# Free Courant and Derived Leibniz Pseudoalgebras

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**Abstract**

We introduce the category of generalized Courant pseudoalgebras and show that it admits a free object on any anchored module over 'functions'. The free generalized Courant pseudoalgebra is built from two components: the generalized Courant pseudoalgebra associated to a symmetric Leibniz pseudoalgebra and the free symmetric Leibniz pseudoalgebra on an anchored module. Our construction is thus based on the new concept of symmetric Leibniz algebroid. We compare this subclass of Leibniz algebroids with the subclass made of Loday algebroids, which were introduced in [GKP13] as geometric replacements of standard Leibniz algebroids. Eventually, we apply our algebro-categorical machinery to associate a differential graded Lie algebra to any symmetric Leibniz pseudoalgebra, such that the Leibniz bracket of the latter coincides with the derived bracket of the former.

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## Contents

# 1 Introduction

The skew-symmetric non-Jacobi Courant bracket [Cou90] on sections of $\mathcal{T}M := TM \oplus T^*M$ was originally introduced by Courant to formulate the integrability condition defining a Dirac structure. However its nature became clear only due to the observation by Liu, Weinstein and Xu [LWX97] that $\mathcal{TM}$ endowed with the Courant bracket plays the role of a 'double' object, in the sense of Drinfeld [Dri86], for a pair of Lie algebroids over $M$. Whereas any Lie bialgebra has a double which is a Lie algebra, the double of a Lie bialgebroid is not a Lie algebroid, but a Courant algebroid – a generalization of $\mathcal{T}M$ equipped with the Courant bracket. There is another way of viewing *Courant algebroids* as a generalization of Lie algebroids. This requires a change in the definition of the Courant bracket and the use of an analog of the non-antisymmetric Dorfman bracket [Dor87]. The traditional Courant bracket then becomes the skew-symmetrization of the new one [Roy02]. This change replaces one defect with another: a version of the Jacobi identity is satisfied, while the bracket is no longer skew-symmetric. Such algebraic structures have been introduced by Loday [Lod93] under the name of Leibniz algebras. Canonical examples of Leibniz algebras arise often as derived brackets introduced by Kosmann-Schwarzbach [Kos96, Kos04]. Since Leibniz brackets appear naturally in Geometry and Physics in the form of 'algebroid brackets', i.e., brackets on sections of vector bundles, there were a number of attempts to formalize the concept of *Leibniz algebroid* [GV11, Gra03, GM01, ILMP99, Hag02, HM02, KS10, MM05, SX08, Wad02]. Note also that a Leibniz algebroid is the horizontal categorification of a Leibniz algebra; vertical categorification leads to Leibniz $n$-algebras and Leibniz $n$-algebroids [AP10, KMP11, KPQ14, DP12, BP12].

It is important to observe that, despite the sheaf-theoretic nature of classical Differential Geometry, most textbooks present it in terms of global sections and morphisms between them.

This global viewpoint is possible, since all morphisms between modules of sections are local – and even differential – operators in all their arguments (indeed, locality allows one to localize the operators in a way that they commute with restrictions). It follows that a map that is not a differential operator in one of its arguments is not a true geometric concept [BPP15, Appendix 3]. However, none of the aforementioned definitions of Leibniz algebroids requires any differentiability condition for the first argument of the bracket and thus *none* of these concepts is geometric. In [GKP13], the authors propose – under the name of *Loday algebroid* – a new notion of Leibniz algebroid, which *is* geometric and includes the vast majority of Leibniz brackets that can be found in the literature, in particular Courant brackets.

In the present article, we introduce the category of generalized Courant pseudoalgebras and show that it admits a free object on any anchored module over 'functions'. Our construction is based on another new concept: symmetric Leibniz algebroids and pseudoalgebras. We compare this subclass of Leibniz algebroids with the subclass made of Loday algebroids. The prototypical example of a generalized Courant pseudoalgebra is the one associated to an arbitrary symmetric Leibniz pseudoalgebra. Both notions are fundamental ingredients of the free generalized Courant pseudoalgebra. As an *application*, we show how the associated generalized Courant pseudoalgebra allows one to prove that any symmetric Leibniz pseudoalgebra bracket can be represented as a universal derived bracket. This formal-noncommutative-geometry-related problem, see Section 7, was one of the motivations for the present work. The possibility to encode numerous types of (homotopy) algebras in a (co)homological vector field of a possibly formal noncommutative manifold [GK94], is an example that emphasizes the importance of free algebras, see Equation (58). The prominence of the latter was recognized already in the late fifties, in particular by the Polish mathematical school. Also free Lie pseudoalgebras (free Lie-Rinehart algebras) were yet successfully used [Kap07]. The construction of the free Leibniz and Courant pseudoalgebras was a second inducement.

Let us emphasize that, whereas, as indicated above, Courant algebroids and, more generally, Loday algebroids are geometric, Leibniz algebroids, symmetric Leibniz algebroids, and generalized Courant algebroids are not. Hence, this paper should be viewed as a *purely algebro-categorical work*. From the categorical standpoint, most definitions are quite obvious. This holds in particular for the notions of free object and of morphism between generalized Courant pseudoalgebras – see Definitions 2 and 17.

The paper is organized as follows. In Section 2, we recall the definitions of the categories of Leibniz algebroids [ILMP99], of Leibniz pseudoalgebras – their algebraic counterpart –, and of modules over them, as well as the classical notion of Courant algebroid. We then describe, in Section 3, two intersecting subclasses of Leibniz algebroids, namely the class of Loday algebroids (Definition 11), which are Leibniz algebroids that admit a generalized right anchor, and the class of symmetric Leibniz algebroids (Definition 13), a new concept, made of Leibniz algebroids that satisfy two symmetry conditions and contain Courant algebroids as a particular example.

Examples of symmetric and nonsymmetric Leibniz and Loday algebroids and pseudoalgebras are given. In Section 4, we motivate the definition of generalized Courant pseudoalgebras (Definitions 15 and 16), which are specific symmetric Leibniz pseudoalgebras. The prototypical example of a generalized Courant pseudoalgebra is the one that is naturally associated to a symmetric Leibniz pseudoalgebra (Theorem 1). This associated Courant pseudoalgebra is one of the two ingredients of the free generalized Courant pseudoalgebra. Moreover, Theorem 1 allows to understand the origin of the definition of symmetric Leibniz pseudoalgebras. The second ingredient is the free symmetric Leibniz pseudoalgebra, which we construct in Section 5 (Theorem 2 and Proposition 5.1). In Section 6, we combine the results of Section 4 and Section 5 to build the free Courant pseudoalgebra (Theorem 3). Finally, we apply, in Section 7, our algebro-categorical constructions to show that any symmetric Leibniz pseudoalgebra bracket is a universal derived bracket implemented by a differential graded Lie algebra that is put up from the associated Courant pseudoalgebra.

# 2 Preliminaries

## 2.1 Notation and Conventions

Unless otherwise specified, manifolds are made of a finite-dimensional smooth structure on a second-countable Hausdorff space.

If $[-,-]$ is a Leibniz bracket, we denote by $-\circ-$ its symmetrization, i.e.,

$$X \circ Y := [X, Y] + [Y, X] \tag{1}$$

for any elements $X, Y$ of the Leibniz algebra.

## 2.2 Anchored Vector Bundles and Anchored Modules

**Definition 1.** *If $M$ is a manifold, an* anchored vector bundle *over $M$ is a vector bundle $E \to M$ with a vector bundle morphism $a\colon E \to TM$, called its* anchor*.*

*If $R$ is a commutative unital ring and $\mathcal{A}$ is a commutative unital $R$-algebra, an* anchored module *over $(R, \mathcal{A})$ is an $\mathcal{A}$-module $\mathcal{E}$ endowed with an* anchor*, i.e., an $\mathcal{A}$-module morphism $a\colon \mathcal{E} \to \mathrm{Der}\mathcal{A}$.*

Of course, here $TM$ is the tangent bundle of $M$ and $\mathrm{Der}\mathcal{A}$ is the $\mathcal{A}$-module of derivations of $\mathcal{A}$. If $a\colon E \to TM$ is an anchor, we still denote by $a\colon \Gamma E \to \Gamma TM = \mathrm{Der}(C^\infty(M))$ the corresponding $C^\infty(M)$-linear map between sections. Obviously, if $E$ is an anchored vector bundle over $M$ with anchor $a$, then its space $\Gamma E$ of sections is an anchored module over $(\mathbb{R}, C^\infty(M))$ with anchor $a$.

Morphisms of anchored vector bundles (resp., anchored modules) over a fixed base $M$ (resp., over a fixed algebra $\mathcal{A}$) are defined in the obvious way, and we obtain categories $\texttt{AncVec}(M)$

and $\texttt{AncMod}(\mathcal{A})$, respectively. The algebroids (resp., pseudoalgebras) we are going to define in this article will be anchored vector bundles (resp., anchored modules) with extra structure. They will form, together with their morphisms, categories that are concrete over $\texttt{AncVec}(M)$ and $\texttt{AncMod}(\mathcal{A})$, i.e., admit a (faithful) forgetful functor to the latter. One of our goals is to define left adjoints to these functors, or, in other words, to define the free algebroid (resp., pseudoalgebra) of a given type on a given anchored vector bundle (resp., anchored module). More generally,

**Definition 2.** *Let* $\texttt{C}$ *and* $\texttt{D}$ *be categories, such that there exists a forgetful functor* $\mathrm{For} : \texttt{C} \to \texttt{D}$*. For any* $D \in \texttt{D}$*, the corresponding* free object *in* $\texttt{C}$ *over* $D$ *is an object* $F \in \texttt{C}$ *equipped with a* $\texttt{D}$*-morphism* $i : D \to F$ *which is universal among all pairs of this type.*

For the different types of pseudoalgebras we are going to define, we will also define *modules* over them, using the following general principle: if $V$ is an $R$-module with extra structure, then a $V$*-module* is an $R$-module $W$ such that $V \oplus W$ is of the same type as $V$ and contains $V$ as a subobject and $W$ as an abelian ideal in an appropriate sense. Similarly, a morphism of modules from the $V$-module $W$ to the $V'$-module $W'$ will be a morphism $V \oplus W \to V' \oplus W'$ sending $V$ to $V'$ and $W$ to $W'$. It is possible to make these statements precise, but we prefer to keep them heuristic here and to work out the details below in the specific cases.

## 2.3 Leibniz Algebroids

In this paper, we consider *left Leibniz brackets*, i.e., bilinear brackets that satisfy the left Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \, . \tag{2}$$

Alternatively, one could work with *right Leibniz brackets*, which are defined similarly, except that one requires the right Jacobi identity

$$[[Y, Z], X] = [[Y, X], Z] + [Y, [Z, X]] \, . \tag{3}$$

We first recall the definition of a Leibniz algebroid given in [ILMP99]. Note that this notion of Leibniz algebroid does not impose any differentiability requirement on the first argument of the bracket and is thus not a geometric concept.

**Definition 3.** *A* Leibniz algebroid *is an anchored vector bundle* $E \to M$ *together with a Leibniz bracket* $[-, -]$ *on its space* $\Gamma E$ *of sections, which satisfy*

$$[X, fY] = f[X, Y] + a(X)(f)Y, \tag{4}$$

*for any* $f \in C^\infty(M)$ *and* $X, Y \in \Gamma E$*.*

It is easily checked that the Leibniz rule (4) and the Jacobi identity imply that $a$ is a Leibniz algebra morphism:

$$a[X, Y] = [a(X), a(Y)], \tag{5}$$

where the RHS bracket is the Lie bracket on $\Gamma TM$.

We will essentially deal with the algebraic counterpart of Leibniz algebroids:

**Definition 4.** *Let $R$ be commutative unital ring and let $\mathcal{A}$ be a commutative unital $R$-algebra. A* Leibniz pseudoalgebra *(or* Leibniz-Rinehart algebra*) over $(R, \mathcal{A})$ is an anchored module $(\mathcal{E}, a)$ over $(R, \mathcal{A})$ endowed with a Leibniz $R$-algebra structure $[-,-]$, such that, for all $f \in \mathcal{A}$ and $X, Y \in \mathcal{E}$,*

- $[X, fY] = f[X,Y] + a(X)(f)\,Y$ *and*
- $a[X,Y] = [a(X), a(Y)]$, *where the RHS is the commutator.*

If the $\mathcal{A}$-module $\mathcal{E}$ is faithful, the last requirement is again a consequence of the Leibniz rule and the Jacobi identity.

The space of sections of a Leibniz algebroid over $M$ is obviously a Leibniz pseudoalgebra over $(\mathbb{R}, C^\infty(M))$.

Of course, if, in Definitions 3 and 4, the Leibniz bracket is antisymmetric, we get a Lie algebroid and a Lie pseudoalgebra, respectively.

Leibniz algebroids over $M$ and Leibniz pseudoalgebras over $(R, \mathcal{A})$ are the objects of categories `LeiOid` $M$ and `LeiPsAlg` $(R, \mathcal{A})$. The morphisms of these categories are defined as follows.

**Definition 5.** *Let $(E_1, [-,-]_1, a_1)$ and $(E_2, [-,-]_2, a_2)$ be two Leibniz algebroids over a same manifold $M$. A* Leibniz algebroid morphism *between them is a bundle map $\phi : E_1 \to E_2$ such that $a_2\,\phi = a_1$ and $\phi[X,Y]_1 = [\phi X, \phi Y]_2$, for any $X, Y \in \Gamma E_1$.*

**Definition 6.** *Let $(\mathcal{E}_1, [-,-]_1, a_1)$ and $(\mathcal{E}_2, [-,-]_2, a_2)$ be two Leibniz pseudoalgebras over the same pair $(R, \mathcal{A})$. A* Leibniz pseudoalgebra morphism *between them is an $\mathcal{A}$-module morphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ such that $a_2\,\phi = a_1$ and $\phi[X,Y]_1 = [\phi X, \phi Y]_2$, for any $X, Y \in \mathcal{E}_1$.*

We now define (bi)modules over Leibniz algebroids and pseudoalgebras.

Recall first the definition of a (bi)module over a Leibniz $R$-algebra $(V, [-,-])$. By the general heuristic described above, this is an $R$-module $W$ with a Leibniz $R$-algebra structure $[-,-]$ on $V \oplus W$ containing $V$ as a subalgebra and $W$ as an abelian ideal. Therefore, this bracket has to be the original bracket on $V \times V$, and 0 on $W \times W$, so it is determined by the values of $[x,w] \in W$ and $[w,x] \in W$, where $x \in V$ and $w \in W$. Setting $\mu^l(x)(w) = [x,w]$ and $\mu^r(x)(w) = [w,x]$, we recover the usual notion: A *(bi)module* over a Leibniz $R$-algebra $(V, [-,-])$ is an $R$-module $W$ together with a left and a right *action* $\mu^l \in \mathrm{Hom}_R(V \otimes_R W, W)$ and $\mu^r \in \mathrm{Hom}_R(W \otimes_R V, W)$, which satisfy the following requirements

$$\mu^r[x,y] = \mu^r(y)\mu^r(x) + \mu^l(x)\mu^r(y)\ , \tag{6}$$

$$\mu^r[x,y] = \mu^l(x)\mu^r(y) - \mu^r(y)\mu^l(x)\ , \tag{7}$$

$$\mu^l[x,y] = \mu^l(x)\mu^l(y) - \mu^l(y)\mu^l(x)\ , \tag{8}$$

for all $x, y \in V$.

In particular, let $\nabla$ be a representation of $(V, [-,-])$ on $W$, i.e., a Leibniz $R$-algebra morphism $V \to \mathrm{End}_R W$. Then $\mu^\ell = \nabla$ and $\mu^r = -\nabla$ is a (bi)module structure over $V$ on $W$.

**Definition 7.** *Let $(E_1, [-,-], a)$ be a Leibniz algebroid over $M$. A* module over the Leibniz algebroid $E_1$ *is a $C^\infty(M)$-module $\mathcal{E}_2$, which is a module $(\mu^\ell, \mu^r)$ over the Leibniz $\mathbb{R}$-algebra $\Gamma E_1$ whose left action satisfies the Leibniz rule*

$$\mu^\ell(X)(fY) = f\mu^\ell(X)(Y) + a(X)(f)Y \ , \tag{9}$$

*for any $f \in C^\infty(M), X \in \Gamma E_1$, and $Y \in \mathcal{E}_2$.*

In this definition, the $C^\infty(M)$-module $\mathcal{E}_2$ is not required to define a locally free sheaf of modules over the function sheaf $C^\infty_M$ of $M$, i.e., it is not required to be a module of sections of a vector bundle. Moreover, just as for the Leibniz bracket on $\Gamma E_1$, we do not impose any differentiability condition on $\mu^r$.

Similarly,

**Definition 8.** *Let $(\mathcal{E}_1, [-,-], a)$ be a Leibniz pseudoalgebra over $(R, \mathcal{A})$. A* module over the Leibniz pseudoalgebra $\mathcal{E}_1$ *is an $\mathcal{A}$-module $\mathcal{E}_2$ (hence an $R$-module), which is a module over the Leibniz $R$-algebra $\mathcal{E}_1$ whose left action $\mu^\ell$ satisfies the Leibniz rule*

$$\mu^\ell(X)(fY) = f\mu^\ell(X)(Y) + a(X)(f)Y \ ,$$

*for any $f \in \mathcal{A}, X \in \mathcal{E}_1$, and $Y \in \mathcal{E}_2$.*

In the case $\mathcal{E}_1 = \Gamma E_1$ and $\mathcal{E}_2 = \Gamma E_2$, where $E_1$ and $E_2$ are vector bundles over $M$, and $(\mu^\ell, \mu^r) = (\nabla, -\nabla)$, where $\nabla$ is a representation of $\Gamma E_1$ on $\Gamma E_2$, we deal with an $\mathbb{R}$-bilinear map $\nabla : \Gamma E_1 \times \Gamma E_2 \to \Gamma E_2$ such that, for any $f \in C^\infty(M)$, $X, X_1, X_2 \in \Gamma E_1$, and $Y \in \Gamma E_2$,

$$\nabla_{[X_1, X_2]} = [\nabla_{X_1}, \nabla_{X_2}]$$

and

$$\nabla_X(fY) = a(X)(f)Y + f\nabla_X Y \ .$$

If $\nabla$ is in addition $C^\infty(M)$-linear in its first argument, the module structure is nothing but a flat $E_1$-connection on $E_2$. For a Lie algebroid $E_1$, we thus recover the classical concept of $E_1$-module.

Note also that, for any Leibniz algebroid $(E_1, [-,-], a)$, the $C^\infty(M)$-module $\mathcal{E}_2 = \Gamma(M \times \mathbb{R}) = C^\infty(M)$ and the actions $(\mu^\ell, \mu^r) = (a, -a)$ define on $C^\infty(M)$ a module structure over the Leibniz algebroid $E_1$.

Finally we define morphisms of modules over Leibniz pseudoalgebras. From the above heuristic, a morphism from the $\mathcal{E}_1$-module $\mathcal{E}_2$ to the $\mathcal{E}'_1$-module $\mathcal{E}'_2$ should be a Leibniz pseudoalgebra morphism from $\mathcal{E}_1 \oplus \mathcal{E}_2$ to $\mathcal{E}'_1 \oplus \mathcal{E}'_2$ sending $\mathcal{E}_1$ to $\mathcal{E}'_1$ and $\mathcal{E}_2$ to $\mathcal{E}'_2$. Unpacking this principle gives the following

**Definition 9.** *Let* $(\mathcal{E}_1, [-,-], a)$ *and* $(\mathcal{E}'_1, [-,-]', a')$ *be two Leibniz pseudoalgebras, and let* $(\mathcal{E}_2, \mu^\ell, \mu^r)$ *and* $(\mathcal{E}'_2, \mu'^\ell, \mu'^r)$ *be an* $\mathcal{E}_1$*-module and an* $\mathcal{E}'_1$*-module, respectively. A* morphism *between these two modules, is a pair* $(\phi_1, \phi_2)$ *made of a morphism* $\phi_1 : \mathcal{E}_1 \to \mathcal{E}'_1$ *of Leibniz pseudoalgebras and an* $\mathcal{A}$*-linear map* $\phi_2 : \mathcal{E}_2 \to \mathcal{E}'_2$*, such that*

$$\mu'^\ell(\phi_1 \times \phi_2) = \phi_2 \, \mu^\ell \;\; \textit{and} \;\; \mu'^r(\phi_2 \times \phi_1) = \phi_2 \, \mu^r \; . \tag{10}$$

## 2.4 Courant Algebroids

As for the definition of Courant algebroids, we refer the reader to [LWX97], [Roy99], [GM03], [Uch02], and [Kos05].

**Definition 10.** *A* Courant algebroid *is an anchored vector bundle* $E \to M$*, with anchor* $a$*, together with a Leibniz bracket* $[-,-]$ *on* $\Gamma E$ *and a bundle map* $(-|-)\colon E \otimes E \to M \times \mathbb{R}$ *that is in each fiber nondegenerate symmetric, called* scalar product*, which satisfy*

$$a(X)(Y|Y) = 2(X|[Y,Y]) \; , \tag{11}$$

$$a(X)(Y|Y) = 2([X,Y]|Y) \; , \tag{12}$$

*for any* $X, Y \in \Gamma E$*.*

The nondegeneracy of the scalar product allows us to identify $E$ with its dual $E^*$, and we will use this identification implicitly in the following. Note that (11) is equivalent to

$$a(X)(Y|Z) = (X|Y \circ Z) \; , \tag{13}$$

where $Y \circ Z$ denotes the symmetrized bracket. Similarly, (12) easily implies the invariance of the scalar product,

$$a(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]) \; , \tag{14}$$

which in turn shows that $a$ is the anchor of the left adjoint map:

$$[X, fY] = f[X,Y] + a(X)(f)Y \; . \tag{15}$$

Hence, a Courant algebroid is a particular Leibniz algebroid. When defining a derivation $D : C^\infty(M) \to \Gamma E$ by

$$(Df|X) = a(X)(f) \; , \tag{16}$$

we get out of (13) that

$$D(Y|Z) = Y \circ Z = [Y,Z] + [Z,Y] \; . \tag{17}$$

The fact that (17) is a consequence of the 'invariance' condition (13) and the nondegeneracy of the scalar product, will be of importance later on. Let us moreover stress that (17) implies a differentiability condition for the first argument of the Leibniz bracket:

$$[fX, Y] = f[X,Y] - a(Y)(f)X + (X|Y)Df \; . \tag{18}$$

It is now clear that

**Proposition 1.** *Courant algebroids* $(E, [-,-], (-|-), a)$ *are exactly the Leibniz algebroids* $(E, [-,-], a)$ *endowed with a scalar product* $(-|-)$*, such that, for any* $X, Y, Z \in \Gamma E$*,*

$$a(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]) \; , \tag{19}$$

$$a(X)(Y|Z) = (X|[Y,Z] + [Z,Y]) \; . \tag{20}$$

As already indicated above, we view the conditions (19) and (20), as well as their consequence

$$([X,Y]|Z) + (Y|[X,Z]) = (X|[Y,Z] + [Z,Y]) \; , \tag{21}$$

as the invariance properties of the scalar product. We will come back to this idea in Subsection 4.2.

# 3 Subclasses of Leibniz Algebroids

## 3.1 Loday Algebroids

In [GKP13], the authors observe that a Leibniz algebroid – in the sense of the present paper – is not a proper geometric concept. They suggest a new notion of Leibniz algebroid, called *Loday algebroid*, which has a right anchor satisfying a condition analogous to (18), and *is* therefore geometric. Moreover, they show that almost all 'Leibniz algebroids' met in the literature are Loday algebroids in their sense.

**Definition 11.** *A* Loday algebroid *is a Leibniz algebroid* $(E, [-,-], a)$ *equipped with a derivation*

$$\mathsf{D} : C^\infty(M) \to \mathrm{Hom}_{C^\infty(M)}(\Gamma(E^{\otimes 2}), \Gamma E) \; ,$$

*such that*

$$[fX, Y] = f[X,Y] - a(Y)(f)X + (\mathsf{D}f)(X,Y) \; , \tag{22}$$

*for any* $X, Y \in \Gamma E$ *and* $f \in C^\infty(M)$*.*

Let us mention that the right anchor $\mathsf{D}$ can be viewed as a bundle map $\mathsf{D} : E \to TM \otimes \mathrm{End}\, E$ (whereas the left anchor is a bundle map $a : E \to TM$). Its local form is

$$(\mathsf{D}f)(X^i e_i, Y^j e_j) = X^i \mathsf{D}_{ij}^{\ell k} \partial_k f \; Y^j e_\ell$$

(whereas the local form of $a$ is

$$a(X^i e_i)(f)Y = X^i a_i^k \partial_k f \; Y^\ell e_\ell \; ) \; .$$

**Example 1** ([GKP13], Section 5)**.** *Leibniz algebra brackets, Courant-Dorfman brackets, twisted Courant-Dorfman brackets, Courant algebroid brackets, brackets associated to contact structures, Grassmann-Dorfman brackets, Grassmann-Dorfman brackets for Lie algebroids, Lie derivative brackets for Lie algebroids, Leibniz algebroid brackets associated to Nambu-Poisson structures... are Loday algebroid brackets.*

For instance, it is clear from what was said above that, in the case of Courant algebroids, the derivation $\mathsf{D}$ is given by

$$\mathsf{D} : C^\infty(M) \times \Gamma E \times \Gamma E \ni (f, X, Y) \mapsto (X|Y)Df \in \Gamma E \ .$$

The algebraic version of Loday algebroids is defined as follows:

**Definition 12.** *A* Loday pseudoalgebra *is a Leibniz pseudoalgebra* $(\mathcal{E}, [-,-], a)$ *over* $(R, \mathcal{A})$ *equipped with a derivation*

$$\mathsf{D} : \mathcal{A} \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{E}) \ ,$$

*such that*

$$[fX, Y] = f[X, Y] - a(Y)(f)X + (\mathsf{D}f)(X, Y) \ . \tag{23}$$

## 3.2 Symmetric Leibniz Algebroids

We now introduce another subclass of Leibniz algebroids, *symmetric Leibniz algebroids*, which also contains Courant algebroids as a particular example. Let us briefly mention the origins of the next definition. The problem, when searching for a free (generalized) Courant algebroid (or, better, pseudoalgebra), or when trying to represent a Leibniz algebroid (pseudoalgebra) bracket by a derived bracket, is the absence of a differentiability condition on the first argument of the involved bracket. It turns out that both issues can be reduced to the two fundamental symmetry conditions (24) and (25).

**Definition 13.** *A* symmetric Leibniz algebroid *is a Leibniz algebroid* $(E, [-,-], a)$ *over* $M$*, such that, for any* $f \in C^\infty(M)$*,* $X, Y \in \Gamma E$*,*

$$X \circ fY - (fX) \circ Y = 0 \ \text{ and} \ , \tag{24}$$

$$[fX, Y \circ Z] - [X, Y] \circ fZ - (fY) \circ [X, Z] = 0 \ . \tag{25}$$

The definition can be equivalently formulated as follows:

**Proposition 2.** *A* symmetric Leibniz algebroid *is a Leibniz algebroid* $(E, [-,-], a)$ *over* $M$*, such that, for any* $f \in C^\infty(M)$*,* $X, Y \in \Gamma E$*,*

$$X \circ fY - (fX) \circ Y = 0 \ \text{ and} \ , \tag{26}$$

$$([fX, Y] - f[X, Y]) \circ Z + Y \circ ([fX, Z] - f[X, Z]) = 0 \ . \tag{27}$$

*Proof.* It suffices to show that, for a Leibniz algebroid that satisfies the first condition, the conditions (25) and (27) are equivalent. Note first that the Jacobi identity implies that $[X, Y \circ Z] = [X, Y] \circ Z + Y \circ [X, Z]$. It now follows that (25) is equivalent to

$$[fX, Y] \circ Z + Y \circ [fX, Z] = [fX, Y \circ Z]$$

$$= [X, Y] \circ fZ + (fY) \circ [X, Z] = (f[X, Y]) \circ Z + Y \circ f[X, Z] \ .$$

□

The first condition (26) means that the symmetrized bracket is $C^\infty(M)$-linear between the two variables $X, Y$. The second condition (27) is a $C^\infty(M)$-linearity condition in a combination of symmetrized products.

**Proposition 3.** *A Loday algebroid $(E, [-,-], a, \mathsf{D})$ over a manifold $M$ is a symmetric Leibniz algebroid if and only if, for all $f \in C^\infty(M)$ and all $X, Y, Z \in \Gamma E$,*

$$(\mathsf{D}f)(X, Y) = (\mathsf{D}f)(Y, X) \quad \textit{and} \tag{28}$$

$$(\mathsf{D}f)(X, [Y, Z] + [Z, Y]) = (\mathsf{D}f)([X, Y], Z) + (\mathsf{D}f)(Y, [X, Z]) \; . \tag{29}$$

*Proof.* It follows from the differentiability properties (4) and (23), and from the antisymmetry $a[X, Y] = -a[Y, X]$ of the bracket of vector fields, that the $C^\infty(M)$-linearity conditions (24) and (25) are equivalent to (28) and (29). □

**Example 2.** *Leibniz algebra brackets, Courant-Dorfman brackets, twisted Courant-Dorfman brackets, Courant algebroid brackets, brackets associated to contact structures, Grassmann-Dorfman brackets, Grassmann-Dorfman brackets for Lie algebroids... are Loday algebroid and symmetric Leibniz algebroid brackets.*

Indeed, since for a Courant algebroid $(E, [-,-], (-|-), a)$ over $M$, with derivation $D : C^\infty(M) \to \Gamma E$, we have $\mathsf{D} = 2(-|-)D$, the $C^\infty(M)$-linearity conditions (28) and (29) are direct consequences of the symmetry and the invariance properties of the scalar product. For definitions concerning the other examples, we refer the reader to [GKP13], Section 5.

**Example 3.** *Lie derivative brackets for Lie algebroids, Leibniz algebroid brackets associated to Nambu-Poisson structures... are Loday algebroid but (usually) nonsymmetric Leibniz algebroid brackets.*

We examine the first example. Let $(E, [-,-]_E, a_E)$ be a Lie algebroid, let $d^E : \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^{\bullet+1} E^*)$ be the Lie algebroid differential, and denote by

$$\mathcal{L}^E : \Gamma E \times \Gamma(\wedge^\bullet E^*) \ni (X, \omega) \mapsto i_X d^E \omega + d^E i_X \omega \in \Gamma(\wedge^\bullet E^*)$$

the Lie algebroid Lie derivative, where $i_X$ is the interior product. There is a Leibniz bracket on sections of the vector bundle $E \oplus \wedge E^*$. Indeed, set, for any $X, Y \in \Gamma E$ and any $\omega, \eta \in \Gamma(\wedge E^*)$,

$$[X + \omega, Y + \eta] = [X, Y]_E + \mathcal{L}^E_X \eta \; . \tag{30}$$

This is a Loday algebroid bracket with left anchor $a(X + \omega) = a_E(X)$ and right anchor

$$(\mathsf{D}h)(X + \omega, Y + \eta) = d^E h \wedge i_Y \omega + a_E(X)(h)\eta \; ,$$

see [GKP13], Section 5.

If this Loday algebroid $E \oplus \wedge E^*$ is symmetric, Condition (28) is satisfied in particular for 0-forms, i.e., we have

$$a_E(X)(h)g = a_E(Y)(h)f \ ,$$

for any $f, g, h \in C^\infty(M)$ and any $X, Y \in \Gamma E$. If we choose $f = 0$ and $g = 1$, we find that $a_E = 0$, so that the considered Lie algebroid $E$ is a Lie algebra bundle (LAB). Conversely, if $E$ is a LAB, we get $(Dh)(X+\omega, Y+\eta) = 0$, since $(d^E h)(-) = a_E(-)(h) = 0$; hence, the conditions (28) and (29) are satisfied and $E \oplus \wedge E^*$ is a symmetric Leibniz algebroid.

Let us close this section with the definition of symmetric Leibniz pseudoalgebras:

**Definition 14.** *A* symmetric Leibniz pseudoalgebra *is a Leibniz pseudoalgebra* $(\mathcal{E}, [-,\, -], a)$ *over some* $(R, \mathcal{A})$*, such that (24) and (25), or, equivalently, (26) and (27), are satisfied for all* $f \in \mathcal{A}$ *and all* $X, Y, Z \in \mathcal{E}$*. We denote by* `SymLeiPsAlg` $(R, \mathcal{A}) \subset$ `LeiPsAlg`$(R, \mathcal{A})$ *the full subcategory of symmetric Leibniz pseudoalgebras.*

**Example 4.** *An example of a symmetric Leibniz pseudoalgebra is the free symmetric Leibniz pseudoalgebra over an anchored module that we describe in Theorem 2 and in Proposition 5.1. Observe that this symmetric Leibniz bracket is not Loday.*

# 4 Generalized Courant Pseudoalgebras

## 4.1 Generalized Courant Pseudoalgebras

To motivate the definition of generalized Courant pseudoalgebras, we sketch a natural approach to the free 'Courant pseudoalgebra'. It seems clear that we should start from the free Leibniz algebra $(\overline{T}\mathcal{E}, [-,-])$ over an $R$-module $\mathcal{E}$ [LP93]. The anchor, say $\mathrm{F}a$, on $\mathcal{F} := \overline{T}\mathcal{E}$ 'must be' implemented by a given anchor $a$ on $\mathcal{E}$ – which has thus to be an anchored $\mathcal{A}$-module. It can be shown that the triple $(\mathcal{F}, [-,-], \mathrm{F}a)$ is the free Leibniz pseudoalgebra over $(\mathcal{E}, a)$. The free 'Courant pseudoalgebra' over $(\mathcal{E}, a)$ should now be obtained by completing the preceding free Leibniz pseudoalgebra by a 'scalar product' $(-|-)$. To get an idea of the latter, recall that the free 'Courant pseudoalgebra' is universal, and consider a classical Courant pseudoalgebra $(\mathcal{E}_0, [-,-]_0, (-|-)_0, a_0)$ over a manifold $M$, together with an anchored $\mathcal{A}$-module map $f : \mathcal{E} \to \mathcal{E}_0$.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ i\ } & \mathcal{F} \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathcal{E}_0 \end{array} \tag{31}$$

The 'Courant pseudoalgebra' 'morphism' $f_1 : \mathcal{F} \to \mathcal{E}_0$ can easily be handled, hence, we do not insist on it here. On the other hand, the searched universal 'scalar product' $(-|-)$ on $\mathcal{F}$ should of course satisfy a condition of the type $(\mathfrak{m}|\mathfrak{m}') = (f_1(\mathfrak{m})|f_1(\mathfrak{m}'))_0 \in C^\infty(M)$, for any $\mathfrak{m}, \mathfrak{m}' \in \mathcal{F}$.

The RHS of this condition is visibly defined on $\mathcal{F}\odot\mathcal{F}$, where $\odot$ is the symmetric tensor product over $\mathcal{A}$, but it does not provide a 'universal product' $(-|-)$. The way out is to compose the map $f_2 : \mathcal{F}\odot\mathcal{F} \to C^\infty(M)$, defined by $f_2(\mathfrak{m}\odot\mathfrak{m}') = (f_1(\mathfrak{m})|f_1(\mathfrak{m}'))_0$, with the 'universal scalar product' $(-|-) : \mathcal{F}\times\mathcal{F}\to\mathcal{F}\odot\mathcal{F}$, given by

$$(\mathfrak{m}|\mathfrak{m}') = \mathfrak{m}\odot\mathfrak{m}' \ . \tag{32}$$

The 'Courant pseudoalgebra' 'morphism' $(f_1, f_2)$ then respects the 'metrics' $(-|-)$ and $(-|-)_0$. To make sure that the 'universal scalar product' satisfies the compatibility condition (21), it is natural to replace the 'product' (32) by the 'product' (we use the same notation as before) defined by

$$(-|-) : \mathcal{F}\times\mathcal{F}\to\mathcal{R}(\mathcal{F}) := \mathcal{F}\odot\mathcal{F}/\langle[\mathfrak{m},\mathfrak{m}']\odot\mathfrak{m}''+\mathfrak{m}'\odot[\mathfrak{m},\mathfrak{m}'']-\mathfrak{m}\odot([\mathfrak{m}',\mathfrak{m}'']+[\mathfrak{m}'',\mathfrak{m}'])\rangle \ . \tag{33}$$

In view of the invariance conditions (19) and (20) in a classical Courant pseudoalgebra, the target $\mathcal{R}(\mathcal{F})$ must be a (bi)module over $\mathcal{F}$. The definitions of $\mu^\ell$ and $\mu^r$ are clear. Hence, the candidate

$$(\mathcal{F}, \mathcal{R}(\mathcal{F}), [-,-], (-|-), \mathrm{F}a, \mu^\ell, \mu^r) \tag{34}$$

for the free generalized Courant pseudoalgebra, as well as the definition of 'generalized'. When trying to prove that the actions $\mu^\ell$ and $\mu^r$ are well-defined on the symmetric tensor product, we discover the first symmetry condition (24) for $\mathcal{F}$. When attempting to show that they are well-defined on the quotient $\mathcal{R}(\mathcal{F})$, one finds the second symmetry condition (25) for $\mathcal{F}$. It then suffices to force these properties in $\mathcal{F}$, i.e., to pass again to the quotient.

Equation (34) explains the following

**Definition 15.** *Let $(\mathcal{E}_1, [-,-], a)$ be a Leibniz pseudoalgebra over $(R,\mathcal{A})$ and let $(\mathcal{E}_2, \mu^\ell, \mu^r)$ be an $\mathcal{E}_1$-module. Assume further that $(-|-) : \mathcal{E}_1\times\mathcal{E}_1\to\mathcal{E}_2$ is a symmetric $\mathcal{A}$-bilinear $\mathcal{E}_2$-valued map, such that, for any $X, Y, Z\in\mathcal{E}_1$, the 'invariance relations'*

$$\mu^\ell(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]) \ , \tag{35}$$

$$-\mu^r(X)(Y|Z) = ([Y,Z]+[Z,Y]|X) \ \ \textit{and} \tag{36}$$

$$([X,Y]|Z) + (Y|[X,Z]) = ([Y,Z]+[Z,Y]|X) \tag{37}$$

*hold true. We refer to such a tuple*

$$(\mathcal{E}_1, \mathcal{E}_2, [-,-], (-|-), a, \mu^\ell, \mu^r)$$

*as a* generalized pre-Courant pseudoalgebra.

**Remark 1.** *In the geometric situation $\mathcal{E}_1 = \Gamma E_1$, where $E_1\to M$ is a vector bundle over a manifold, $R=\mathbb{R}$ and $\mathcal{A} = C^\infty(M)$, we can take $(\mathcal{E}_2, \mu^\ell, \mu^r) = (C^\infty(M), a, -a)$, which is actually an $E_1$-module. If we now assume in addition that $(-|-)$ is nondegenerate, the generalized pre-Courant pseudoalgebra is a classical Courant algebroid $(E_1, [-,-], (-|-), a)$.*

**Proposition 4.** *Any generalized pre-Courant pseudoalgebra with nondegenerate scalar product* $(-|-)$ *is a Loday pseudoalgebra and a symmetric Leibniz pseudoalgebra.*

It is clear that, for any $X \in \mathcal{E}_1$, we have $(X|-) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_1, \mathcal{E}_2)$. By nondegenerate scalar product, we mean here that any $\Delta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_1, \mathcal{E}_2)$ reads $\Delta = (X|-)$, and that the map $Y \mapsto (Y|-)$ is injective, so that $X$ is unique. Indeed, in the aforementioned geometric case, nondegeneracy in each fiber, see Definition 10, implies these requirements.

*Proof.* We use the above notation; in particular $f \in \mathcal{A}$ and $X, Y, Z \in \mathcal{E}_1$. In view of the invariance relations and Leibniz rule for the left action, we get

$$-\mu^r(X)(f(Y|Z)) = f([Y,Z] + [Z,Y]|X) + a(X)(f)(Y|Z) . \tag{38}$$

On the other hand, the Leibniz rule for the bracket $[-,-]$ gives

$$-\mu^r(X)(fY|Z) = ([fY,Z] + [Z,fY]|X) = ([fY,Z]|X) + (f[Z,Y] + a(Z)(f)Y|X) . \tag{39}$$

Note now that $a(-)(f)(Y|Z) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_1, \mathcal{E}_2)$, so that there is a unique $(\mathsf{D}f)(Y,Z) \in \mathcal{E}_1$ such that

$$((\mathsf{D}f)(Y,Z)|-) = a(-)(f)(Y|Z) . \tag{40}$$

The properties of the anchor and the scalar product imply that $\mathsf{D}$ is a derivation

$$\mathsf{D} : \mathcal{A} \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_1, \mathcal{E}_1) .$$

It now follows from (38), (39), (40), and nondegeneracy that

$$[fY,Z] = f[Y,Z] - a(Z)(f)Y + (\mathsf{D}f)(Y,Z) ,$$

i.e., that $(\mathcal{E}_1, [-,-], a)$ is a Loday pseudoalgebra.

Furthermore, the latter is a symmetric Leibniz pseudoalgebra if and only if the conditions (28) and (29) are satisfied – the proof in the geometric situation remains valid in the present algebraic case –. It is easily seen that these requirements are fulfilled due to the invariance relation (37). □

A priori it might seem natural to define generalized Courant pseudoalgebras as generalized pre-Courant pseudoalgebras endowed with a nondegenerate scalar product $(-|-)$. In view of the candidate (34), (33), we choose however the following more general (see Proposition 4) definition:

**Definition 16.** *A* generalized Courant pseudoalgebra *is a symmetric generalized pre-Courant pseudoalgebra.*

Generalized Courant pseudoalgebras extend the canonical algebraic counterpart of Courant algebroids, which we called Courant pseudoalgebras above. A variant of Courant pseudoalgebras has been introduced and studied by D. Roytenberg under the name of Courant-Dorfman algebras [Roy09].

Whereas Courant algebroids are Leibniz algebroids endowed with a scalar product, such that some invariance conditions are satisfied, generalized Courant pseudoalgebras are symmetric Leibniz pseudoalgebras that are endowed with a 'scalar product' (a symmetric $\mathcal{A}$-bilinear map) valued in a 'representation' (a module over the Leibniz pseudoalgebra) and satisfy similar invariance conditions. In other words, the representation $(C^\infty(M), a, -a)$ is replaced by a 'representation' $(\mathcal{E}_2, \mu^\ell, \mu^r)$, the $C^\infty(M)$-valued scalar product is replaced by an $\mathcal{E}_2$-valued 'scalar product', and the symmetry of the Leibniz pseudoalgebra substitutes for the nondegeneracy of the classical scalar product.

**Example 5.** *In particular, a generalized Courant pseudoalgebra over a point, is a Leibniz algebra $(L, [-,-])$ over $\mathbb{R}$ (in the geometric context $R = \mathbb{R}$ and $\mathcal{A} = C^\infty(M)$, and, if $M = \{\mathrm{pt}\}$, then $\mathcal{A} = \mathbb{R}$ – the symmetry condition is then a consequence of $R$-linearity), together with a symmetric $\mathbb{R}$-bilinear map $(-|-)$ with values in a vector space $V$ over $\mathbb{R}$ endowed with a module structure over $L$, such that the usual invariance conditions hold. If the actions $\mu^\ell, \mu^r$ of $L$ on $V$ are opposite, the $L$-module structure is a representation $(V, \rho)$ and the invariance conditions reduce to*

$$\rho(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]) = (X|Y \circ Z) .$$

*In the case of the trivial representation $(\mathbb{R}, 0)$ and a nondegenerate map $(-|-)$, we get a Lie $\mathbb{R}$-algebra equipped with an invariant scalar product, i.e., a quadratic Lie algebra, or, still, a Courant algebroid over a point.*

Generalized Courant pseudoalgebras are a full subcategory `CrtPsAlg` of the category `PrCrtPsAlg` of generalized pre-Courant pseudoalgebras.

**Definition 17.** *A* morphism *between two generalized (pre-)Courant pseudoalgebras*

$$(\mathcal{E}_1, \mathcal{E}_2, [-,-], (-|-), a, \mu^\ell, \mu^r) \;\; \text{and} \;\; (\mathcal{E}_1', \mathcal{E}_2', [-,-]', (-|-)', a', \mu'^\ell, \mu'^r)$$

*over the same pair $(R, \mathcal{A})$, is a morphism $(\phi_1, \phi_2)$ from the $\mathcal{E}_1$-module $\mathcal{E}_2$ to the $\mathcal{E}_1'$-module $\mathcal{E}_2'$, such that*

$$(-|-)' \, (\phi_1 \times \phi_1) = \phi_2 \, (-|-) . \tag{41}$$

**Remark 2.** The preceding definition of morphisms between Courant pseudoalgebras (certain bracket and 'metric' respecting module morphisms) is different from Courant morphisms as defined in [BIS09], [AX01] (certain Dirac structures). Indeed, there are Courant morphisms that do not correspond to module morphisms. However, Definition 17 is completely natural from the categorical point of view adopted in the present algebro-categorical text. Moreover, only the chosen definition leads to the forgetful functor from `CrtPsAlg` to `AncMod` that is needed for the free Courant pseudoalgebra that we study later on.

### 4.2 Generalized Courant Pseudoalgebra Associated to a Symmetric Leibniz Pseudoalgebra

The next theorem describes this generalized Courant pseudoalgebra, which is actually the prototypical example. It is also the motivation for the introduction of symmetric Leibniz algebroids. Moreover, it will turn out that the generalized Courant pseudoalgebra associated to a symmetric Leibniz pseudoalgebra is one of the two components of the free Courant pseudoalgebra – see Subsection 4.1, introductory remark.

**Theorem 1.** *Let $(\mathcal{E}, [-,-], a)$ be a symmetric Leibniz pseudoalgebra over $(R, \mathcal{A})$. Denote by $\odot$ the symmetric tensor product over $\mathcal{A}$, take the subset*

$$\mathrm{Inv} = \{[X,Y] \odot Z + Y \odot [X,Z] - X \odot ([Y,Z] + [Z,Y]) : X, Y, Z \in \mathcal{E}\} \tag{42}$$

*of the $\mathcal{A}$-module $\mathcal{E}^{\odot 2}$, and let $\langle \mathrm{Inv} \rangle$ be the $\mathcal{A}$-submodule of $\mathcal{E}^{\odot 2}$ generated by* Inv. *The quotient*

$$\mathcal{R}(\mathcal{E}) = \mathcal{E}^{\odot 2} / \langle \mathrm{Inv} \rangle$$

*is an $\mathcal{E}$-module with actions $\tilde{\mu}^\ell$ and $\tilde{\mu}^r$ induced by*

$$\mu^\ell(X)(Y_1 \odot Y_2) = [X, Y_1] \odot Y_2 + Y_1 \odot [X, Y_2] \tag{43}$$

*and*

$$-\mu^r(X)(Y_1 \odot Y_2) = (Y_1 \circ Y_2) \odot X \ . \tag{44}$$

*These data, together with the* universal scalar product

$$(-|-) : \mathcal{E} \times \mathcal{E} \ni (X, Y) \mapsto (X \odot Y)^{\sim} \in \mathcal{R}(\mathcal{E}) \ , \tag{45}$$

*define a generalized Courant pseudoalgebra*

$$\mathcal{C}(\mathcal{E}) := (\mathcal{E}, \mathcal{R}(\mathcal{E}), [-,-], (-|-), a, \tilde{\mu}^\ell, \tilde{\mu}^r) \ . \tag{46}$$

**Remark 3.** *The associated generalized Courant pseudoalgebra is a very natural construction over a symmetric Leibniz pseudoalgebra, whose scalar product is the universal scalar product given by the symmetric tensor product and whose actions are the 'invariant' Courant actions.*

**Example 6.** *All the examples of symmetric Leibniz brackets described in Example 2 and Example 4 thus give rise to generalized Courant pseudoalgebras. For instance, if we pass from a classical Courant algebroid $(\mathcal{E}, [-,-]_{\mathrm{C}}, (-,-)_{\mathrm{C}}, a)$ ( $\mathcal{E}$ is the module of sections of a vector bundle $E \to M$ ) to its associated generalized Courant pseudoalgebra, we replace the original scalar product $(-,-)_{\mathrm{C}}$ valued in the $\mathcal{E}$-module $C^\infty(M)$ by the universal 'scalar product' $\odot^{\sim}$ valued in the $\mathcal{E}$-module $\mathcal{E}^{\odot 2}/\langle \mathrm{Inv} \rangle$, where the quotient forces the invariance of this 'scalar product' ( i.e., of the symmetric tensor product $\odot$ ) with respect to the Leibniz bracket $[-,-]_{\mathrm{C}}$ ( i.e., forces a certain sum of tensor products to vanish ).*

Let us now come to the proof of Theorem 1. We first examine the following

**Lemma 1.** *The $\mathcal{A}$-module $\mathcal{E}^{\odot 2}$ is an $\mathcal{E}$-module for the actions $\mu^\ell$ and $\mu^r$.*

*Proof.* (i) We first show that $\mu^\ell(X)$ and $\mu^r(X)$ are well-defined on $\mathcal{E}^{\odot 2}$ (note that we do of course not intend to show that they are $\mathcal{A}$-linear on $\mathcal{E}^{\odot 2}$; indeed, they are visibly only $R$-linear). Since the RHSs of (43) and (44) are symmetric in $Y_1, Y_2$, it suffices to prove that they respect the 'defining relations' of the tensor product over $\mathcal{A}$. The only nonobvious condition is that $Y_1 \odot fY_2 = (fY_1) \odot Y_2$ be preserved. And indeed, we have

$$\begin{aligned}
\mu^\ell(X)(Y_1 \odot fY_2) &= [X, Y_1] \odot fY_2 + Y_1 \odot [X, fY_2] \\
&= [X, Y_1] \odot fY_2 + Y_1 \odot f[X, Y_2] + Y_1 \odot a(X)(f)Y_2 \\
&= (f[X, Y_1]) \odot Y_2 + (fY_1) \odot [X, Y_2] + (a(X)(f)Y_1) \odot Y_2 \\
&= [X, fY_1] \odot Y_2 + (fY_1) \odot [X, Y_2] \\
&= \mu^\ell(X)((fY_1) \odot Y_2)
\end{aligned}$$

and

$$\begin{aligned}
\mu^r(X)(Y_1 \odot fY_2) &= -(Y_1 \circ fY_2) \odot X \\
&= -((fY_1) \circ Y_2) \odot X \\
&= \mu^r(X)((fY_1) \odot Y_2) \; .
\end{aligned}$$

(ii) It remains to check the 'Leibniz morphism conditions' (6), (7), and (8), as well as the Leibniz rule (9). The Leibniz rule

$$\mu^\ell(X)(Y_1 \odot fY_2) = f\mu^\ell(X)(Y_1 \odot Y_2) + a(X)(f)(Y_1 \odot Y_2)$$

is clear from (i). The morphism conditions are also straightforwardly checked. To verify for instance

$$\mu^r[X, Z] = \mu^r(Z)\mu^r(X) + \mu^\ell(X)\mu^r(Z) \; ,$$

note first that the right adjoint action $[-, X]$ on a symmetrized product vanishes:

$$[[Y_1, Y_2] + [Y_2, Y_1], X] = [Y_1, [Y_2, X]] - [Y_2, [Y_1, X]] + [Y_2, [Y_1, X]] - [Y_1, [Y_2, X]] = 0 \; .$$

We now get

$$\mu^r[X, Z](Y_1 \odot Y_2) = -(Y_1 \circ Y_2) \odot [X, Z] \; ,$$

$$\begin{aligned}
\mu^r(Z)\mu^r(X)(Y_1 \odot Y_2) &= -\mu^r(Z)((Y_1 \circ Y_2) \odot X) \\
&= ((Y_1 \circ Y_2) \circ X) \odot Z \\
&= ([Y_1 \circ Y_2, X] + [X, Y_1 \circ Y_2]) \odot Z \\
&= [X, Y_1 \circ Y_2] \odot Z \; ,
\end{aligned}$$

and

$$\begin{aligned}\mu^\ell(X)\mu^r(Z)(Y_1\odot Y_2) &= -\mu^\ell(X)((Y_1\circ Y_2)\odot Z)\\ &= -[X, Y_1\circ Y_2]\odot Z-(Y_1\circ Y_2)\odot[X,Z]\ .\end{aligned}$$

Hence, the result. □

The symmetric $\mathcal{A}$-bilinear map

$$\langle -|-\rangle : \mathcal{E}^{\times 2}\ni (X,Y)\mapsto X\odot Y\in\mathcal{E}^{\odot 2}$$

satisfies

$$\mu^\ell(X)\langle Y|Z\rangle = \langle [X,Y]|Z\rangle + \langle Y|[X,Z]\rangle \tag{47}$$

and

$$-\mu^r(X)\langle Y|Z\rangle = \langle X|[Y,Z]+[Z,Y]\rangle\ , \tag{48}$$

which are similar to (19) and (20). Since (21) does however not hold in general, we consider the quotient $\mathcal{A}$-module

$$\mathcal{R}(\mathcal{E}) = \mathcal{E}^{\odot 2}/\langle \mathrm{Inv}\rangle\ .$$

**Lemma 2.** *The $\mathcal{A}$-module $\mathcal{R}(\mathcal{E})$ is an $\mathcal{E}$-module for the actions $\tilde{\mu}^\ell$ and $\tilde{\mu}^r$ induced by $\mu^\ell$ and $\mu^r$.*

*Proof.* It suffices to show that the actions descend to the quotient; indeed, the induced maps then inherit the required properties.

(i) Left action. Let $I(X,Y,Z)$, or just $I$, be any element in $\mathrm{Inv}\subset\mathcal{E}^{\odot 2}$, and let $f\in\mathcal{A}$ and $W\in\mathcal{E}$. Since

$$\mu^\ell(W)(fI) = f\mu^\ell(W)(I) + a(W)(f)I\ ,$$

we have $\mu^\ell(W)\langle\mathrm{Inv}\rangle\subset\langle\mathrm{Inv}\rangle$, if $\mu^\ell(W)\,\mathrm{Inv}\subset\langle\mathrm{Inv}\rangle$. The latter actually holds true:

$$\begin{gathered}\mu^\ell(W)I(X,Y,Z)\\ =[W,[X,Y]]\odot Z+[X,Y]\odot[W,Z]+[W,Y]\odot[X,Z]+Y\odot[W,[X,Z]]\\ -[W,X]\odot(Y\circ Z)-X\odot([W,Y]\circ Z)-X\odot(Y\circ[W,Z])\\ =[[W,X],Y]\odot Z+[X,[W,Y]]\odot Z+[X,Y]\odot[W,Z]+[W,Y]\odot[X,Z]\\ +Y\odot[[W,X],Z]+Y\odot[X,[W,Z]]-[W,X]\odot(Y\circ Z)-X\odot([W,Y]\circ Z)-X\odot(Y\circ[W,Z])\\ =I([W,X],Y,Z)+I(X,[W,Y],Z)+I(X,Y,[W,Z])\ .\end{gathered}$$

(ii) Right action. In view of the annihilation of symmetrized products by right adjoint actions and due to the symmetry condition (25),

$$[fX, Y\circ Z] = [X,Y]\circ fZ + (fY)\circ[X,Z]\ ,$$

we get

$$\begin{aligned}&\mu^r(W)(fI(X,Y,Z))\\ &= \mu^r(W)\left([X,Y]\odot fZ + (fY)\odot [X,Z] - (fX)\odot (Y\circ Z)\right)\\ &= -([X,Y]\circ fZ + (fY)\circ [X,Z] - [fX, Y\circ Z] - [Y\circ Z, fX])\odot W\\ &= 0\ .\end{aligned}$$

□

It follows from (47) and (48) that $(\mathcal{E}, [-,-], a)$, $(\mathcal{R}(\mathcal{E}), \tilde{\mu}^\ell, \tilde{\mu}^r)$, and the symmetric $\mathcal{A}$-bilinear 'universal scalar product'

$$(-|-) : \mathcal{E}^{\times 2} \ni (X,Y) \mapsto \langle X|Y\rangle^{\sim} \in \mathcal{R}(\mathcal{E})$$

define a generalized Courant pseudoalgebra. This completes the proof of Theorem 1.

# 5 Free Symmetric Leibniz Pseudoalgebra

The free symmetric Leibniz pseudoalgebra is the second ingredient needed for the construction of the free Courant pseudoalgebra.

## 5.1 Leibniz Pseudoalgebra Ideals

**Definition 18.** *Let $(\mathcal{E}, [-,-], a)$ be a Leibniz pseudoalgebra over $(R, \mathcal{A})$. A* Leibniz pseudoalgebra ideal *is an $\mathcal{A}$-submodule $\mathcal{I} \subset \mathcal{E}$, which is a two-sided Leibniz $R$-algebra ideal, i.e., $[\mathcal{I}, \mathcal{E}] \subset \mathcal{I}$ and $[\mathcal{E}, \mathcal{I}] \subset \mathcal{I}$, and which is contained in the kernel of the anchor, i.e., $\mathcal{I} \subset \ker a$.*

**Proposition 5.** *The quotient of a Leibniz pseudoalgebra by a Leibniz pseudoalgebra ideal is a Leibniz pseudoalgebra for the induced bracket and anchor.*

*Proof.* Obvious. □

**Lemma 3.** *Let $(\mathcal{E}, [-,-], a)$ be a Leibniz pseudoalgebra over $(R, \mathcal{A})$, let*

$$J_1 = \{X \circ fY - (fX)\circ Y : f \in \mathcal{A},\ X, Y \in \mathcal{E}\} \ \ \textit{and} \tag{49}$$

$$J_2 = \{[fX, Y\circ Z] - [X,Y]\circ fZ - (fY)\circ [X,Z] : f \in \mathcal{A},\ X,Y,Z \in \mathcal{E}\}\ , \tag{50}$$

*and denote by $\langle J_1\rangle$ ( resp., $\langle J_2\rangle$ ) the $\mathcal{A}$-module generated by $J_1$ ( resp., $J_2$ ). The $\mathcal{A}$-module $(J_1 + J_2) := \langle J_1\rangle + \langle J_2\rangle$ is an ideal of the Leibniz pseudoalgebra $\mathcal{E}$, so that the quotient $\mathcal{E}/(J_1+J_2)$ inherits a symmetric Leibniz pseudoalgebra structure.*

*Proof.* The left adjoint action $[W, -]$, $W \in \mathcal{E}$, satisfies the Leibniz rule with respect to the Leibniz bracket $(X, Y) \mapsto [X, Y]$, the symmetrized bracket $(X, Y) \mapsto X \circ Y$, and the $\mathcal{A}$-module structure $(f, X) \mapsto fX$. It follows that

$$[W, X \circ fY - (fX) \circ Y] =$$

$$[W, X] \circ fY + X \circ f[W, Y] + X \circ\, a(W)(f)Y -$$

$$(f[W, X]) \circ Y -\, (a(W)(f)X) \circ Y - (fX) \circ [W, Y] \in \langle J_1 \rangle\ ,$$

and similarly for $J_2$.

As for the right action $[-, W]$, $W \in \mathcal{E}$, recall that it vanishes on every symmetrized bracket. Since the first term $[fX, Y \circ Z] = [fX, Y] \circ Z + Y \circ [fX, Z]$ of an element of $J_2$ is symmetric as well, the sets $J_1$ and $J_2$ vanish under the right action.

For any $f \in \mathcal{A}$, $W \in \mathcal{E}$, and $Q \in J_1$, we have now

$$Q \circ fW - (fQ) \circ W = [Q, fW] + [fW, Q] - [fQ, W] - [W, fQ] \in J_1\ ,$$

with $[Q, fW] = 0$, $[fW, Q] \in \langle J_1 \rangle$, and

$$[W, fQ] = f[W, Q] + a(W)(f)Q \in \langle J_1 \rangle\ . \tag{51}$$

Hence,

$$[fQ, W] \in \langle J_1 \rangle\ . \tag{52}$$

Similarly, if $Q \in J_2$,

$$W \circ fQ - (fW) \circ Q = [W, fQ] + [fQ, W] - [fW, Q] - [Q, fW] \in J_1\ ,$$

with

$$[W, fQ] = f[W, Q] + a(W)(f)Q \in \langle J_2 \rangle\ , \tag{53}$$

$[fW, Q] \in \langle J_2 \rangle$, and $[Q, fW] = 0$. Therefore,

$$[fQ, W] \in J_1 + \langle J_2 \rangle\ . \tag{54}$$

Equations (51), (52), (53), and (54) imply that the $\mathcal{A}$-submodule $\langle J_1 \rangle + \langle J_2 \rangle \subset \mathcal{E}$ is a Leibniz $R$-algebra ideal.

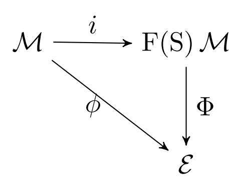
To see that $\langle J_1 \rangle + \langle J_2 \rangle$ is a Leibniz pseudoalgebra ideal, it now suffices to recall that $a$ is $\mathcal{A}$-linear and that any symmetrized bracket belongs to $\ker a$. □

### 5.2 Free Leibniz Pseudoalgebra

There exists a forgetful functor

$$\mathrm{For} : (\texttt{Sym})\texttt{LeiPsAlg}\,(R,\mathcal{A}) \to \texttt{AncMod}(\mathcal{A})\ .$$

We write for short $\texttt{C} := (\texttt{Sym})\texttt{LeiPsAlg}\,(R,\mathcal{A})$ and $\texttt{D} := \texttt{AncMod}(\mathcal{A})$. Therefore, for any $\mathcal{M} \in \texttt{D}$, we can define the free (symmetric) Leibniz pseudoalgebra over $\mathcal{M}$, see Definition 2. It is made of an object $\mathrm{F(S)}\,\mathcal{M} \in \texttt{C}$ and a $\texttt{D}$-morphism $i : \mathcal{M} \to \mathrm{F(S)}\,\mathcal{M}$, such that, for any object $\mathcal{E} \in \texttt{C}$ and any $\texttt{D}$-morphism $\phi : \mathcal{M} \to \mathcal{E}$, there is a unique $\texttt{C}$-morphism $\Phi : \mathrm{F(S)}\,\mathcal{M} \to \mathcal{E}$, such that $\Phi\, i = \phi$ :




(55)

We first recall the construction of the free Leibniz algebra over an $R$-module [LP93]. Let $V$ be an $R$-module and let $\overline{T}V = \bigoplus_{k\geq 1} V^{\otimes_R k}$ be the reduced tensor $R$-module over $V$. The *universal Leibniz bracket* $[-,-]$ is defined by the requirement

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k = [v_1, [v_2, ..., [v_{k-2}, [v_{k-1}, v_k]]...]]\ ,$$

$v_i \in V$.

For instance,

$$[v_1, v_2] = v_1 \otimes v_2 \in V^{\otimes_R 2}\ ,$$

$$[v_1, v_2 \otimes v_3] = [v_1, [v_2, v_3]] = v_1 \otimes v_2 \otimes v_3 \in V^{\otimes_R 3}\ ,$$

$$[v_1 \otimes v_2, v_3] = [[v_1, v_2], v_3] = [v_1, [v_2, v_3]] - [v_2, [v_1, v_3]] = v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3 \in V^{\otimes_R 3}\ .$$

The next theorem has been conjectured at the beginning of Subsection 4.1.

**Theorem 2.** *Let $(\mathcal{M}, a)$ be an anchored $\mathcal{A}$-module. The free Leibniz $(R,\mathcal{A})$-pseudoalgebra $\mathrm{F}\,\mathcal{M}$ over $\mathcal{M}$ is the triple $(\overline{T}\mathcal{M}, [-,-]_{\mathrm{Lei}}, \mathrm{F}\,a)$, where $\overline{T}\mathcal{M}$ is endowed with the $\mathcal{A}$-module structure defined inductively by*

$$f(m_1 \otimes m_2 \otimes \ldots \otimes m_n) := m_1 \otimes f(m_2 \otimes \ldots \otimes m_n) - a(m_1)(f)(m_2 \otimes \ldots \otimes m_n) \tag{56}$$

*$(f \in \mathcal{A}, m_i \in \mathcal{M})$, where $[-,-]_{\mathrm{Lei}}$ is the universal Leibniz bracket on $\overline{T}\mathcal{M}$, and where*

$$\mathrm{F}\,a : \overline{T}\mathcal{M} \to \mathrm{Der}\,\mathcal{A}$$

*is induced by $a : \mathcal{M} \to \mathrm{Der}\,\mathcal{A}$ .*

Note that the $\mathcal{A}$-module structure on $\overline{T}\mathcal{M}$ is necessarily given by (56), due to the needed Leibniz property

$$[m_1, f(m_2 \otimes \ldots \otimes m_n)]_{\mathrm{Lei}} = f[m_1, m_2 \otimes \ldots \otimes m_n]_{\mathrm{Lei}} + a(m_1)(f)(m_2 \otimes \ldots \otimes m_n)$$

and the fact that

$$[m_1, m_2 \otimes \ldots \otimes m_n]_{\mathrm{Lei}} = m_1 \otimes m_2 \otimes \ldots \otimes m_n \ .$$

*Proof.* We denote by $\mathrm{F}^n \mathcal{M} = \mathcal{M}^{\otimes_R n}$ (resp., $\mathrm{F}_n \mathcal{M} = \bigoplus_{1 \le k \le n} \mathcal{M}^{\otimes_R k}$) the grading (resp., the filtration) of $\mathrm{F}\,\mathcal{M}$.

(i) Module structure. Equation (56) provides a well-defined $\mathcal{A}$-module structure on $\mathrm{F}_n \mathcal{M}$, if we are given a well-defined $\mathcal{A}$-module structure on $\mathrm{F}_{n-1} \mathcal{M}$, $n \ge 2$. Since the RHS of (56) is $R$-multilinear, the 'action' is well-defined from $\mathrm{F}^n \mathcal{M}$ into $\mathrm{F}_n \mathcal{M}$. We extend it by linearity to $\mathrm{F}_n \mathcal{M} = \mathrm{F}^n \mathcal{M} \oplus \mathrm{F}_{n-1} \mathcal{M}$. It is now straightforwardly checked that this extension satisfies all the $\mathcal{A}$-module requirements, except, maybe, the condition $f(g\mu) = (fg)\mu$, where $f, g \in \mathcal{A}$ and $\mu \in \mathrm{F}_n \mathcal{M}$. As for the latter, note first that, if $f, g \in \mathcal{A}$, $m, m_i \in \mathcal{M}$, and $\mathbf{m} = m_2 \otimes \ldots \otimes m_n$, we have

$$f(m \otimes g\mathbf{m}) = m \otimes f(g\mathbf{m}) - a(m)(f)(g\mathbf{m}) \ ,$$

since $g\mathbf{m}$ is a finite sum $g\mathbf{m} = \sum_{i \le n-1} \mathbf{m}_i$, where $\mathbf{m}_i \in \mathrm{F}^i \mathcal{M}$ is a decomposed tensor. Thus,

$$\begin{aligned} f(g(m \otimes \mathbf{m})) &= f(m \otimes g\mathbf{m}) - f(a(m)(g)\mathbf{m}) \\ &= m \otimes (fg)\mathbf{m} - (a(m)(f)g)\mathbf{m} - (fa(m)(g))\mathbf{m} \\ &= m \otimes (fg)\mathbf{m} - a(m)(fg)\mathbf{m} \\ &= (fg)(m \otimes \mathbf{m}) \ . \end{aligned}$$

The $\mathcal{A}$-module structures on the filters $\mathrm{F}_n \mathcal{M}$, $n \ge 1$, naturally induce an $\mathcal{A}$-module structure on $\mathrm{F}\,\mathcal{M}$.

(ii) Universal anchor map. Since $\mathrm{F}\,\mathcal{M}$ is the free Leibniz algebra over $\mathcal{M}$, the map $a : \mathcal{M} \to \mathrm{Der}\,\mathcal{A}$ factors through the inclusion $\mathcal{M} \to \mathrm{F}\,\mathcal{M}$:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathrm{F}\,\mathcal{M} \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{F}\,a} \\ \mathcal{M} & \xrightarrow{\ a\ } & \mathrm{Der}\,\mathcal{A} \end{array}$$

The Leibniz algebra morphism $\mathrm{F}\,a$ is actually $\mathcal{A}$-linear. Indeed, in view of the decomposition $f\mathbf{m} = \sum_{i \le n-1} \mathbf{m}_i$, where $\mathbf{m}_i \in \mathrm{F}^i \mathcal{M}$ is a decomposed tensor, we obtain first $m \otimes f\mathbf{m} = [m, f\mathbf{m}]$, where the notation is the same as above. Since, by induction, $\mathrm{F}\,a$ is $\mathcal{A}$-linear on $\mathrm{F}^{n-1} \mathcal{M}$, we then have

$$\begin{aligned}
\mathrm{F}\,a(f(m\otimes\mathbf{m})) &= \mathrm{F}\,a(m\otimes f\mathbf{m}) - \mathrm{F}\,a(a(m)(f)\mathbf{m})\\
&= \mathrm{F}\,a[m, f\mathbf{m}] - a(m)(f)\,\mathrm{F}\,a(\mathbf{m})\\
&= [a(m), f\,\mathrm{F}\,a(\mathbf{m})] - a(m)(f)\,\mathrm{F}\,a(\mathbf{m})\\
&= f[a(m), \mathrm{F}\,a(\mathbf{m})]\\
&= f\,\mathrm{F}\,a(m\otimes\mathbf{m})\ .
\end{aligned}$$

(iii) Leibniz pseudoalgebra conditions. To see that $(\mathrm{F}\,\mathcal{M}, [-,-]_{\mathrm{Lei}}, \mathrm{F}\,a)$ (in the following we omit subscript Lei) is a Leibniz pseudoalgebra, it now suffices to check that (4) is satisfied. We proceed by induction and assume that, for any $f\in\mathcal{A}$, $\mathbf{m}\in\mathrm{F}_{n-1}\,\mathcal{M}$ and $\mathbf{m}'\in\mathrm{F}\,\mathcal{M}$, $n\geq 2$, the bracket $[\mathbf{m}, f\mathbf{m}']$ satisfies Condition (4). Indeed, for $n=2$, we have

$$[m, f\mathbf{m}'] = m\otimes f\mathbf{m}' = f(m\otimes\mathbf{m}') + a(m)(f)\mathbf{m}' = f[m,\mathbf{m}'] + \mathrm{F}\,a(m)(f)\mathbf{m}'\ .$$

It is easily seen that (4) is then also satisfied in $\mathrm{F}_n\,\mathcal{M}$:

$$\begin{aligned}
[m\otimes\mathbf{m}, f\mathbf{m}'] &= [[m,\mathbf{m}], f\mathbf{m}']\\
&= [m,[\mathbf{m}, f\mathbf{m}']] - [\mathbf{m},[m, f\mathbf{m}']]\\
&= [m, f[\mathbf{m},\mathbf{m}']] + [m, \mathrm{F}\,a(\mathbf{m})(f)\mathbf{m}'] - [\mathbf{m}, f[m,\mathbf{m}']] - [\mathbf{m}, a(m)(f)\mathbf{m}']\\
&= \cdots\\
&= f[m,[\mathbf{m},\mathbf{m}']] - f[\mathbf{m},[m,\mathbf{m}']] + [\mathrm{F}\,a(m), \mathrm{F}\,a(\mathbf{m})](f)\mathbf{m}'\\
&= f[m\otimes\mathbf{m},\mathbf{m}'] + \mathrm{F}\,a(m\otimes\mathbf{m})(f)\mathbf{m}'.
\end{aligned}$$

(iv) Freeness. It remains to show that the Leibniz $(R,\mathcal{A})$-pseudoalgebra $(\mathrm{F}\,\mathcal{M}, [-,\,-], \mathrm{F}\,a)$ (we omit Lei), together with the anchored $\mathcal{A}$-module morphism $i:\mathcal{M}\hookrightarrow\mathrm{F}\,\mathcal{M}$, is free. Let thus $(\mathcal{E}, [-,-]', a')$ be any Leibniz $(R,\mathcal{A})$-pseudoalgebra and let $\phi:\mathcal{M}\to\mathcal{E}$ be any anchored $\mathcal{A}$-module morphism. Since $(\mathrm{F}\,\mathcal{M}, [-,-], i)$ is the free Leibniz $R$-algebra over the $R$-module $\mathcal{M}$, the $R$-linear map $\phi$ extends uniquely to a Leibniz $R$-algebra map $\mathrm{F}\,\phi:\mathrm{F}\,\mathcal{M}\to\mathcal{E}$. When assuming that $a'\,\mathrm{F}\,\phi = \mathrm{F}\,a$ (resp., that $\mathrm{F}\,\phi$ is $\mathcal{A}$-linear) on $\mathrm{F}_{n-1}\,\mathcal{M}$, the usual proof based on the observation that $m\otimes f\mathbf{m} = [m, f\mathbf{m}]$ (resp., on this observation combined with (56) and (4)) allows to see that the same property holds on $F_n\mathcal{M}$. □

## 5.3 Free Symmetric Leibniz Pseudoalgebra

**Proposition 5.1.** *Let $J_1$, $J_2$ be the ideals (49) and (50) associated to the free Leibniz $(R,\mathcal{A})$-pseudoalgebra $(\mathrm{F}\,\mathcal{M}, [-,-]_{\mathrm{Lei}}, \mathrm{F}\,a)$ over an anchored $\mathcal{A}$-module $(\mathcal{M}, a)$. The quotient $\mathrm{FS}\,\mathcal{M} := \mathrm{F}\,\mathcal{M}/(J_1+J_2)$, with induced bracket, anchor and 'inclusion', is the free symmetric Leibniz pseudoalgebra over the anchored module $\mathcal{M}$.*

**Remark 4.** *The free symmetric Leibniz pseudoalgebra over an anchored module* $(\mathcal{M}, a)$ *is the natural quotient of the free Leibniz pseudoalgebra over* $(\mathcal{M}, a)$*. The latter is the reduced tensor* $R$*-module* $\mathrm{F}\,\mathcal{M} = \overline{T}\mathcal{M}$ *over* $\mathcal{M}$*, endowed with an* $\mathcal{A}$*-module structure that encodes the anchor* $a$*, the universal Leibniz bracket* $[-,-]_{\mathrm{Lei}}$*, and the induced anchor* $\mathrm{F}\,a$*. In the geometric case, when* $\mathcal{M} = \Gamma(E)$*, with* $E \to M$ *an anchored vector bundle over a manifold, the module* $\overline{T}\mathcal{M}$ *is not a space of sections, since the tensor product in* $\overline{T}\mathcal{M}$ *is over* $R$*.*

*Proof.* We characterize the classes and the mentioned induced data by the symbol 'tilde'. It has already been said that $(\mathrm{FS}\,\mathcal{M}, [-,-]^{\tilde{}}_{\mathrm{Lei}}, (\mathrm{F}\,a)^{\tilde{}})$ is a symmetric Leibniz pseudoalgebra (as usual we will omit Lei). On the other hand, it is clear from the definition of $(\mathrm{F}\,a)^{\tilde{}}$ that $\tilde{\imath} : \mathcal{M} \ni m \to \tilde{m} \in \mathrm{FS}\,\mathcal{M}$ is an anchored $\mathcal{A}$-module map.

As for freeness, let $(\mathcal{E}, [-,-]', a')$ be a symmetric Leibniz pseudoalgebra. Any anchored module map $\phi : \mathcal{M} \to \mathcal{E}$ uniquely extends to a Leibniz pseudoalgebra map $\mathrm{F}\,\phi : \mathrm{F}\,\mathcal{M} \to \mathcal{E}$. To see that $\mathrm{F}\,\phi$ descends to $\mathrm{FS}\,\mathcal{M}$, it suffices to show that it vanishes on $J_1$ and $J_2$. Observe first that, for any $\mu, \nu \in \mathrm{F}\,\mathcal{M}$, we have

$$\mathrm{F}\,\phi(\mu \circ \nu) = \mathrm{F}\,\phi(\mu) \circ' \mathrm{F}\,\phi(\nu) \; .$$

It follows now from the $\mathcal{A}$-linearity of $\mathrm{F}\,\phi$ and the symmetry of $\mathcal{E}$ that $\mathrm{F}\,\phi$ annihilates $J_1$ and $J_2$. It is also straightforwardly checked that the induced map $(\mathrm{F}\,\phi)^{\tilde{}} : \mathrm{FS}\,\mathcal{M} \to \mathcal{E}$ is a map of Leibniz pseudoalgebras such that $(\mathrm{F}\,\phi)^{\tilde{}}\,\tilde{\imath} = \phi$. As for uniqueness of this extension, note that any Leibniz pseudoalgebra morphism $\mathrm{FS}\,\phi : \mathrm{FS}\,\mathcal{M} \to \mathcal{E}$ that extends $\phi$, implements a Leibniz pseudoalgebra morphism

$$(\mathrm{FS}\,\phi)^{\bar{}} : \mathrm{F}\,\mathcal{M} \ni \mu \mapsto \mathrm{FS}\,\phi(\tilde{\mu}) \in \mathcal{E}$$

that extends $\phi$; hence, $(\mathrm{FS}\,\phi)^{\bar{}} = \mathrm{F}\,\phi$ and $\mathrm{FS}\,\phi = (\mathrm{F}\,\phi)^{\tilde{}}$. □

# 6 Free Courant Pseudoalgebra

There is a forgetful functor $\mathrm{For} : \texttt{CrtPsAlg} \to \texttt{AncMod}$ between the categories of generalized Courant $(R, \mathcal{A})$-pseudoalgebras and anchored $\mathcal{A}$-modules.

**Theorem 3.** *The free Courant pseudoalgebra over an anchored module* $(\mathcal{M}, a)$ *is the generalized Courant pseudoalgebra*

$$\mathcal{C}(\mathrm{FS}\,\mathcal{M}) = (\mathrm{FS}\,\mathcal{M}, \mathcal{R}(\mathrm{FS}\,\mathcal{M}), [-,-]^{\tilde{}}_{\mathrm{Lei}}, (-|-), (\mathrm{F}\,a)^{\tilde{}}, \tilde{\mu}^{\ell}, \tilde{\mu}^{r}) \; ,$$

*associated to the free symmetric Leibniz pseudoalgebra over* $\mathcal{M}$*, together with the anchored module map* $\tilde{\imath} : \mathcal{M} \to \mathrm{FS}\,\mathcal{M}$*. In other words, for any generalized Courant pseudoalgebra*

$$\mathcal{C} = (\mathcal{E}_1, \mathcal{E}_2, [-,-]', (-|-)', a', \mu'^{\ell}, \mu'^{r})$$

*and any anchored module map* $\phi : \mathcal{M} \to \mathcal{E}_1$*, there exists a unique morphism of generalized Courant pseudoalgebras* $(\phi_1, \phi_2)$ *from* $\mathcal{C}(\mathrm{FS}\,\mathcal{M})$ *to* $\mathcal{C}$*, such that* $\phi_1\,\tilde{\imath} = \phi$*.*

*Proof.* Let $\mathcal{C}$ and $\phi$ be as in the statement of the theorem. Due to freeness of $\mathrm{FS}\,\mathcal{M}$, the anchored module map $\phi$ uniquely factors through $\mathrm{FS}\,\mathcal{M}$, thus leading to a unique Leibniz pseudoalgebra morphism $\phi_1 : \mathrm{FS}\,\mathcal{M} \to \mathcal{E}_1$ such that $\phi_1\,\tilde{\imath} = \phi$. As for $\phi_2$, note that the $\mathcal{A}$-linear map

$$\phi_2 = (-|-)'(\phi_1 \odot \phi_1) : \mathrm{FS}\,\mathcal{M} \odot \mathrm{FS}\,\mathcal{M} \to \mathcal{E}_1 \odot \mathcal{E}_1 \to \mathcal{E}_2$$

descends to the quotient $\mathcal{R}(\mathrm{FS}\,\mathcal{M})$. Indeed, if $I(\tilde{\mu}, \tilde{\nu}, \tilde{\tau}) \in \mathrm{Inv}$ (in the following we omit the symbol 'tilde'), we have

$$\phi_2(I(\mu, \nu, \tau)) =$$

$$([\phi_1\mu, \phi_1\nu]'|\phi_1\tau)' + (\phi_1\nu|[\phi_1\mu, \phi_1\tau]')' - (\phi_1\mu|[\phi_1\nu, \phi_1\tau]' + [\phi_1\tau, \phi_1\nu]')' = 0\ ,$$

since $\mathcal{C}$ is a generalized Courant pseudoalgebra. The resulting $\mathcal{A}$-linear map $\mathcal{R}(\mathrm{FS}\,\mathcal{M}) \to \mathcal{E}_2$ will still be denoted by $\phi_2$. Since

$$(\mu|\nu) = (\mu \odot \nu)^{\tilde{}}\ , \tag{57}$$

it is clear that the requirement (41), i.e.,

$$(-|-)'(\phi_1 \times \phi_1) = \phi_2(-|-)\ ,$$

is satisfied. It now suffices to check that the conditions (10), i.e.,

$$\mu'^{\ell}(\phi_1 \times \phi_2) = \phi_2\,\tilde{\mu}^{\ell} \ \text{ and } \ \mu'^{r}(\phi_2 \times \phi_1) = \phi_2\,\tilde{\mu}^{r}\ ,$$

hold as well. Let us detail the last case. Let $\tilde{\mu}, \tilde{\nu}, \tilde{\tau} \in \mathrm{FS}\,\mathcal{M}$ (and omit again the 'tilde'). Since

$$\tilde{\mu}^{r}(\mu)(\nu \odot \tau)^{\tilde{}} = (\mu^{r}(\mu)(\nu \odot \tau))^{\tilde{}} = (-(\nu \circ \tau) \odot \mu)^{\tilde{}}\,,$$

the application of $\phi_2$ leads to

$$-(\phi_1\nu \circ' \phi_1\tau|\phi_1\mu)'\ .$$

The latter coincides with the value of $\mu'^{r}(\phi_2 \times \phi_1)$ on the arguments $(\nu \odot \tau)^{\tilde{}}$ and $\mu$. Finally, uniqueness of $\phi_1$ was already mentioned, and uniqueness of $\phi_2$ is a consequence of (41) and (57). □

**Example 7.** *As seen in Example 6, the difference between a classical Courant algebroid*

$$(\mathcal{E}, [-,-]_{\mathrm{C}}, (-|-)_{\mathrm{C}}, a)$$

*and its associated generalized Courant pseudoalgebra resides in the substitution of the universal 'scalar product' to the original product. On the other hand, the free generalized Courant pseudoalgebra over a classical Courant algebroid, viewed as anchored module* $(\mathcal{E}, a)$*, is completely different from the initial one, since it is given by*

$$\mathcal{C}(\mathrm{FS}\,\mathcal{E}) = (\mathrm{FS}\,\mathcal{E}, (\mathrm{FS}\,\mathcal{E})^{\odot 2}/\langle \mathrm{Inv}\rangle, [-,-]^{\tilde{}}_{\mathrm{Lei}}, \odot^{\tilde{}}, (\mathrm{F}\,a)^{\tilde{}}, \tilde{\mu}^{\ell}, \tilde{\mu}^{r})\ ,$$

*where*

$$\mathrm{FS}\,\mathcal{E} = \overline{T}\mathcal{E}/(\langle J_1\rangle + \langle J_2\rangle)\ .$$

*All the ingredients of this free object are* implicit*: the free Leibniz algebra bracket, the $\mathcal{A}$-module structure, and the anchor on $\overline{T}\mathcal{E}$ are defined* inductively*, whereas the module* $\mathrm{FS}\,\mathcal{E}$ *and its 'representation' module are* quotients *by the abstract symmetry conditions (24) and (25) and by the abstract invariance condition (42), respectively. It follows that the associated and the free generalized Courant pseudoalgebras are important rather by their existence than by their description in concrete situations – see Section 7. Moreover, the free generalized Courant bracket is not geometric, in the sense that it is not Loday. This can be quite easily checked by an argument to absurdity.*

# 7 Symmetric Leibniz Pseudoalgebra Bracket as Universal Derived Bracket

Many algebraic and algebro-geometric concepts can be encoded in a (co)homological vector field of a possibly formal and noncommutative manifold.

For instance, if $P$ is a quadratic Koszul operad, a $P_\infty$–structure on a finite-dimensional graded vector space $V$ over a field $\mathbb{K}$ of characteristic zero, is essentially a sequence $\ell_n$ of $n$-ary brackets of degree $2-n$ on $V$, which satisfy a sequence $R_n$ of defining relations, $n \in \{1, 2, \ldots\}$. These structures are 1:1 with cohomological vector fields

$$\delta \in \mathrm{Der}_1(\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*)) \tag{58}$$

of the 'manifold' with function algebra $\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*)$ – the free graded algebra over the Koszul dual operad $P^!$ of $P$ on the suspended linear dual $sV^*$ of $V$.

In the case $P = \mathrm{Lie}$, the latter is the graded symmetric tensor algebra $\overline{\odot}sV^*$ without unit. The $n$-ary brackets $\ell_n$ of the Lie infinity structure on $V$ are obtained, up to (de)suspension, as the transposes of the projections to $\odot^n sV^*$ of the restriction of $\delta$ to $sV^*$. T. Voronov uses an alternative method and constructs a Lie infinity structure on $sV$, starting, in the main, from a cohomological vector field $\delta$ of a Lie superalgebra, and using higher derived brackets [Vor05].

The higher derived brackets modus operandi goes through in the geometric situation of Lie $n$-algebroids, $n \geq 1$, [BP12], in particular, as well-known, for Lie algebroids.

Another geometric context, where this technique can be applied, is the case of Loday algebroids: if $E$ denotes a vector bundle, there is a 1:1 correspondence between Loday algebroid structures on $E$ and equivalence classes of cohomological vector fields $\delta \in \mathrm{Der}_1(\mathrm{D}^\bullet(E), \pitchfork)$ [GKP13]. The latter is nonobvious and far from the known solution in the purely algebraic case $P = \mathrm{Lei}$ of Leibniz infinity or just Leibniz structures: we have to consider specific derivations of specific multidifferential operators $\mathrm{D}^\bullet(E)$, as well as the symmetrization $\pitchfork$ of the half-shuffle or Zinbiel multiplication. In particular, the Loday algebroid bracket is the derived bracket induced by the graded Lie algebra $\mathrm{Der}(\mathrm{D}^\bullet(E), \pitchfork)$ and its interior derivation implemented by the cohomological field $\delta$.

In the present paper, we investigate which Leibniz algebroid or Leibniz pseudoalgebra brackets can be viewed as derived brackets. The difficulty, in the passage from algebras to algebroids or pseudoalgebras, is the replacement of scalars (in a field or ring) by functions (in an algebra over this ring). Additional obstruction comes – in the Leibniz pseudoalgebra setting – from the absence of a differentiability condition on the first argument of the bracket. This is one of the origins of the subclass of symmetric Leibniz pseudoalgebras. We will show that each symmetric Leibniz pseudoalgebra bracket can be universally represented by a derived bracket.

Let $(\mathcal{E}, [-,-], a)$ be a symmetric Leibniz pseudoalgebra (over $(R, \mathcal{A})$) and let

$$\mathcal{C}(\mathcal{E}) = (\mathcal{E}, \mathcal{R}(\mathcal{E}), [-,-], (-|-), a, \tilde{\mu}^\ell, \tilde{\mu}^r)$$

be the associated generalized Courant pseudoalgebra. Recall that $\mathcal{R}(\mathcal{E})$ is the universal representation $\mathcal{A}$-module $\mathcal{E}^{\odot 2}/\langle \mathrm{Inv} \rangle$, where the denominator is the $\mathcal{A}$-submodule induced by

$$\mathrm{Inv} = \{[X,Y] \odot Z + Y \odot [X,Z] - X \odot (Y \circ Z) : X, Y, Z \in \mathcal{E}\} \,.$$

We denote by $\mathcal{E}_{\mathrm{Lie}}$ the Lie $R$-algebra $\mathcal{E}/I$ obtained as the quotient of the Leibniz $R$-algebra $(\mathcal{E}, [-,-])$ by the Leibniz $R$-ideal $I = \langle X \circ Y \rangle$. Further, we introduce the graded $R$-module

$$\mathcal{D}^\bullet(\mathcal{E}) := \mathcal{D}^0(\mathcal{E}) \oplus \mathcal{D}^1(\mathcal{E}) \oplus \mathcal{D}^2(\mathcal{E}), \text{ where } \mathcal{D}^0(\mathcal{E}) := \mathcal{E}_{\mathrm{Lie}},\ \mathcal{D}^1(\mathcal{E}) := \mathcal{E}, \text{ and } \mathcal{D}^2(\mathcal{E}) := \mathcal{R}(\mathcal{E}) \,.$$

We denote its degree by $|-|$, its elements of degree 1 (resp., degree 0, degree 2) by $X, Y, \ldots$ (resp., by $\bar{X}, \bar{Y}, \ldots$ , by $(X \odot Y)\tilde{}, \ldots$), and its elements of arbitrary degree by $a, b, \ldots$

We will endow this graded module with a differential graded Lie $R$-algebra (DGLA) structure $(\mathcal{D}^\bullet(\mathcal{E}), \{-,-\}, \delta)$, such that the induced derived Leibniz $R$-bracket $\{-,-\}_\delta$ on $\mathcal{D}^\bullet(\mathcal{E})$ coincides on $\mathcal{E}$ with the original Leibniz bracket $[-,-]$. More precisely, the graded Lie bracket $\{-,-\}$ will be of degree 0, the differential $\delta$ of degree $-1$, the derived bracket

$$\{a,b\}_\delta := (-1)^{|a|+1}\{\delta a, b\}$$

will thus be of degree $-1$ as well, and the result $s[X,Y] = \{sX, sY\}_\delta$ , where $s : \mathcal{E} \hookrightarrow \mathcal{D}^\bullet(\mathcal{E})$ is the suspension operator, will hold.

As $\{-,-\}$ must be of degree 0 and graded antisymmetric, it is naturel to set ($s$ is understood, wherever possible):

$$\{\bar{X}, \bar{Y}\} := \overline{[X,Y]}, \quad \{\bar{X}, Y\} = -\{Y, \bar{X}\} := [X,Y],$$

$$\{\bar{X}, (Y \odot Z)\tilde{}\} = -\{(Y \odot Z)\tilde{}, \bar{X}\} := \tilde{\mu}^\ell(X)(Y \odot Z)\tilde{}, \quad \text{and} \quad \{X, Y\} := (X|Y) = (X \odot Y)\tilde{},$$

whereas the brackets of pairs of elements of degrees $(1,2),(2,1)$, and $(2,2)$ are of degree $> 2$ and must therefore vanish.

The bracket $\{-,-\}$ is obviously of degree 0 and graded skew-symmetric.

Of course, we have to check well-definedness. For the first bracket, of degree $(0,0)$ elements, remember that $\overline{[T \circ U, Y]} = 0$ (even without passing to $\mathcal{E}_{\text{Lie}}$ – in view of the Jacobi identity) and note that

$$\overline{[X, V \circ W]} = \overline{[X,V] \circ W} + \overline{V \circ [X,W]} = 0 \; .$$

Well-definedness is now also clear for degree $(0,1)$ and $(1,0)$ elements. As for degree $(0,2)$ and $(2,0)$ elements, we already proved above that $\tilde{\mu}^\ell(X)$ is well-defined on the quotient $\mathcal{R}(\mathcal{E})$, for any $X \in \mathcal{E}$. Concerning the argument $\bar{X}$, it suffices to observe that

$$\tilde{\mu}^\ell(T \circ U)(Y \odot Z)^{\tilde{}} = ([T \circ U, Y] \odot Z)^{\tilde{}} + (Y \odot [T \circ U, Z])^{\tilde{}} = 0 \; .$$

Regarding the graded Jacobi identity, it is easily seen that, if it holds for three elements of some given degrees, it also holds for elements whose degrees are any permutation of the initial ones. Therefore, it suffices to check the identity for the degrees $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, and $(0,1,1)$. Indeed, all other cases are permutations or their sum of degrees $> 2$. In the four relevant cases, the graded Jacobi identity is a direct consequence of the definitions.

If the abovementioned degree $-1$ derivation $\delta$ does exist, it sends $X \in \mathcal{E}$ to $\delta X \in \mathcal{E}_{\text{Lie}}$. A naturel choice is

$$\delta X := \bar{X} \; .$$

Further, since $\delta(X \odot Y)^{\tilde{}} = \delta\{X, Y\}$, the graded derivation property shows that the latter is equal to

$$\{\delta X, Y\} - \{X, \delta Y\} = [X,Y] + [Y,X] = X \circ Y \; .$$

When adopting this definition

$$\delta(X \odot Y)^{\tilde{}} := X \circ Y \; ,$$

we have to check that $\delta$ is well-defined. Well-definedness on $\mathcal{E}^{\odot 2}$ is a direct consequence of the first symmetry condition, whereas well-definedness on the quotient $\mathcal{R}(\mathcal{E})$ requires that

$$\delta(f([X,Y] \odot Z + Y \odot [X,Z] - X \odot (Y \circ Z))) = 0 \; ,$$

for any $f \in \mathcal{A}\,, X, Y, Z \in \mathcal{E}$. However, the LHS of the latter reads

$$\delta([X,Y] \odot (fZ) + (fY) \odot [X,Z] - (fX) \odot (Y \circ Z)) =$$

$$[X,Y] \circ (fZ) + (fY) \circ [X,Z] - [fX, Y \circ Z] - [Y \circ Z, fX] \; ,$$

where the sum of the first three terms vanishes due to the second symmetry condition, while the last term is of the form 'right adjoint action on a symmetrized product' and thus vanishes – as recalled above. Finally, the map $\delta$ is a well-defined degree $-1$ map on the $R$-module $\mathcal{D}^\bullet(\mathcal{E})$, which is visibly $R$-linear and of square 0.

It now suffices to prove that $\delta$ is a graded derivation for $\{-,-\}$. If the graded derivation property holds for degree $(i,j)$ elements, it is also valid for degree $(j,i)$ elements. Hence, we only examine the degrees $(0,0)$, $(0,1)$, $(0,2)$, $(1,1)$, and $(1,2)$. These verifications are straightforward, at least if one remembers the second symmetry condition (in fact, here we do not even need the symmetry assumption: we use the second symmetry condition with $f = 1 \in \mathcal{A}$, and in this case it is satisfied in any Leibniz algebra).

Eventually, the triple $(\mathcal{D}^\bullet(\mathcal{E}), \{-,-\}, \delta)$ is a differential graded Lie algebra. The chosen definitions imply that, as announced, the original Leibniz bracket can be represented as the derived bracket coming from this DGLA: $s[X,Y] = \{sX, sY\}_\delta$ . In fact, the suspension $s : \mathcal{E} \hookrightarrow \mathcal{D}^\bullet(\mathcal{E})$ is a derived bracket representation of $(\mathcal{E}, [-,-])$. More generally,

**Definition 19.** *A* derived bracket representation *of a graded Leibniz algebra* $(L^\bullet, [-,-])$ *with degree 0 bracket, is a degree* 1 *graded Leibniz algebra morphism*

$$\xi : (L^\bullet, [-,-]) \to (K^\bullet, \{\!\{-,-\}\!\}_\Delta) ,$$

*whose target is the graded Leibniz algebra with the degree* $-1$ *derived bracket implemented by a DGLA* $(K^\bullet, \{\!\{-,-\}\!\}, \Delta)$*, whose bracket is of degree 0 and whose differential has degree* $-1$*.*

In this definition, all algebras are over $R$. In the following, we consider representations of the Leibniz algebra $(\mathcal{E}, [-,-])$ (concentrated in degree 0), and we restrict ourselves to representations $\xi$, such that $\xi(\mathcal{E})$ (resp., $\{\!\{\xi(\mathcal{E}), \xi(\mathcal{E})\}\!\}$) is an $\mathcal{A}$-module, $\xi : \mathcal{E} \to \xi(\mathcal{E})$ (resp., $\{\!\{-,-\}\!\} : \xi(\mathcal{E}) \times \xi(\mathcal{E}) \to \{\!\{\xi(\mathcal{E}), \xi(\mathcal{E})\}\!\}$) is $\mathcal{A}$-linear (resp., $\mathcal{A}$-bilinear), and such that $\xi(\mathcal{E})$ is Lie 3-nilpotent, in the sense that

$$\{\!\{\xi(\mathcal{E}),\ \{\!\{\xi(\mathcal{E}), \xi(\mathcal{E})\}\!\}\ \}\!\} = 0 .$$

In other words, all brackets of the type $\{\!\{\xi(X),\ \{\!\{\xi(Y), \xi(Z)\}\!\}\ \}\!\}$ vanish. We refer to such representations as *$\mathcal{A}$-linear nilpotent* representations. The above suspension, for instance, is $\mathcal{A}$-linear and nilpotent.

In fact, the suspension is the best possible representation:

**Theorem 4.** *For any symmetric Leibniz pseudoalgebra* $(\mathcal{E}, [-,-], a)$*, the suspension*

$$s : (\mathcal{E}, [-,-]) \to (\mathcal{D}^\bullet(\mathcal{E}), \{-,-\}_\delta)$$

*is an $\mathcal{A}$-linear nilpotent derived bracket representation ( implemented by the DGLA* $(\mathcal{D}^\bullet(\mathcal{E}), \{-,-\}, \delta)$ *) that is universal among all representations of this type. More precisely, for any $\mathcal{A}$-linear nilpotent derived bracket representation*

$$\xi : (\mathcal{E}, [-,-]) \to (K^\bullet, \{\!\{-,-\}\!\}_\Delta) ,$$

*( where the target is the graded Leibniz algebra with the degree* $-1$ *derived bracket implemented by a DGLA* $(K^\bullet, \{\!\{-,-\}\!\}, \Delta)$*, whose bracket is of degree 0 and whose differential has degree* $-1$*, ) there exists a unique DGLA-morphism*

$$\Xi : (\mathcal{D}^\bullet(\mathcal{E}), \{-,-\}, \delta) \to (K^\bullet, \{\!\{-,-\}\!\}, \Delta)$$

*(which induces a degree 0 morphism between the corresponding graded Leibniz algebras), such that* $\Xi \circ s = \xi$ .

*Proof.* Let $\xi$ be as described. If $\Xi$ exists, we have necessarily

$$\Xi(X) = \Xi(sX) = \xi(X), \quad \Xi(\bar{X}) = \Xi(\delta X) = \Delta(\Xi(X)) = \Delta(\xi(X)), \quad \text{and}$$

$$\Xi(X \odot Y)^{\tilde{}} = \Xi\{X, Y\} = \{\!\{\Xi(X), \Xi(Y)\}\!\} = \{\!\{\xi(X), \xi(Y)\}\!\} .$$

Hence, the uniqueness of $\Xi$ .

We adopt these definitions. Clearly, the map $\Xi$ is of degree 0, but we must verify that it is well-defined. In the case of $\bar{X}$, we get

$$\Delta\left(\xi([Y, Z] + [Z, Y])\right) = \Delta\{\!\{\Delta(\xi(Y)), \xi(Z)\}\!\} + \Delta\{\!\{\Delta(\xi(Z)), \xi(Y)\}\!\} = 0 ,$$

in view of the DGLA-properties. For $(X \odot Y)^{\tilde{}}$, the RHS of the definition is $\mathcal{A}$-bilinear, and we have

$$\Xi\left([X, Y] \odot (fZ) + (fY) \odot [X, Z] - (fX) \odot (Y \circ Z)\right) =$$

$$\{\!\{\xi[X, Y], \xi(fZ)\}\!\} + \{\!\{\xi(fY), \xi[X, Z]\}\!\} - \{\!\{\xi(fX), \xi(Y \circ Z)\}\!\} =$$

$$f\left(\{\!\{\xi[X, Y], \xi(Z)\}\!\} + \{\!\{\xi(Y), \xi[X, Z]\}\!\} - \{\!\{\xi(X), \xi([Y, Z] + [Z, Y])\}\!\}\right) =$$

$$f\left(\{\!\{\{\!\{\Delta(\xi(X)), \xi(Y)\}\!\}, \xi(Z)\}\!\} + \{\!\{\xi(Y), \{\!\{\Delta(\xi(X)), \xi(Z)\}\!\}\}\!\}\right) -$$

$$f\{\!\{\xi(X), \{\!\{\Delta(\xi(Y)), \xi(Z)\}\!\} - \{\!\{\xi(Y), \Delta(\xi(Z))\}\!\}\}\!\} =$$

$$f\{\!\{\Delta(\xi(X)), \{\!\{\xi(Y), \xi(Z)\}\!\}\}\!\} - f\{\!\{\xi(X), \Delta\{\!\{\xi(Y), \xi(Z)\}\!\}\}\!\} =$$

$$f\Delta\{\!\{\xi(X), \{\!\{\xi(Y), \xi(Z)\}\!\}\}\!\} = 0 ,$$

due to the 3-nilpotency assumption.

It remains to prove that $\Xi$ respects the brackets and the differentials, and actually induces a degree 0 morphism of graded Leibniz algebras.

It suffices to examine the brackets of two elements of degree $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 1)$, $(1, 2)$, and $(2, 2)$. We get:

$$\Xi\{\bar{X}, \bar{Y}\} = \Xi\overline{[X, Y]} = \Delta\{\!\{\Delta(\xi(X)), \xi(Y)\}\!\} = \{\!\{\Xi(\bar{X}), \Xi(\bar{Y})\}\!\} ,$$

$$\Xi\{\bar{X}, Y\} = \{\!\{\Delta(\xi(X)), \xi(Y)\}\!\} = \{\!\{\Xi(\bar{X}), \Xi(Y)\}\!\} ,$$

$$\Xi\{\bar{X}, (Y \odot Z)^{\tilde{}}\} = \Xi\left([X, Y] \odot Z\right)^{\tilde{}} + \Xi\left(Y \odot [X, Z]\right)^{\tilde{}} = \{\!\{\xi[X, Y], \xi(Z)\}\!\} + \{\!\{\xi(Y), \xi[X, Z]\}\!\} =$$

$$\{\!\{\{\!\{\Xi(\bar{X}), \xi(Y)\}\!\}, \xi(Z)\}\!\} + \{\!\{\xi(Y), \{\!\{\Xi(\bar{X}), \xi(Z)\}\!\}\}\!\} =$$

$$\{\!\{\Xi(\bar{X}), \Xi(Y \odot Z)^{\tilde{}}\}\!\} ,$$

$$\Xi\{X, Y\} = \Xi(X \odot Y)^{\tilde{}} = \{\!\{\Xi(X), \Xi(Y)\}\!\} .$$

In the two last cases, the LHS of the Leibniz algebra morphism condition vanishes. The RHS is given by

$$\{\!\{\Xi(X), \Xi(Y \odot Z)^{\sim}\}\!\} = \{\!\{\xi(X), \{\!\{\xi(Y), \xi(Z)\}\!\}\}\!\} = 0$$

and

$$\{\!\{\Xi(X \odot Y)^{\sim}, \Xi(U \odot V)^{\sim}\}\!\} = \{\!\{\{\!\{\xi(X), \xi(Y)\}\!\}, \{\!\{\xi(U), \xi(V)\}\!\}\}\!\} =$$

$$\{\!\{\{\!\{\xi(X), \{\!\{\xi(U), \xi(V)\}\!\}\}\!\}, \xi(Y)\}\!\} + \{\!\{\xi(X), \{\!\{\xi(Y), \{\!\{\xi(U), \xi(V)\}\!\}\}\!\}\}\!\} = 0 \ ,$$

respectively.

The morphism $\Xi$ also intertwines the differentials $\delta$ and $\Delta$. This is straightforwardly checked in all degrees. Eventually, for any $a, b \in \mathcal{D}^\bullet(\mathcal{E})$ ($a$ homogeneous), we get

$$\Xi\{a, b\}_\delta = (-1)^{|a|+1}\Xi\{\delta a, b\} = (-1)^{|a|+1}\{\!\{\Delta(\Xi(a)), \Xi(b)\}\!\} = \{\!\{\Xi(a), \Xi(b)\}\!\}_\Delta \ .$$

□

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