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Mathematics Research Unit (RMATH)

A global description of the fine Simpson moduli space of 1-dimensional sheaves supported on plane quartics

1. Introduction

LET $X \subseteq \mathbb{P}_N$ be a projective variety, let $\mathcal{O}_X(1)$ be the corresponding ample invertible sheaf.

FOR a coherent sheaf $\mathcal{F}$ on X, the function

$$m \mapsto \sum_i (-1)^i \cdot \dim H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m))$$

is a numeric polynomial $P(\mathcal{F}, m) \in \mathbb{Q}[m]$ called the *Hilbert polynomial* of $\mathcal{F}$. The degree d of $P(\mathcal{F}, m)$ equals the dimension of the support of $\mathcal{F}$ and

$$P(\mathcal{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \cdot \frac{m^i}{i!}, \quad \alpha_d(\mathcal{F}) \in \mathbb{Z}_{>0}.$$

THE sheaf $\mathcal{F}$ is called (semi-)stable if for every proper subsheaf $\mathcal{E} \subsetneq \mathcal{F}$

$$\alpha_d(\mathcal{F}) \cdot P(\mathcal{E}, m) < (\leqslant) \alpha_d(\mathcal{E}) \cdot P(\mathcal{F}, m), \quad m \gg 0.$$

Theorem (C. Simpson). *For a fixed $P(m) \in \mathbb{Q}[m]$ there exists a coarse projective moduli space $M := M_P(X)$ of semi-stable sheaves (their s-equivalence classes) on X with Hilbert polynomial P.*

IF X is a surface and $P(m)$ is linear, then the sheaves from $M = M_P(X)$ are supported on curves and M may be seen as a compactification of a certain moduli space of vector bundles on curves in X by torsion-free sheaves (on support).

LET $X = \mathbb{P}V = \mathbb{P}_2$, $\dim V = 3$, be a projective plane, let $P(m) = am + b$, $a \in \mathbb{Z}_{>0}, b \in \mathbb{Z}$. Denote $M_{am+b} = M_{am+b}(\mathbb{P}_2)$. Twisting with $\mathcal{O}_{\mathbb{P}_2}(1)$ gives an isomorphism

$$M_{am+b} \cong M_{am+b+a}. \tag{1}$$

BY [3] there is also an isomorphism

$$M_{am+b}(\mathbb{P}_2) \cong M_{am+(a-b)}. \tag{2}$$

IF $\gcd(a, b) = 1$, every semi-stable sheaf is stable and M_{am+b} is a fine moduli space.

2. First examples

FOR $a = 1$, M_{m+1} consists of the isomorphism classes of the structure sheaves of lines $L \subseteq \mathbb{P}_2$, hence isomorphic to $\mathbb{P}_2^* = \mathbb{P}V^*$. By (1) $M_{m+1} \cong M_{m+b}$, $b \in \mathbb{Z}$.

FOR $a = 2$, M_{2m+1} consists of the isomorphism classes of the structure sheaves of conics $C \subseteq \mathbb{P}_2$, hence isomorphic to $\mathbb{P}_5$. By (1) $M_{2m+1} \cong M_{2m+b}$, $b \in \mathbb{Z}$, $\gcd(2, b) = 1$.

FOR $a = 3$, M_{3m-1} consists of the isomorphism classes of the ideal sheaves of a point p on a cubic curve C

$$0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{O}_{\{p\}} \to 0,$$

hence isomorphic to the universal cubic plane curve (cf. [2]). By (1) and (2) $M_{3m-1} \cong M_{3m+b}$, $b \in \mathbb{Z}$, $\gcd(3, b) = 1$.

3. Sheaves on plane quartics

FOR $a = 4$ and $\gcd(4, b) = 1$, $M_{4m+b} \cong M_{4m-1}$. By [1] $M = M_{4m-1}$ is a disjoint union of two strata M_1 and M_0.

Closed stratum.

THE closed stratum M_1, which is isomorphic to the universal plane quartic, is a closed subvariety of M of codimension 2 given by the condition $h^0(\mathcal{E}) \neq 0$. It consists of the sheaves that are non-trivial extensions

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_{\{p\}} \to 0, \tag{3}$$

where C is a plane quartic and $p \in C$ a point on it.

Open stratum.

THE open stratum M_0 is the complement of M_1 given by the condition $h^0(\mathcal{E}) = 0$, it consists of the isomorphism classes of the cokernels of the injective morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} 3\mathcal{O}_{\mathbb{P}_2}(-1) \tag{4}$$

such that the (2×2)-minors of the linear part $\left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$ of $A = \begin{pmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}$ are linear independent.

Twisted ideals of 3 points on C. If the maximal minors of the linear part of A are coprime, then the cokernel $\mathcal{E}_A$ of (4) is the twisted ideal $\mathcal{I}_Z(1)$, where $\mathcal{I}_Z$ is the ideal sheaf of the zero dimensional subscheme $Z \subseteq C$ of length 3 defined by the maximal minors of the linear submatrix of A.

$$0 \to \mathcal{E}_A \to \mathcal{O}_C(1) \to \mathcal{O}_Z \to 0.$$

THE subvariety M_{00} of such sheaves is open in M_0.

Extensions. If the maximal minors of the linear part of A have a linear common factor, say l, then $\det(A) = l \cdot h$ and $\mathcal{E}_A$ is in this case a non-split extension

$$0 \to \mathcal{O}_L(-2) \to \mathcal{E}_A \to \mathcal{O}_{C'} \to 0, \tag{5}$$

where $L = Z(l)$, $C' = Z(h)$.

THE subvariety M_{01} of such sheaves is closed in M_0 and locally closed in M. Its boundary consists of the sheaves from M_1 as in (3) such that C has a linear component $L \subseteq C$ and $p \in L$.

FOR fixed L and C', the subscheme of the isomorphism classes of non-trivial extensions from M_{01} as in (5) can be identified with $\Bbbk^2$, its closure in M can be seen as $\mathbb{P}_2$ with line at infinity being identified with L.

M_0 as a geometric quotient. M_0 is an open subvariety in the geometric quotient B of the variety of stable matrices as in (4) by the group

$$\operatorname{Aut}(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2)) \times \operatorname{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1)).$$

ITS complement in B is a closed subvariety B' corresponding to the matrices with zero determinant.

Description of B. B is a $\mathbb{P}_{11}$-vector bundle B over the moduli space $N = N(3; 2, 3)$ of stable (2×3) Kronecker modules, i. e., over the quotient of stable (2×3)-matrices of linear forms on $\mathbb{P}_2$ by $\operatorname{Aut}(2\mathcal{O}_{\mathbb{P}_2}(-2)) \times \operatorname{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1))$.

THE projection $B \to N$ is induced by $\begin{pmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix} \mapsto \left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$.

The base N. The subvariety $N' \subseteq N$ corresponding to the matrices whose minors have a common linear factor is isomorphic to $\mathbb{P}_2^* = \mathbb{P}V^*$, the space of lines in $\mathbb{P}_2$, such that a line corresponds to the common linear factor of the minors of the corresponding Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$.

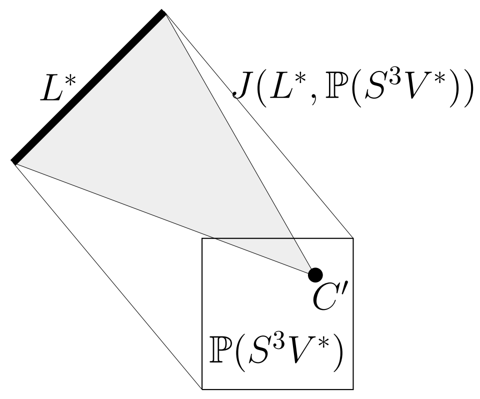
THE blow up of N along N' is isomorphic to the Hilbert scheme $H = \mathbb{P}_2^{[3]}$ of 3 points in $\mathbb{P}_2$. The exceptional divisor $H' \subseteq H$ is a $\mathbb{P}_3$-bundle over N', whose fibre over $L \in \mathbb{P}_2^*$ is the Hilbert scheme $L^{[3]}$ of 3 points on L. The class in N of a Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$ with coprime minors corresponds to the subscheme of 3 non-collinear points in $\mathbb{P}_2$ defined by the minors of the matrix.

The fibres of $B \to N$. A fibre over a point from $N \setminus N'$ can be seen as the space of plane quartics through the corresponding subscheme of 3 non-collinear points.

A FIBRE over $L \in N'$ can be seen as the join $J(L^*, \mathbb{P}S^3V^*) \cong \mathbb{P}_{11}$ of $L^* \cong \mathbb{P}_1$ and the space of plane cubic curves $\mathbb{P}(S^3V^*) \cong \mathbb{P}_9$.

$J(L^*, \mathbb{P}(S^3V^*)) \setminus L^*$ is a rank 2 vector bundle over $\mathbb{P}(S^3V^*)$, whose fibre over a cubic curve $C' \in \mathbb{P}S^3V^*$ corresponds to the projective plane joining C' with L^* inside the join $J(L^*, \mathbb{P}(S^3V^*))$.

The points of $J(L^*, \mathbb{P}(S^3V^*)) \setminus L^*$ parameterize the extensions (5) from M_{01} with fixed L.



Description of B'. B' is a union of lines L^* from each fibre over N' (as explained above), it is isomorphic to the tautological $\mathbb{P}_1$-bundle over $N' = \mathbb{P}_2^*$

$$\{(L, x) \in \mathbb{P}_2^* \times \mathbb{P}_2 \mid L \in \mathbb{P}_2^*, x \in L\}. \tag{6}$$

THE fibre $\mathbb{P}_1$ of B' over, say, line $L = Z(x_0) \subseteq \mathbb{P}_2$ can be identified with the space of classes of matrices (4) with zero determinant

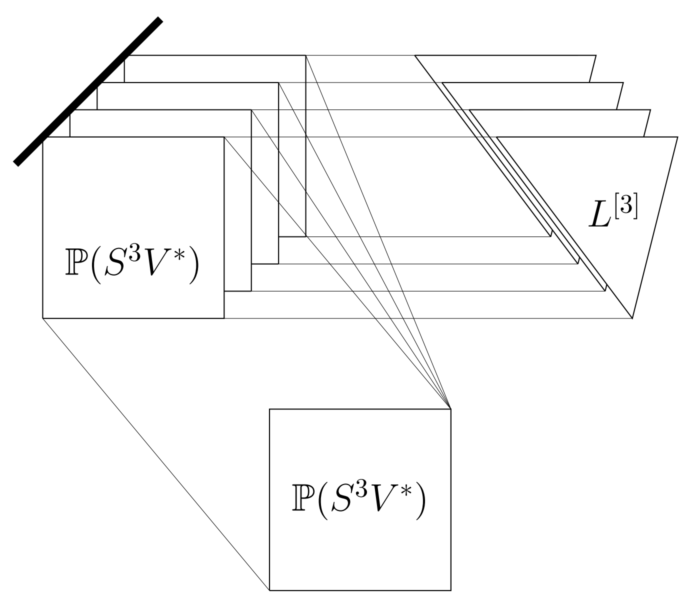
$$\begin{pmatrix} \xi \cdot x_2 & -\xi \cdot x_1 & 0 \\ x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}, \quad \xi = \alpha x_1 + \beta x_2, \quad \langle \alpha, \beta \rangle \in \mathbb{P}_1.$$

4. Main result

Theorem. $\operatorname{Bl}_{M_1} M = \operatorname{Bl}_{B'} B$.

BLOWING up B along B' substitutes B' by the projective conormal bundle of B'. So, each point (L, x) of B' is substituted by $\mathbb{P}_{13}$, which can be seen as the join $\mathbb{P}_{13} = J(L^{[3]}, \mathbb{P}(S^3V^*))$ of $\mathbb{P}_3 = L^{[3]}$ (normal directions along N) and $\mathbb{P}_9 = \mathbb{P}(S^3V^*)$ (normal directions along the fibre $J(L^*, \mathbb{P}(S^3V^*))$).

THE fibre of $B \to N$ over $L \in N'$ is substituted by the fibre that consists of two components: the first component is the blow-up of $J(L^*, \mathbb{P}(S^3V^*))$ along L^*, the second one is $\mathbb{P}_{13} = J(L^{[3]}, \mathbb{P}(S^3V^*))$, the components intersect along $L^* \times \mathbb{P}(S^3V^*)$.



FOR a fixed point $x \in L$, $J(L^{[3]}, \mathbb{P}(S^3V^*))$ can be interpreted as the space of quartics through x.

EVERY point x in $\mathbb{P}_2$ defines a projective line $l_x = \{(x, L) \mid x \in L\}$ in B', every two points (L, x) and (L', x) of $l_x \subseteq B'$ are substituted by the projective spaces $J(L^{[3]}, \mathbb{P}(S^3V^*))$ and $J(L'^{[3]}, \mathbb{P}(S^3V^*))$ respectively, which are naturally identified with the space of quartics through x.

CONTRACTING the exceptional divisor of $\operatorname{Bl}_{B'} B$ along all lines l_x, one gets the universal quartic M_1.

THIS describes the blow up of M along M_1, whose exceptional divisor is a $\mathbb{P}_1$-bundle over M_1 with fibres being the lines l_x.

References

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