 representing some cohomology class $[\gamma] \in \mathfrak{grt}_1$, we have

$$e^{ad_{\hat{\mathfrak{s}}(\gamma)}}\breve{\mathsf{A}}_0 = \left(\circ\circ + \begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix} - \delta_{\circ\circ}\gamma_\circ + \mathcal{O}(\gamma^2)\ ,\ \bullet\!\!-\!\!\bullet\right) \tag{6.3}$$

so that the first corrections to the standard wedge multiplication, $\circ\circ$, in polyvector fields is given by the following graph

$$\circ\circ - \delta_{\circ\circ}\gamma^{max}_{\circ\ldots\circ} + \mathcal{O}(\gamma^2)$$

As $\gamma^{max}_{\circ\ldots\circ}$ has at least four white vertices, we conclude that the universal $\mathcal{A}ss_\infty$ structure corresponding to $e^{\mathrm{ad}_{\hat{\mathsf{s}}(\gamma)}}\breve{\mathsf{A}}_0$ has operations μ_1 and μ_2 unchanged at the price of non-trivial higher homotopy operations $\mu_{\bullet\geq 4} \neq 0$. We have

$$e^{\mathrm{ad}_{\hat{\mathsf{s}}(\gamma)}}\breve{\mathsf{A}}_0 = e^{\mathrm{ad}_{\hat{\mathsf{s}}(\gamma)}}e^{-\mathrm{ad}_{\mathsf{H}^\gamma}}\breve{\mathsf{A}}_0^\gamma = e^{-\mathrm{ad}(H,0)}\breve{\mathsf{A}}_0^\gamma$$

for some $H \in \mathsf{Def}(\mathcal{A}ss_\infty \to f\mathcal{G}raphs)$. Thus the universal $\mathcal{A}ss_\infty$ structures corresponding to Maurer–Cartan elements (6.2) and (6.3) are $\mathcal{A}ss_\infty$ isomorphic. This proves our Main Theorem for the case $M = \mathbb{R}^d$, the affine space.

6.4.1. Globalization to any Poisson manifold. Let M be a finite-dimensional smooth manifold. A torsion-free affine connection on M defines an isomorphism of sheaves of algebras between the sheaf of jets of functions, $J^\infty C^\infty_M$, and the completed symmetric bundle $\hat{S}(T^*_M)$ of the cotangent bundle. Similarly, the sheaf of jets of polyvector fields, $J^\infty(S(T_M[-1]))$, becomes isomorphic to the sheaf $\mathfrak{T} := \hat{S}(T^*_M \oplus T_M[-1])$. The canonical jet bundle connection defines, *via* this isomorphism, a Maurer–Cartan element $B \in \Omega(M, \mathfrak{T})$ of the dg Lie algebra of differential forms on M with values in $\mathfrak{T}$. Taking jets (with respect to the affine connection) is a quasi-isomorphism

$$j : (\mathcal{T}_{poly}(M), \wedge, [\,,\,]_S) \hookrightarrow (\Omega(M, \mathfrak{T}), d_{dR} + [B,\,]_S, \wedge, [\,,\,]_S)$$

of Gerstenhaber algebras. The space on the right was used, e.g., in [2], to globalize Kontsevich's formality morphism. The action of degree 0 cocycles of Kontsevich's graph complex GC_2 by $\mathcal{L}ie_\infty$-derivations of the polyvector fields on affine $\mathbb{R}^d$ defines (essentially, because of equivariance with respect to linear coordinate changes) $\mathcal{L}ie_\infty$-derivations of the dg Lie algebra $(\Omega(M, \mathfrak{T}), d_{dR}, [\,,\,]_S)$. Let now π be a Poisson bivector on M. The jet $j(\pi)$ is then a Maurer–Cartan element of $\Omega(M, \mathfrak{T})$ and, because $\mathcal{L}ie_\infty$ morphisms can be twisted by Maurer–Cartan elements, any degree 0 graph cocycle γ will define a $\mathcal{L}ie_\infty$ morphism

$$\begin{aligned} e^\gamma : (\Omega(M, \mathfrak{T}), d_{dR} + [j(\pi) + B,\,]_S, [\,,\,]_S) \\ \to (\Omega(M, \mathfrak{T}), d_{dR} + [\gamma(j(\pi) + B),\,]_S, [\,,\,]_S). \end{aligned}$$

Define $\delta := d_{dR} + [B,\,]_S$. Because $\gamma(j(\pi) + B) = j(\gamma(\pi)) + B$, the γ on the right referring to the globalized automorphism of polyvector fields on M (see [2] for the arguments), the above is a morphism

$$(\Omega(M, \mathfrak{T}), \delta + d_{j(\pi)}, [\,,\,]_S) \to (\Omega(M, \mathfrak{T}), \delta + d_{j(\gamma(\pi))}, [\,,\,]_S).$$

Our formula for the morphism of associative Poisson cohomology algebras defines a morphism of associative algebras

$$F^\gamma : H((\Omega(M, \mathfrak{T}), \delta, \wedge), d_{j(\pi)}) \to H((\Omega(M, \mathfrak{T}), \delta, \wedge), d_{j(\gamma(\pi))}).$$

Since taking jets is a quasi-isomorphism of associative algebras,

$$(\mathcal{T}_{poly}(M), d_\pi, \wedge) \to (\Omega(M, \mathfrak{T}), \delta + d_{j(\pi)}, \wedge,),$$

this shows that the morphism F^γ globalizes.

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