Universal plane curves and Simpson moduli spaces of 1-dimensional sheaves.

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Motivation and abstract

Simpson showed in [7] that for an arbitrary smooth projective variety $\mathfrak{X}$ and for an arbitrary numerical polynomial $P \in \mathbb{Q}[m]$ there is a coarse moduli space $M_P(\mathfrak{X})$ of semi-stable sheaves on $\mathfrak{X}$ with Hilbert polynomial P, which turns out to be a projective variety.
It is known (see [5], [1]) that the universal plane cubic curve may be identified with the fine Simpson moduli space of stable coherent sheaves on $\mathbb{P}_2$ with Hilbert polynomial $3m+1$. It has been shown in [3] that the blowing-up of the universal plane cubic curve along its universal singular locus may be seen as a construction which substitutes the sheaves which are not vector bundles (on their 1-dimensional support) by vector bundles (on support).

One can ask whether there is a similar interpretation of the universal plane curve of arbitrary fixed degree.
It turns out that the universal plane curve M of degree $d \geqslant 3$ is a closed subvariety of codimension $\frac{d(d-3)}{2}$ in the Simpson moduli space of semistable plane sheaves with Hilbert polynomial

$$dm + \frac{d(3-d)}{2} + 1.$$

The universal singular locus M', i. e., the subvariety of the pairs $p \in C$, where C is a plane curve of degree d and p is its singular point, coincides with the subvariety of M consisting of sheaves that are not locally free on their support.
It turns out that the blow up $\mathrm{Bl}_{M'} M$ may be naturally seen as a compactification of $M_B = M \setminus M'$ by vector bundles (on support).

Some general notions

Reminder (Hilbert polynomial & stability)

$\mathfrak{X} \subseteq \mathbb{P}_N$ — a projective variety, $\Longrightarrow$ an ample invertible sheaf $\mathcal{O}_{\mathfrak{X}}(1)$.
$\mathcal{F}$ — coherent sheaf on $\mathfrak{X}$ $\Longrightarrow$

- $P(\mathcal{F},m) = P(m) = \sum\limits_i (-1)^i \cdot \dim H^i(\mathfrak{X}, \mathcal{F} \otimes \mathcal{O}_{\mathfrak{X}}(m)) \in \mathbb{Q}[m]$ *is the **Hilbert polynomial** of $\mathcal{F}$.*

$d = \dim \operatorname{Supp} \mathcal{F} \Longrightarrow$

- $P(\mathcal{F},m) = \sum\limits_{i=0}^{d} \alpha_i(\mathcal{F}) \cdot \frac{m^i}{i!}, \quad \alpha_d(\mathcal{F}) \in \mathbb{Z}, \quad \alpha_d(\mathcal{F}) > 0.$

$\mathcal{F}, \quad \dim(\operatorname{Supp} \mathcal{F}) = d \Longrightarrow$

- *$\mathcal{F}$ is (semi-)stable $\overset{def}{\Longleftrightarrow}$ for every proper subsheaf $\mathcal{E} \subseteq \mathcal{F}$*

$$\alpha_d(\mathcal{F}) \cdot P(\mathcal{E},m) < (\leqslant) \alpha_d(\mathcal{E}) \cdot P(\mathcal{F},m), \quad \text{for } m \gg 0$$

Fix $\mathfrak{X}$, $P \in \mathbb{Q}[m] \Longrightarrow$

Theorem (Carlos Simpson, [7])

There is a moduli space $M = M_P(\mathfrak{X})$ of semi-stable sheaves with Hilbert polynomial P.

- $M = M_P(\mathfrak{X})$ is a projective variety.
- M contains generally speaking sheaves that are not locally free on their support, so called (**singular sheaves**).
- The subvariety $M' \subseteq M$ of singular sheaves is in general non-empty!
- Hence one can consider the complement of the subvariety of singular sheaves $M \setminus M'$ as a space of vector bundles (on support). This space is **non-compact**.
- Since M is a projective variety, it can be seen as a compactification of $M \setminus M'$.
- If $\operatorname{codim}_M M' > 1$ (M' is not a divisor) $\Longrightarrow$ M is not "maximal"

General question

- Find a maximal compactification of $M \setminus M'$.
- Find a compactification of $M \setminus M'$ by vector bundles (on support).

Partial answer

For $\mathfrak{X} = \mathbb{P}_2$, $P(m) = 3m+1$:

- M is the universal plane cubic curve (cf. [5], [1]).
- M' is the universal singular locus, $\operatorname{codim}_M M' = 2$.
- $\widetilde{M} = \mathrm{Bl}_{M'} M$ is an answer to both questions (cf. [4] and [3]).

Another question

- Is there a similar interpretation for the universal curve of any degree d?

Universal plane curve, definition

The space of plane curves of degree d can be identified with the projective space $\mathbb{P}_N$, $N = \frac{(d+2)(d+1)}{2} - 1$.

Indeed, let V be a vector space over $\Bbbk$, $\dim_{\Bbbk} V = 3$. Then $\mathbb{P}_2 = \mathbb{P}V$ is a projective plane. The dual space $V^* = \mathrm{Hom}(V, \Bbbk)$ is identified with $\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. Its d-th symmetric power S^dV^* is naturally identified with $\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. For a fixed basis $\{x_0, x_1, x_2\}$ of V^* (projective coordinates) the space S^dV^* consists of homogeneous polynomials in x_0, x_1, x_2 of degree d. Since a plane curve of degree d is uniquely defined, up to a multiplication by a non-zero constant, by a homogeneous polynomial of degree d in x_0, x_1, x_2, one sees that $\mathbb{P}(S^dV^*)$ is the space of all plane curves of degree d.

Definition (Universal plane curve of degree d)

$M = \{(C,p) \mid p \in C\} = \{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_N \times \mathbb{P}_2 \mid f(x) = 0\}.$

Remark

M is smooth, projective, $\dim M = N + 1 = \frac{(d+2)(d+1)}{2}$.

Parameter space of the universal curve

Let X be the space of matrices $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$, where $z_1, z_2 \in V^*$ are linear forms and $q_1, q_2 \in S^{d-1}V^*$ are forms of degree $d-1$ such that:

- z_1 and z_2 linear independent;
- $\det A \neq 0$

Then X is a quasi-affine variety and can be realized as an open subvariety in $\Bbbk^{d^2+d+6}$.

$$X \xrightarrow{\nu} M, \quad \left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right) \mapsto (\langle z_1q_2 - z_2q_1 \rangle, z_1 \wedge z_2),$$

$z_1 \wedge z_2 \in \mathbb{P}_2$ common zero of z_1 and z_2.

Lemma

- *ν is surjective.*
- *$\nu(A_1) = \nu(A_2) \Longleftrightarrow gA_1h = A_2$ for $g \in \mathrm{GL}_2(\Bbbk)$, $h = \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \Bbbk^*$, q homogeneous of degree $d-2$.*

The universal curve as a geometric quotient

Therefore, one considers the algebraic group

$$G = \mathrm{GL}_2(\Bbbk) \times H,$$

where

$$H = \{\begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix} \mid \lambda, \mu \in \Bbbk^*, q \text{ homogeneous of degree } d-2\}.$$

There is a natural group action

$$G \times X \to X, \quad (g,h) \cdot A = gAh^{-1}.$$

Lemma

M is the orbit space of this action:
points of M are in one to one correspondence with the orbits of the action.

Since every $A \in X$ has the same stabilizer

$$St = \{(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right), (\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)) \mid \lambda \in \Bbbk^*\},$$

one can consider the projectivization $\mathbb{P}G := G/St$ and obtain a free action of $\mathbb{P}G$ on X.

- Locally over M there exists a section $M \supseteq U \to X$ of $X \xrightarrow{\nu} M$.
- Therefore, by Zariski's main theorem X is a $\mathbb{P}G$-principal bundle over M.
- In particular M is a geometric quotient of X.

Universal singular locus

Definition (Universal singular locus)

The subvariety $M' = \{(C,p) \mid p \in \mathrm{Sing}(C)\}$. of the universal curve M is called the universal singular locus.

- M' is smooth,
- $\operatorname{codim}_M M' = 2$.
- For $A \in X$, $\quad A = \left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)$, $\quad p = z_1 \wedge z_2$ the image of A in M belong to M' iff $q_1(p) = q_2(p) = 0$.
- Then $X' \subseteq X$ is the corresponding parameter space of M'.
- X' is a global complete intersection in X (defined by two global equations).

Universal curve as a space of sheaves

- Elements $A \in X$ can be identified with the injections $2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}$.
- Therefore, X is parameter space of sheaves with resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0. \tag{1}$$

- Since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) = 0$, the morphisms of sheaves as in (1) are in one to one correspondence with the morphisms of resolutions.

$$\begin{array}{ccccccccc} 0 & \to & 2\mathcal{O}_{\mathbb{P}_2}(-d+1) & \to & \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} & \to & \mathcal{F}_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 2\mathcal{O}_{\mathbb{P}_2}(-d+1) & \to & \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} & \to & \mathcal{F}_1 & \to & 0. \end{array}$$

$\Longrightarrow$

Proposition

- *Points of $M \overset{1:1}{\longleftrightarrow}$ isomorphism classes of sheaves with (1).*
- *Points of $M' \overset{1:1}{\longleftrightarrow}$ isomorphism classes of singular sheaves with (1)*

Universal curve and Simpson moduli spaces

- All sheaves from M are stable.
- There is a closed embedding

$$M \to M_{dm+\frac{d(3-d)}{2}+1}(\mathbb{P}_2), \quad [\mathcal{F}] \mapsto [\mathcal{F}].$$

- Hence M is a closed subvariety of Simpson moduli space of semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm + \frac{d(3-d)}{2} + 1$ (contained in the stable locus) of codimension $\frac{d(d-3)}{2}$.

Examples for different d

- $d = 3 \Longrightarrow M = M_{3m+1}(\mathbb{P}_2)$.
- $d = 4 \Longrightarrow M \subseteq M_{4m-1}(\mathbb{P}_2), \quad \mathrm{codim} = 2.$
- $d = 5 \Longrightarrow M \subseteq M_{5m-4}(\mathbb{P}_2), \quad \mathrm{codim} = 5.$
- $d = 6 \Longrightarrow M \subseteq M_{6m-8}(\mathbb{P}_2), \quad \mathrm{codim} = 9.$

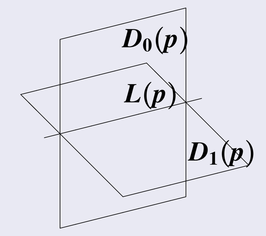
New objects, construction

One dimesional sheaves on surfaces $D(p)$

Surfaces $D(p)$ (subvarieties of $\mathbb{P}_2 \times \mathbb{P}_2$)

Fix $p \in \mathbb{P}_2$. Then $D(p) = D_0(p) \cup D_1(p)$, where

- $D_0(p) = \mathrm{Bl}_p(\mathbb{P}_2)$ is the blowing up of p in $\mathbb{P}_2$, $L(p)$ its exceptional divisor
- $D_1(p) = \mathbb{P}_2$ with $D_0 \cap D_1 = L$.



Picard-Group of $D(p)$

- $D(p) \subseteq \mathbb{P}_2 \times \mathbb{P}_2$, $\mathcal{O}_{\mathbb{P}_2}(1)$ on each $\mathbb{P}_2 \Longrightarrow$
- $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,0) = pr_1^*(\mathcal{O}_{\mathbb{P}_2}(1))$, $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(0,1) = pr_2^*(\mathcal{O}_{\mathbb{P}_2}(1))$
- restrictions to $D(p)$:

$$\mathcal{O}_{D(p)}(1,0) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,0)|_{D(p)}, \quad \mathcal{O}_{D(p)}(0,1) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(0,1)|_{D(p)}.$$

- $\mathrm{Pic}\, D(p) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators:

$$\mathcal{O}_{D(p)}(1,0), \quad \mathcal{O}_{D(p)}(0,1).$$

Sheaves on $D(p)$

For $[\mathcal{F}] \in M'$, $t \in T_{[\mathcal{F}]}M$ there is a way to construct a sheaf $\mathcal{E} = \mathcal{E}([\mathcal{F}], t)$ on $D(p)$ with resolution

$$0 \to 2\mathcal{O}_{D(p)}(-d+2,-1) \xrightarrow{\Phi([\mathcal{F}],\, t)} \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \to \mathcal{E} \to 0. \tag{2}$$

Lemma

$\mathcal{E}([\mathcal{F}], t)$ is locally free on its one-dimensional support iff $B \in T_{[\mathcal{F}]}M \setminus T_{[\mathcal{F}]}M'$.

Definition

R-bundles $\overset{def}{\Longleftrightarrow}$ sheaves $\mathcal{E}([\mathcal{F}], t)$ for $[\mathcal{F}] \in M'$, $t \in T_{[\mathcal{F}]}X \setminus T_{[\mathcal{F}]}M'$.

Remark

- *R-bundles are flat limits of non-singular sheaves defined by points of the universal plane curve.*
- *Morphisms of sheaves $\overset{1:1}{\longleftrightarrow}$ morphisms of resolutions of the type (2).*

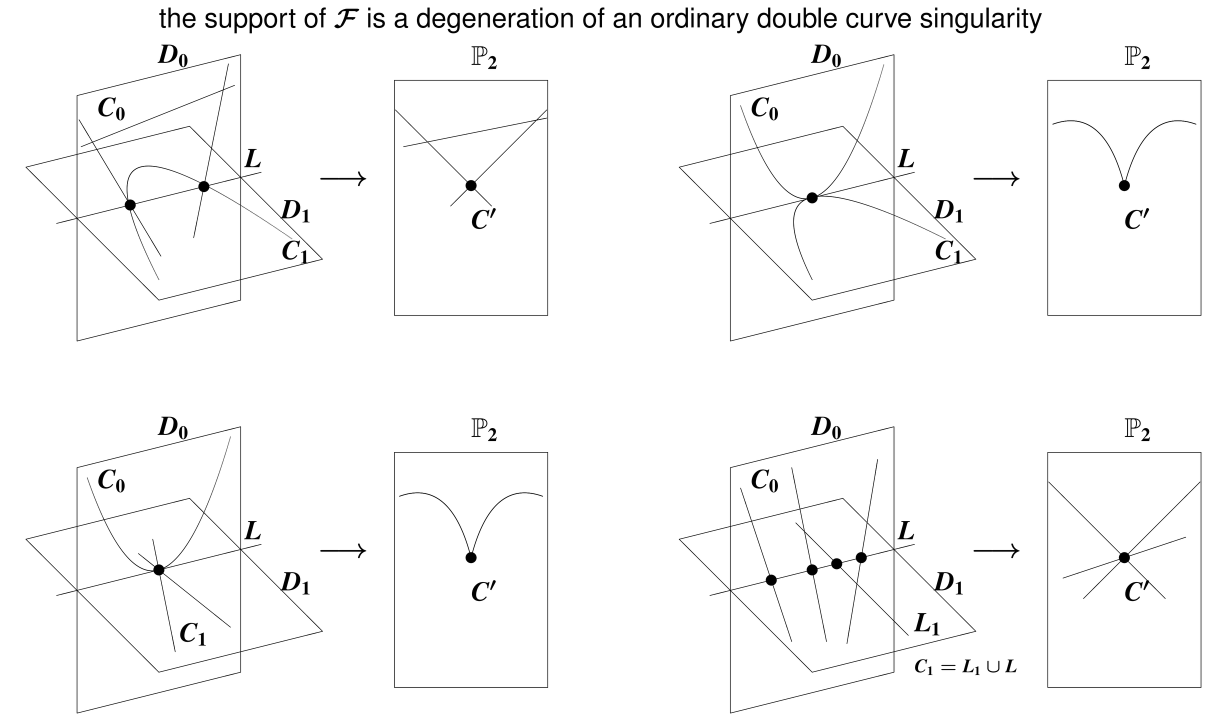
Proposition

R-bundles are supported on reducible curves of the type $C_0 \cup C_1$, where C_0 is a curve in D_0, a partial normalization of the curve $C' = \operatorname{Supp} \mathcal{F}$, i. e., of a plane curve of degree d, and C_1 is a conic in $D_1 = \mathbb{P}_2$.

- *restriction to D_0: structure sheaf of C_0;*
- *restriction to D_1: bundle of degree 1 on C_1.*

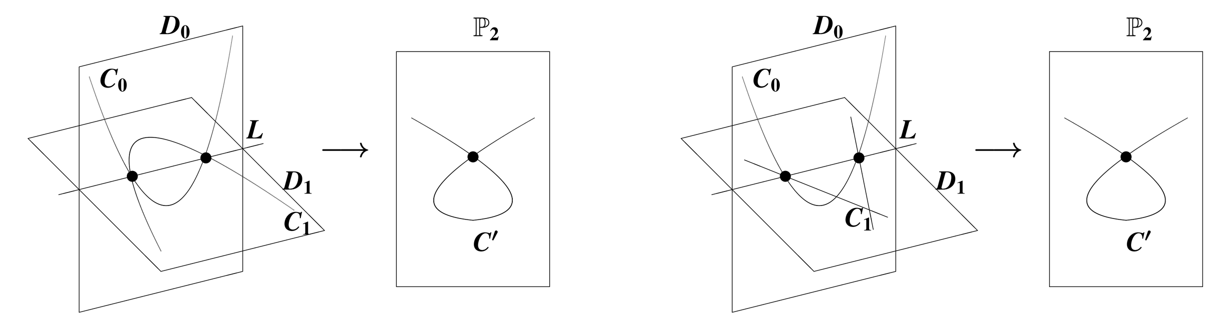
Some illustrations (supports of R-bundles)

Some degeneration cases:
the support of $\mathcal{F}$ is a degeneration of an ordinary double curve singularity



Some illustrations (supports of R-bundles)

Generic case: ordinary double curve singularity of the support of $\mathcal{F}$



Equivalence of R-bundles

Definition (Motivated by Nagaraj and Seshadri, [6])

R-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on $D(p)$ constructed at $[\mathcal{F}] \in M'$
$\mathcal{E}_1$ and $\mathcal{E}_2$ equivalent $\overset{def}{\Longleftrightarrow}$ $\exists$ an automorphism ϕ of $D(p)$, identical on D_0, $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.

Theorem (cf. [2])

Equivalence classes of R-bundles constructed at $[\mathcal{F}] \in M' \overset{1:1}{\leftrightarrow}$ points of $\mathbb{P}N_{[\mathcal{F}]} = \mathbb{P}(T_{[\mathcal{F}]}(M)/T_{[\mathcal{F}]}(M'))$.

Corollary

- *$\widetilde{M} = \mathrm{Bl}_{M'} M$ the space whose points are all the isomorphism classes of non-singular sheaves on $\mathbb{P}_2$ and also all the equivalence classes of R-bundles.*

Idea of the proof.

First of all get tangent equations to M' at $[\mathcal{F}] \in M'$ using the parameter space X and the global defining equations of X' in X.

"$\Longrightarrow$"

Let $\mathcal{E}_1$, $\mathcal{E}_2$ two equivalent R-bundles at $[\mathcal{F}] \in M'$ defined by $t_1, t_2 \in T_{[\mathcal{F}]}M$.

- One obtains resolutions of the type (2)

$$0 \to 2\mathcal{O}_{D(p)}(-d+2,-1) \xrightarrow{\Phi([\mathcal{F}],\, t)} \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \to \mathcal{E}_i \to 0.$$

- Write down the conditions for $\mathcal{E}_1 \sim \mathcal{E}_2$.
- One obtains an automorphism ϕ of $D(p)$ as in the definition above such that $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.
- Since $\phi^*\mathcal{O}_{D(p)}(a,b) \cong \mathcal{O}_{D(p)}(a,b)$, one obtains a resolution of the type (2)

$$0 \to 2\mathcal{O}_{D(p)}(-d+2,-1) \xrightarrow{\Phi([\mathcal{F}],\, t)} \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \to \phi^*(\mathcal{E}_1) \to 0.$$

- Therefore, the isomorphism $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$ induces a commutative diagram

$$\begin{array}{ccc} 2\mathcal{O}_{D(p)}(-d+2,-1) & \longrightarrow & \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \\ \downarrow & & \downarrow \\ 2\mathcal{O}_{D(p)}(-d+2,-1) & \longrightarrow & \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \end{array}$$

- The horizontal arrows depend on t_1 and t_2
- Commutativity of the diagram implies that $t_1 - \lambda t_2$ satisfies the tangent equations of M' for some $\lambda \in \Bbbk^*$.
- Hence $[\bar{t}_1] = [\bar{t}_2]$ in $\mathbb{P}N_{[\mathcal{F}]}$, where $\bar{t}_i \in N_{[\mathcal{F}]} = T_{[\mathcal{F}]}M/T_{[\mathcal{F}]}M'$, $[\bar{t}_i] \in \mathbb{P}N_{[\mathcal{F}]}$.

"$\Longleftarrow$"

If two tangent vectors $t_1, t_2 \in T_{[\mathcal{F}]}M$ define the same point in $\mathbb{P}N_{[\mathcal{F}]}$, then $t_1 - \lambda t_2$ satisfies tangent equations of M' for some $\lambda \in \Bbbk^*$. Using t_1, t_2 and λ one constructs then an automorphism ϕ of $D(p)$ such that $\phi^*(\mathcal{E}([\mathcal{F}], t_1)) \cong \mathcal{E}([\mathcal{F}], t_2)$. □

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