

# Higher Supergeometry

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# Outline \*

Motivations and applications

$\mathbb{Z}_2^n$  Berezinian

$\mathbb{Z}_2^n$  Manifolds and  $\mathbb{Z}_2^n$  morphisms

$\mathbb{Z}_2^n$  Batchelor-Gawedzki theorem

$\mathbb{Z}_2^n$  integral calculus

Outlook

\* Joint with T. Covo, J. Grabowski, V. Ovsienko, S. Kwok, ...

# MOTIVATIONS AND APPLICATIONS

# Higher gradings and modified sign rule

Supergeometry: coordinates

$x$  of degree 0

$\xi$  of degree 1

# Higher gradings and modified sign rule

$\mathbb{Z}_2^2$ -Supergeometry: coordinates

$x$  of degree  $(0, 0)$      $y$  of degree  $(1, 1)$

$\xi$  of degree  $(0, 1)$      $\eta$  of degree  $(1, 0)$

# Higher gradings and modified sign rule

$\mathbb{Z}_2^3$ -Supergeometry: coordinates

$x$  of degree  $(0, 0, 0)$      $y$  of degree  $(0, 1, 1)$     ...

$\xi$  of degree  $(0, 0, 1)$      $\eta$  of degree  $(0, 1, 0)$     ...

$$y \cdot \eta = (-1)^{\langle (0,1,1), (0,1,0) \rangle} \eta \cdot y$$

# Higher gradings and modified sign rule

$\mathbb{Z}_2^3$ -Supergeometry: coordinates

$x$  of degree  $(0, 0, 0)$      $y$  of degree  $(0, 1, 1)$     ...

$\xi$  of degree  $(0, 0, 1)$      $\eta$  of degree  $(0, 1, 0)$     ...

$$y \cdot \eta = (-1)^{\langle (0,1,1), (0,1,0) \rangle} \eta \cdot y$$

**New features:**

Even coordinates may anticommute:

$$(-1)^{\langle (1,1,0), (1,0,1) \rangle} = -1$$

Odd coordinates may commute:

$$(-1)^{\langle (1,0,0), (0,1,0) \rangle} = +1$$

Non-zero degree coordinates may not be nilpotent:

$$(-1)^{\langle (1,1,0), (1,1,0) \rangle} = +1$$

# Examples

- Physics:**
- Parastatistical supersymmetry
  - String orbifolds
  - Anyons

- Algebra:**
- Super differential forms  $(n = 2)$

$$\alpha \wedge \beta = (-1)^{\deg(\alpha) \deg(\beta) + p(\alpha) p(\beta)} \beta \wedge \alpha$$

- Quaternions  $\mathbb{H}$   $(n = 3)$
- Clifford algebras  $\mathcal{C}\ell_{p,q}$   $(n = p + q + 1)$

- Geometry:**
- Superized higher vector bundles, e.g.,  $TTM, T^*TM \dots$
  - Tangent bundle of a supermanifold



# Tangent bundle to a supermanifold

Supermanifold  $\mathcal{M} : (x, \xi)$



$$(x, \xi, dx, d\xi) \rightsquigarrow (0, 1, 1 + 0, 1 + 1)$$

+ usual sign rule

$T\mathcal{M} :$  supermanifold

$$C^\infty(x, d\xi)[\xi, dx]$$

$$(x, \xi, dx, d\xi) \rightsquigarrow ((0, 0), (0, 1), (1, 0), (1, 1))$$

+ new sign rule

$T\mathcal{M} :$   $\mathbb{Z}_2^2$ -manifold

$$C^\infty(x)[[\xi, dx, d\xi]]$$

# Coherent differential calculus

$\mathbb{Z}_2^2$ -manifold:

$$(0, 0), (1, 1), (0, 1), (1, 0)$$

$$\phi : \{x, y, \xi, \eta\} \mapsto \{x', y', \xi', \eta'\}$$

$$x' = x + y^2, \quad y' = y, \quad \xi' = \xi, \quad \eta' = \eta$$

$$F(x') = F(x + y^2) = \sum_{\alpha} \frac{1}{\alpha!} (\partial_x^{\alpha} F)(x) y^{2\alpha}$$

Local model:

$$(\mathbb{R}^p, \mathcal{C}^{\infty}(U)[[y, \xi, \eta]])$$

$$\begin{cases} x' = \sum_r f_r^{x'}(x) y^{2r} + \sum_r g_r^{x'}(x) y^{2r+1} \xi \eta \\ y' = \sum_r f_r^{y'}(x) y^{2r+1} + \sum_r g_r^{y'}(x) y^{2r} \xi \eta \\ \xi' = \sum_r f_r^{\xi'}(x) y^{2r} \xi + \sum_r g_r^{\xi'}(x) y^{2r+1} \eta \\ \eta' = \sum_r f_r^{\eta'}(x) y^{2r} \eta + \sum_r g_r^{\eta'}(x) y^{2r+1} \xi \end{cases}$$

# Key-concept

$$\partial_{(x,y,\xi,\eta)}(x', y', \xi', \eta') = \begin{pmatrix} (0,0) & (1,1) & (0,1) & (1,0) \\ (1,1) & (0,0) & (1,0) & (0,1) \\ (1,0) & (0,1) & (0,0) & (1,1) \\ (0,1) & (1,0) & (1,1) & (0,0) \end{pmatrix}$$

Big diagonal blocks: even  $\mathbb{Z}_2^n$ -degrees

Small diagonal blocks: degree zero

# HIGHER TRACE AND BEREZINIAN OF MATRICES OVER A CLIFFORD ALGEBRA

Journal of Geometry and Physics (2012), 62(11), 2294-2319

# Berezinian

$\mathcal{A}$ : supercommutative algebra

## Theorem

$\exists!$  group morphism

$$\text{Ber} : \text{GL}^0(\mathcal{A}) \rightarrow (\mathcal{A}^0)^\times$$

such that

$$\diamond \text{ Ber} \left( \begin{array}{c|c} \mathbf{A} & \\ \hline & \mathbf{D} \end{array} \right) = \det(\mathbf{A}) \det^{-1}(\mathbf{D})$$

$$\diamond \text{ Ber} \left( \begin{array}{c|c} \mathbb{I} & \mathbf{B} \\ \hline \mathbf{C} & \mathbb{I} \end{array} \right) = 1 = \text{Ber} \left( \begin{array}{c|c} \mathbb{I} & \\ \hline \mathbf{C} & \mathbb{I} \end{array} \right)$$

It is defined by

$$\text{Ber} \left( \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right) = \det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) \det^{-1}(\mathbf{D})$$

# $\mathbb{Z}_2^n$ -Berezinian

$\mathcal{A}$ :  $\mathbb{Z}_2^n$ -commutative algebra

## Theorem

$\exists!$  group morphism

$$\mathbb{Z}_2^n \text{Ber} : \text{GL}^0(\mathcal{A}) \rightarrow (\mathcal{A}^0)^\times$$

such that

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \begin{array}{c} \color{orange}{\#} \\ \color{orange}{\#} \\ \color{orange}{\#} \end{array} & \begin{array}{c} \color{blue}{\#} \\ \color{blue}{\#} \\ \color{blue}{\#} \end{array} \\ \hline \color{blue}{\#} & \color{blue}{\#} \end{array} \right) = \color{red}{\det?} \left( \begin{array}{c} \color{orange}{\#} \\ \color{orange}{\#} \\ \color{orange}{\#} \end{array} \right) \color{red}{\det^{-1}?} \left( \begin{array}{c} \color{blue}{\#} \\ \color{blue}{\#} \\ \color{blue}{\#} \end{array} \right)$$

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{I} & \begin{array}{c} \color{green}{\#} \\ \color{green}{\#} \\ \color{green}{\#} \end{array} \\ \hline \color{green}{\#} & \text{I} \end{array} \right) = 1 = \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{I} & \\ \hline \color{green}{\#} & \text{I} \end{array} \right)$$

It is defined by

$$\mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array} \right) = ?$$

# $\mathbb{Z}_2^n$ -Determinant

$\mathcal{A}$ :  $(\mathbb{Z}_2^n)_{\text{even}}$ -commutative algebra

## Theorem

1.  $\exists!$  algebra morphism

$$\mathbb{Z}_2^n \det : \mathfrak{gl}^0(\mathcal{A}) \rightarrow \mathcal{A}^0$$

such that

$$\diamond \quad \mathbb{Z}_2^n \det \left( \begin{array}{ccc|ccc} \star & & & & & \\ \hline & \star & & & & \\ \hline & & \star & & & \\ \hline & & & \star & & \\ \hline & & & & \star & \\ \hline & & & & & \star \end{array} \right) = \prod \det(\star)$$

$$\diamond \quad \mathbb{Z}_2^n \det \left( \begin{array}{ccc|ccc} \mathbb{I} & \star & \star & \star & & \\ \hline & \mathbb{I} & \star & \star & & \\ \hline & & \mathbb{I} & \star & & \\ \hline & & & \mathbb{I} & \star & \\ \hline & & & & \mathbb{I} & \star \end{array} \right) = 1 = \mathbb{Z}_2^n \det \left( \begin{array}{ccc|ccc} \mathbb{I} & & & & & \\ \hline \star & \mathbb{I} & & & & \\ \hline \star & \star & \mathbb{I} & & & \\ \hline \star & \star & \star & \mathbb{I} & & \end{array} \right)$$

2.  $\mathbb{Z}_2^n \det(X)$  is linear in the entries of  $X$

# Quasideterminants (I.Gelfand and V.Retakh)

$$\text{Ber} \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det(A - BD^{-1}C) \det^{-1}(D)$$



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$$\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|_{11} = A - BD^{-1}C$$

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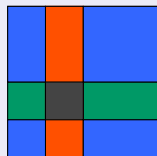
$$\left| \begin{array}{cc} A & B \\ C & D \end{array} \right|_{11} = A - BD^{-1}C$$

## Definition

For a square matrix  $X$  with entries in a ring  $R$ ,

$$|X|_{ij} := x_{ij} - r_i^j (X^{ij})^{-1} c_j^i \in R$$

j-th column



← i-th row

# Examples



$$X = \begin{pmatrix} x & a & b \\ c & y & d \\ e & f & z \end{pmatrix}$$

$$|X|_{11} = x - bz^{-1}e - (a - bz^{-1}f)(y - dz^{-1}f)^{-1}(c - dz^{-1}e)$$



$$X = \left( \begin{array}{cc|c} x & a & b \\ c & y & d \\ e & f & z \end{array} \right)$$

$$|X|_{11} = \begin{pmatrix} x - bz^{-1}e & a - bz^{-1}f \\ c - dz^{-1}e & y - dz^{-1}f \end{pmatrix}$$

Quasi-determinants are **rational** functions

# $\mathbb{Z}_2^n$ -Determinant

$\mathcal{A}$ :  $(\mathbb{Z}_2^n)_{\text{even}}$ -commutative algebra

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$$\diamond \quad \mathbb{Z}_2^n \det \left( \begin{array}{ccc|ccc} \mathbb{I} & \star & \star & \star & & \\ \hline & \mathbb{I} & \star & \star & & \\ \hline & & \mathbb{I} & \star & & \\ \hline & & & \mathbb{I} & \star & \\ \hline & & & & \mathbb{I} & \star \end{array} \right) = 1 = \mathbb{Z}_2^n \det \left( \begin{array}{ccc|ccc} \mathbb{I} & & & & & \\ \hline \star & \mathbb{I} & & & & \\ \hline \star & \star & \mathbb{I} & & & \\ \hline \star & \star & \star & \mathbb{I} & & \end{array} \right)$$

2.  $\mathbb{Z}_2^n \det(X)$  is linear in the entries of  $X$

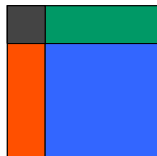
# UDL decomposition

$$X = UDL$$

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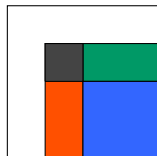
$$D = \begin{pmatrix} |X|_{11} & & & \\ & |X^{1:1}|_{22} & & \\ & & \ddots & \\ & & & X_{qq} \end{pmatrix}$$



# UDL decomposition

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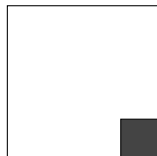




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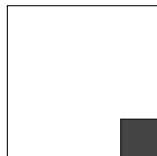
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$$D = \begin{pmatrix} |X|_{11} & & & \\ & |X^{1:1}|_{22} & & \\ & & \ddots & \\ & & & X_{qq} \end{pmatrix}$$



## Theorem

$$\mathbb{Z}_2^n \det(X) = \det(|X|_{11}) \det(|X^{1:1}|_{22}) \dots \det(X_{rr})$$

# $\mathbb{Z}_2^n$ -Berezinian

## Theorem

$\exists!$  group morphism

$$\mathbb{Z}_2^n \text{Ber} : \text{GL}^0(\mathcal{A}) \rightarrow (\mathcal{A}^0)^\times$$

such that

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{III} & \text{III} \\ \hline & \text{III} \end{array} \right) = \begin{pmatrix} \text{III} & \\ & -1(\text{III}) \end{pmatrix}$$

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{II} & \text{III} \\ \hline & \text{II} \end{array} \right) = 1 = \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{II} & \\ \hline \text{III} & \text{II} \end{array} \right)$$

It is defined by

$$\mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) =$$

# $\mathbb{Z}_2^n$ -Berezinian

## Theorem

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such that

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{orange grid} & \text{white grid} \\ \hline \text{white grid} & \text{blue grid} \end{array} \right) = \mathbb{Z}_2^n \det(\text{orange grid}) \mathbb{Z}_2^n \det^{-1}(\text{blue grid})$$

$$\diamond \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{I} & \text{green grid} \\ \hline \text{white grid} & \text{I} \end{array} \right) = 1 = \mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{I} & \text{white grid} \\ \hline \text{green grid} & \text{I} \end{array} \right)$$

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$$\mathbb{Z}_2^n \text{Ber} \left( \begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array} \right) = \mathbb{Z}_2^n \det(A - BD^{-1}C) \mathbb{Z}_2^n \det^{-1}(D)$$

For  $n = 1$ ,  $\mathbb{Z}_2^n \text{Ber} = \text{Ber}$ ; for  $\mathcal{A} = \mathbb{H}$ ,  $\mathbb{Z}_2^n \text{Ber} = |\text{Ddet}|$ ; Liouville formula

# THE CATEGORY OF $\mathbb{Z}_2^n$ -SUPERMANIFOLDS

JOURNAL OF MATHEMATICAL PHYSICS (2016), 57(7)

# Functor of points approach

$$P \in \mathbb{C}[\mathbb{C}^n], \quad V = \{z \in \mathbb{C}^n : P(z) = 0\} \in \mathbf{Aff}, \quad \mathbb{C}[V] = \mathbb{C}[\mathbb{C}^n]/(P) \in \mathbf{CA}$$

$$\mathrm{Sol}_P : \mathbf{CA} \ni A \mapsto \mathrm{Sol}_P(A) = \{a \in A^n : P(a) = 0\} \in \mathbf{Set}$$

$$\mathrm{Sol}_P = \mathrm{Hom}_{\mathbf{CA}}(\mathbb{C}[V], -) \in \mathbf{Fun}(\mathbf{CA}, \mathbf{Set})$$

$$\mathrm{Hom}_{\mathbf{Aff}}(-, V) \in \mathbf{Fun}(\mathbf{Aff}^{\mathrm{op}}, \mathbf{Set})$$

$$\underline{\bullet} : \mathbf{C} \ni c \mapsto \underline{c} := \mathrm{Hom}_{\mathbf{C}}(-, c) \in \mathbf{FunSh}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$$

$$\mathrm{Lim} \underline{c} \simeq \underline{\mathrm{Lim} c}$$

$$\mathrm{Colim} \underline{c} \rightarrow \underline{\mathrm{Colim} c}$$

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$$\mathrm{Lim} \underline{c} \simeq \underline{\mathrm{Lim} c}$$

$$\mathrm{Colim} \underline{c} \rightarrow \underline{\mathrm{Colim} c}$$



# Functor of points approach

Representable  $\text{Sh}(\mathbb{C})$ : trivial spaces

$\text{Sh}(\mathbb{C})$ : spaces

Locally representable  $\text{Sh}(\mathbb{C})$ : varieties or manifolds  $\text{Var}(\mathbb{C})$

Examples:

$\text{Var}(\text{Aff})$ : schemes

$\text{Var}(\mathbb{Z}_2^n\text{-Domain})$ :  $\mathbb{Z}_2^n$ -manifolds

# Locally ringed space approach

## Definition

A  $\mathbb{Z}_2^n$ -manifold is a  $\mathbb{Z}_2^n$ -graded locally ringed space  $(M, \mathcal{A}_M)$  that is *locally modeled on*

$$(\mathbb{R}^p, \mathcal{C}_{\mathbb{R}^p}^\infty(-)[[\xi^1, \dots, \xi^q]]) ,$$

where the  $\xi^a$  are  $\mathbb{Z}_2^n$ -commutative.

# Reconstruction theorem

## Proposition

A topological space that is covered by  $\mathbb{Z}_2^n$ -graded  $\mathbb{Z}_2^n$ -commutative coordinate systems

$$(x, y, \dots, \xi, \eta, \dots)$$

and is endowed with  $\mathbb{Z}_2^n$ -degree preserving coordinate transformations

$$\varphi_{\beta\alpha} : (x, y, \dots, \xi, \eta, \dots) \mapsto (x', y', \dots, \xi', \eta', \dots)$$

that satisfy the cocycle condition

$$\varphi_{\gamma\beta} \varphi_{\beta\alpha} = \varphi_{\gamma\alpha} ,$$

defines a  $\mathbb{Z}_2^n$ -manifold.

# Nilpotency – Formal series

Invertibility of superfunctions:

$$f \in \mathcal{C}^\infty(U)[\xi^1, \dots, \xi^q] \text{ invertible} \Leftrightarrow f_0 \in \mathcal{C}^\infty(U) \text{ invertible}$$

Proof: **Nilpotency**

# Nilpotency – Formal series

Invertibility of  $\mathbb{Z}_2^n$ -functions:

$$f \in \mathcal{C}^\infty(U)[[\xi^1, \dots, \xi^q]] \text{ invertible} \Leftrightarrow f_0 \in \mathcal{C}^\infty(U) \text{ invertible}$$

Proof: **Formal power series**

# $\mathbb{Z}_2^n$ -morphisms

**$\mathbb{Z}_2^n$ -morphism:** ringed space morphism  $(\psi, \psi^*) : (M, \mathcal{A}) \rightarrow (N, \mathcal{B})$

Commutation with base projections:

$$\begin{array}{ccc}
 \mathcal{B}(V) & \xrightarrow{\psi^*} & \mathcal{A}(\psi^{-1}(V)) \\
 \varepsilon_V \downarrow & \circlearrowleft & \downarrow \varepsilon_{\psi^{-1}(V)} \\
 \mathcal{C}_N^\infty(V) & \xrightarrow{\psi^*} & \mathcal{C}_M^\infty(\psi^{-1}(V))
 \end{array}$$

$\psi^*$  is  $C^0$  with respect to the  $\mathcal{J}$ -adic topology,  $\mathcal{J} = \ker \varepsilon$

$\mathcal{A}$  is Hausdorff-complete for the  $\mathcal{J}$ -adic topology

# Fundamental $\mathbb{Z}_2^n$ -Morphism Theorem

## Theorem

A  $\mathbb{Z}_2^n$ -morphism

$$(\psi, \psi^*) : (M, \mathcal{A}_M) \rightarrow (V, \mathcal{C}_V^\infty [[\xi^1, \dots, \xi^q]])$$

is completely and uniquely defined by the pullbacks

$$\psi^* x^i \quad \text{and} \quad \psi^* \xi^a$$

of the base coordinates  $x^i$  and the formal coordinates  $\xi^a$

“The result that makes  $\mathbb{Z}_2^n$ -Geometry a reasonable theory”

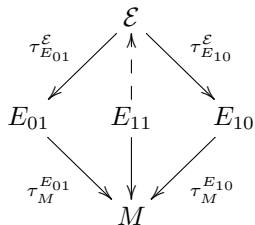
# SPLITTING THEOREM FOR $\mathbb{Z}_2^n$ -MANIFOLDS

JOURNAL OF GEOMETRY AND PHYSICS (2016), 110



# Double vector bundles

Definition 1:



$$E_{11} = \ker \tau_{E_{01}}^E \cap \ker \tau_{E_{10}}^E$$

Trivial example:

$$E_{01} \oplus E_{10} \oplus E_{11}$$

$$\mathcal{E} \simeq E_{01} \oplus E_{10} \oplus E_{11} \quad \text{and} \quad \Gamma(E_{01}^* \otimes E_{10}^* \otimes E_{11})$$

# Double vector bundles

Definition 2: Pair of commuting 'Euler' vector fields on a manifold.

Definition 3: Locally trivial fiber bundle with standard fiber  $V_{01} \oplus V_{10} \oplus V_{11}$  s.th.

$$\begin{cases} \xi'^a &= f_u^a(x)\xi^u \\ \eta'^b &= g_v^b(x)\eta^v \\ y'^c &= h_w^c(x)y^w + k_{u,v}^c(x)\xi^u\eta^v \end{cases}$$

# Split $\mathbb{Z}_2^n$ -manifolds

## Vector bundle:

$$E \rightarrow M \quad \dashrightarrow \quad \Pi E := E[1]$$

$$\mathcal{A}(\Pi E) = \Gamma(\wedge E^*) = \bigoplus_{k=0}^r \Gamma(\odot^k (\Pi E)^*)$$

$$\text{Supermanifold: } \mathcal{M} = (M, \mathcal{A}(\Pi E))$$

## Graded vector bundle:

$$E = E_{01} \oplus E_{10} \oplus E_{11} \rightarrow M \quad \dashrightarrow \quad \Pi E := E_{01}[01] \oplus E_{10}[10] \oplus E_{11}[11]$$

$$\mathcal{A}(\Pi E) = \prod_{k \geq 0} \Gamma(\odot^k (\Pi E)^*)$$

$$\mathbb{Z}_2^2\text{-manifold: } \mathcal{M} = (M, \mathcal{A}(\Pi E))$$

# Non-split $\mathbb{Z}_2^n$ -manifold

**Double vector bundle:  $\mathcal{E}$**

$$\begin{cases} \xi'^a &= f_u^a(x)\xi^u \\ \eta'^b &= g_v^b(x)\eta^v \\ y'^c &= h_w^c(x)y^w + k_{u,v}^c(x)\xi^u\eta^v \end{cases}$$

---> Superization, coherence, cocycle condition

**$\mathbb{Z}_2^2$ -manifold – NOT canonically split**

# $\mathbb{Z}_2^n$ -Batchelor-Gawedzki Theorem

Batchelor, Gawedzki, Kirillov and Rudakov

Smooth and real analytic, but not holomorphic

## Theorem

*Any  $\mathbb{Z}_2^n$ -manifold  $(M, \mathcal{A})$  is non-canonically split, i.e., there exists a non-canonical isomorphism*

$$\mathcal{A} \simeq \mathcal{A}(\Pi E) ,$$

*where  $E$  is a  $\mathbb{Z}_2^n \setminus \{0\}$ -vector bundle.*

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*) \quad ?$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*) \quad ?$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \underset{\text{Sh}}{\simeq} \mathcal{A} \quad ?$$



# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \underset{\text{Stalks}}{\simeq} \mathcal{A}$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \xrightarrow{\text{Sh}} \mathcal{A} \quad ?$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{A}$$

$$\Rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \mathcal{A} \quad ?$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{A}$$

$$\Rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \mathcal{A}$$

$$\Rightarrow 0 \rightarrow \mathcal{J}^2 \rightarrow \mathcal{J} \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow 0 \quad ?$$

# Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\text{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k \geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{C}^\infty \rightarrow 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k \geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{A}$$

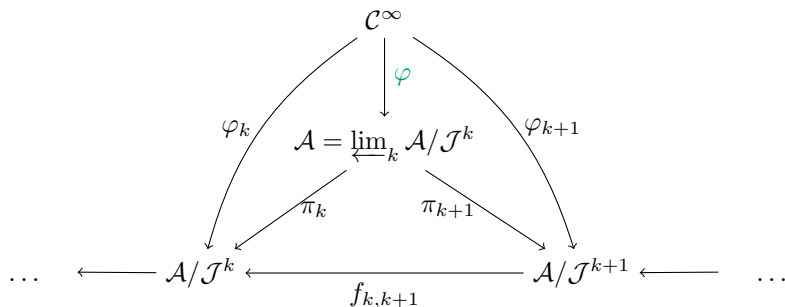
$$\rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \mathcal{A}$$

$$\rightarrow 0 \rightarrow \mathcal{J}^2 \rightarrow \mathcal{J} \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow 0$$

$$\rightarrow \mathcal{C}^\infty \xrightarrow{\varphi} \mathcal{A} \quad ?$$

# Sketch of proof (II)

Step 2:



$$\varphi_{k+1,\Omega} : \mathcal{C}^\infty(\Omega) \rightarrow \mathcal{A}(\Omega)/\mathcal{J}^{k+1}(\Omega)$$

$$\varphi_{k+1,U} : \mathcal{C}^\infty(U) \rightarrow \mathcal{A}(U)/\mathcal{J}^{k+1}(U)$$

$$\varphi_{k+1,U}|_{U \cap V} = \varphi_{k+1,V}|_{U \cap V}$$

## Sketch of proof (III)

$$\omega_{k+1,UV} := \varphi_{k+1,U}|_{U \cap V} - \varphi_{k+1,V}|_{U \cap V}$$

$$\omega_{k+1,UV} \in \text{Der}(\mathcal{C}^\infty(U \cap V), \Gamma(U \cap V, \odot^k(\Pi E)^*))$$

## Sketch of proof (III)

$$\omega_{k+1,UV} := \varphi_{k+1,U}|_{U \cap V} - \varphi_{k+1,V}|_{U \cap V}$$

$$\omega_{k+1,UV} \in \Gamma(U \cap V, TM \otimes \odot^k(\Pi E)^*)$$

$$\omega_{k+1} \in \check{Z}^1 = \check{B}^1$$

$$\varphi_{k+1,U}|_{U \cap V} - \varphi_{k+1,V}|_{U \cap V} = \eta_{k+1,V}|_{U \cap V} - \eta_{k+1,U}|_{U \cap V}$$

$$\varphi_{k+1,U} + \eta_{k+1,U} \text{ consistent (correction of } \varphi_{k+1,U})$$



# COHOMOLOGICAL APPROACH TO THE GRADED BEREZINIAN

Journal of Noncommutative Geometry, 9 (2015), 543–565

# Determinant module

$\mathcal{A}$  a commutative algebra

$M$  a free  $\mathcal{A}$ -module, rank  $r$ , bases  $(e_i), (e'_i)$ ,  $e_j = e'_i B_j^i$

$\text{Det}(M) = \bigwedge^r M$  : rank 1  $\mathcal{A}$ -module with

$$e_1 \wedge \dots \wedge e_r = e'_1 \wedge \dots \wedge e'_r \cdot \det(B)$$

# $\mathbb{Z}_2^n$ -Berezinian module

$\mathcal{A}$  a  $\mathbb{Z}_2^n$ -commutative algebra

$M$  a free  $\mathbb{Z}_2^n$ -graded  $\mathcal{A}$ -module, total rank  $r$ , bases  $(e_i), (e'_i)$ ,  $e_j = e'_i B_j^i$

## Definition

$\mathbb{Z}_2^n \text{Ber}(M)$  is a *rank 1  $\mathcal{A}$ -module* on which  $B \in \text{GL}^0(\mathcal{A})$  acts as  $-\cdot \mathbb{Z}_2^n \text{Ber}(B)$

$\mathbb{Z}_2^n$ -Berezinian module

$$\mathcal{K} = \odot_{\mathcal{A}} \Pi M \otimes \odot_{\mathcal{A}} M^*, \quad d = \sum_i \Pi e_i \varepsilon^i$$

$$H(\mathcal{K}) = H^r(\mathcal{K}) = [\omega] \cdot \mathcal{A}$$

$$\Phi : B \in \mathrm{GL}^0(\mathcal{A}) \mapsto \Phi_B \simeq (B, \mathbb{Z}_2^n t B^{-1}) \in \mathrm{Aut}^0(H(\mathcal{K})) \simeq (\mathcal{A}^0)^\times$$

$$\Phi_B = - \cdot \mathbb{Z}_2^n \mathrm{Ber}(B)$$

$$\omega = \omega' \cdot \mathbb{Z}_2^n \mathrm{Ber}(B)$$

$\omega$  : algebraic  $\mathbb{Z}_2^n$  Berezinian volume

# INTEGRATION ON $\mathbb{Z}_2^n$ -MANIFOLDS

IN PROGRESS – ORBILU [HTTP://HDL.HANDLE.NET/10993/27319](http://hdl.handle.net/10993/27319)

# Local Berezinian section

$$(U, X = (x, y, \xi, \eta)) : ((0, 0), (1, 1), (0, 1), (1, 0))$$

$$M = \Omega^1(\mathcal{M})(U) : (dx, dy, d\xi, d\eta)$$

$$M^* = T(\mathcal{M})(U) : (\partial_x, \partial_y, \partial_\xi, \partial_\eta)$$

# Local Berezinian section

$$(U, X = (x, y, \xi, \eta)) : \quad ((0, 0), (1, 1), (0, 1), (1, 0))$$

$$M = \Omega^1(\mathcal{M})(U) : \quad (dx, dy, d\xi, d\eta)$$

$$M^* = T(\mathcal{M})(U) : \quad (\partial_x, \partial_y, \partial_\xi, \partial_\eta)$$

$$\omega = dx dy \otimes \partial_\xi \partial_\eta$$

# Local Berezinian section

$$(U, X = (x, y, \xi, \eta)) : \quad ((0, 0), (1, 1), (0, 1), (1, 0))$$

$$M = \Omega^1(\mathcal{M})(U) : \quad (dx, dy, d\xi, d\eta)$$

$$M^* = T(\mathcal{M})(U) : \quad (\partial_x, \partial_y, \partial_\xi, \partial_\eta)$$

$$\omega = dx dy \otimes \partial_\xi \partial_\eta$$

$$\omega(X) = \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X)$$

$$\omega(X) f(X) = \omega(X') f(X(X')) \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X) =: \omega(X') f'(X')$$



# Global Berezinian section

## Definition

A **Berezinian section** of a  $\mathbb{Z}_2^n$ -manifold with an oriented base is a family

$$\omega(X)f(X), \omega(X')f'(X'), \dots,$$

whose components transform according to the rule

$$f'(X') = f(X(X')) \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X)$$

# $\mathbb{Z}_2^2$ -integral I

$\beta$ : **Berezinian section supported in a  $\mathbb{Z}_2^2$ -domain**  $X = (x, y, \xi, \eta)$

$$\beta = \omega(X)f(X) = (dx dy \otimes \partial_\xi \partial_\eta) \sum_{k=0}^{\infty} \sum_{a=0}^1 \sum_{b=0}^1 f_{kab}(x) y^k \xi^a \eta^b$$

## Definition

$$\int \beta := \int dx \int dy \partial_\xi \partial_\eta f(x, y, \xi, \eta) =$$

$$\int dx \int dy \sum_{k=0}^{\infty} f_{k11}(x) y^k := \int dx f_{011}(x) \in \mathbb{R}$$

$\beta$ : **arbitrary** Berezinian section  $\rightsquigarrow$  partition of unity

## $\mathbb{Z}_2^N$ -integral II

$\sigma$ : **generalized Berezinian section** supported in a  $\mathbb{Z}_2^N$ -domain  $X = (x, y, \xi, \eta)$

$$\sigma = \omega(X) L(X) = (dx dy \otimes \partial_\xi \partial_\eta) \sum_{k=-N}^{\infty} \sum_{a=0}^1 \sum_{b=0}^1 f_{kab}(x) y^k \xi^a \eta^b$$

### Definition

$$\int \sigma = \int dx \int dy \sum_{k=-N}^{\infty} f_{k11}(x) y^k := \int dx f_{-111}(x) \in \mathbb{R}$$

$\sigma$ : **arbitrary** generalized Berezinian section  $\rightsquigarrow$  partition of unity

# Change-of-variables formula

$\sigma$ : generalized Berezinian section supported in two domains  $\mathcal{U}, \mathcal{U}' (X, X')$

$$\sigma = \omega(X) L(X) = \omega(X') L'(X')$$

# Change-of-variables formula

$\sigma$ : generalized Berezinian section supported in two domains  $\mathcal{U}, \mathcal{U}'$  ( $X, X'$ )

$$\sigma = \omega(X) L(X) = \omega(X') L'(X') = \omega(X') L(X(X')) \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X)$$

# Change-of-variables formula

$\sigma$ : generalized Berezinian section supported in two domains  $\mathcal{U}, \mathcal{U}'$  ( $X, X'$ )

$$\sigma = \omega(X) L(X) = \omega(X') L'(X') = \omega(X') L(X(X')) \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X)$$

## Theorem

$$\int_{\mathcal{U}} \omega(X) L(X) = \int_{\mathcal{U}'} \omega(X') L(X(X')) \mathbb{Z}_2^n \text{Ber}(\partial_{X'} X)$$

## $\mathbb{Z}_2^2$ -integral III

$\Delta$ : **distributional Berezinian section** supported in a  $\mathbb{Z}_2^2$ -domain  $X = (x, y, \xi, \eta)$

$$\Delta = \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) =$$

$$(\mathrm{d}x \mathrm{d}y \otimes \partial_\xi \partial_\eta) \sum_{\ell \leq N} \left( \sum_{k=0}^{\infty} \sum_{a=0}^1 \sum_{b=0}^1 f_{kab;\ell}(x) y^k \xi^a \eta^b \right) \delta^{(\ell)}(y)$$

### Definition

$$\int \Delta = \int \mathrm{d}x \int \mathrm{d}y \sum_{\ell \leq N} \left( \sum_{k=0}^{\infty} f_{k11;\ell}(x) y^k \right) \delta^{(\ell)}(y) =$$

$$\int \mathrm{d}x \int \mathrm{d}y \sum_{\ell \leq N} (-1)^\ell \partial_y^\ell \left( \sum_{k=0}^{\infty} f_{k11;\ell}(x) y^k \right) \delta(y) = \int \mathrm{d}x \sum_{\ell \leq N} (-1)^\ell \ell! f_{\ell 11;\ell}(x) \in \mathbb{R}$$

$$\int \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \stackrel{?}{=} \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi) \Phi^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y)$$



$$\begin{aligned}
\int \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) &\stackrel{?}{=} \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi) \Phi^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial (\Phi_2 \circ \Phi_1)) (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y)
\end{aligned}$$

$$\begin{aligned}
\int \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) &\stackrel{?}{=} \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi) \Phi^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial (\Phi_2 \circ \Phi_1)) (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi_2) \mathbb{Z}_2^n \text{Ber}(\partial \Phi_1) (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y)
\end{aligned}$$

$$\begin{aligned}
\int \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) &\stackrel{?}{=} \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi) \Phi^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial (\Phi_2 \circ \Phi_1)) (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi_2) \mathbb{Z}_2^n \text{Ber}(\partial \Phi_1) (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) \\
&= \int \omega(X') \mathbb{Z}_2^n \text{Ber}(\partial \Phi_2) \mathbb{Z}_2^n \text{Ber}(\partial \Phi_1) \Phi_1^* \Phi_2^* \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y)
\end{aligned}$$

# $\mathbb{Z}_2^n$ -integral IV

Laurent series  $\rightsquigarrow$  generalized fractions  $\rightsquigarrow$   $\mathcal{J}$ -local cohomology classes

# OUTLOOK

# Current and upcoming work

- $\mathbb{Z}_2^n$ -differential calculus ✓  
arXiv:1608.00949
- $\mathbb{Z}_2^n$ -versions of inverse & implicit function, constant rank, Frobenius ✓  
arXiv:1608.00961
- $\mathbb{Z}_2^n$ -integral calculus ✓  
ORBitu:<http://hdl.handle.net/10993/27319>
- Categorical  $\mathbb{Z}_2^n$ -Geometry and Molotkov's work ✓
- Functional analytic issues in  $\mathbb{Z}_2^n$ -Geometry ✓
- Applications in Physics ✓



THANK YOU FOR YOUR ATTENTION