ON SYMBOLIC COMPUTATIONS WITH CHERN CLASSES: REMARKS ON THE LIBRARY CHERN.LIB FOR SINGULAR

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ABSTRACT. We provide an informal overview of the algorithms used for computing with Chern classes in the library `chern.lib` for the computer algebra system SINGULAR.

CONTENTS

INTRODUCTION

The main aim of this note is to present in an informal way the library `chern.lib` [10] for the computer algebra system SINGULAR. We comment on how to perform basic computations with Chern classes without knowing much about vector bundles and complex or algebraic geometry. Though the methods we are using are straightforward and rather obvious for people acquainted with the theory of Chern classes, it may possibly be useful to give an introduction to the theory that is accessible for people with basic mathematical education who are just comfortable enough with polynomials in many variables.

This note was partially motivated by two master theses of my students. The first thesis [17] is an introductory text to the theory of Chern classes. The second one [16] deals in particular with Gröbner bases and elimination theory. The library `chern.lib` can be seen as a link between these two subjects.

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Chern classes are certain cohomological invariants of (complex) vector bundles. Chern classes may be seen as elementary symmetric polynomials in the so called Chern roots. The existence of the Chern roots provides a rather simple theoretical way of computing the Chern classes of vector bundles that were obtained by performing linear algebra operations (such as tensor product, direct sum, etc.) on given vector bundles with known Chern classes.

The formula one is looking for is usually a symmetric expression in the Chern roots of given vector bundles. However, though their existence is granted, the Chern roots of a given vector bundle are almost never explicitly known. From the theoretical point of view it is not a problem since every symmetric polynomial in a given set of variables can be written as a polynomial in the elementary symmetric functions in those variables.

In practice however it is a rather time consuming task to get the required formulas in terms of the Chern classes. Therefore, in order to perform the elimination of the Chern roots, it is reasonable to use a computer.

In Section 1 we give in a very informal way the most important definitions. Sections 2, 3, and 4 concentrate on some computational aspects related to Chern classes. Section 5 briefly addresses the Hirzebruch-Riemann-Roch theorem. In Section 6 the particular situation of complex projective spaces is considered. Appendix A relates the elimination theory and the fundamental theorem of symmetric functions.

Though the elimination approach gives a universal way to work with Chern classes, it is extremely inefficient one since it involves computations of Gröbner bases (see [5] for an informal introduction to the topic by the inventor of the theory). Therefore, if possible, one should use approaches avoiding Gröbner bases computations. This is indeed possible in the cases of Chern characters (Section 2), Todd classes (Section 3), Chern classes of tensor products (Subsection 4.3), second exterior and symmetric powers (Subsection 4.5).

All formulas and computations mentioned in this note are implemented as a library for Singular (cf. [6], [7]). The resulting library `chern.lib` [10] is a basic and by no means complete toolbox for symbolic computations with Chern classes. Though the algorithms from [2] are implemented in the library, we do not discuss them as they require a profound knowledge of algebraic geometry. For other examples of algorithms that could be implemented in the library in the future see [3].

1. Basic definitions

Let

$$A = A^0 \oplus A^1 \oplus \cdots \oplus A^N, \quad A^0 = \mathbb{Z},$$

be a graded commutative ring.

One should think of A as of a ring of invariants associated to a space X. One could think, for example, about the even part of the integer cohomology ring of a complex manifold X of dimension N, i. e.,

$$A = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus \cdots \oplus H^{2N}(X, \mathbb{Z}).$$

If one prefers the category of algebraic varieties (schemes), the ring A may be seen as the Chow ring of an algebraic variety X of dimension N over a field $\Bbbk$.

The ring A is also equipped with a so called degree homomorphism

$$\deg\colon A^N \to \mathbb{Z} \tag{1}$$

of abelian groups, which should be thought about either as a composition of the Poincaré duality isomorphism and the augmentation map (in this case it is an isomorphism)

$$H^{2N}(X,\mathbb{Z}) \xrightarrow{\cong} H_0(X,\mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z}$$

or as the degree homomorphism

$$A^N(X) \to \mathbb{Z}, \quad [\sum_{x\in X} a_x \cdot x] \mapsto \sum_{x\in X} a_x$$

from the highest Chow group of X to $\mathbb{Z}$, which maps the rational equivalence class of a cycle of maximal codimension to its degree.

For a vector bundle E on X there is a way to construct the so called Chern classes $c_i = c_i(E) \in A^i$ (cf. [8]).

Let $r = \operatorname{rank} E$ be the rank of E. Then the splitting principle ensures that possibly after embedding A into a bigger graded ring A' (free module over A), the Chern classes $c_i = c_i(E)$ can be expressed as the elementary symmetric polynomials in certain elements $a_1, \dots, a_r$ of A' of degree 1, so called Chern roots of E.

From now on we shall assume that (for a vector bundle E as above) there is an embedding of graded rings $A \subset A'$ and elements $a_1, \dots, a_r \in A'$ of degree 1 called the Chern roots (of E).

Definition 1.1. For a non-negative integer k, the k-th Chern class c_k is the k-th elementary symmetric polynomial $e_k(a_1, \dots, a_r)$ in $a_1, \dots, a_r$.

The total Chern c class (of E) is defined to be

$$c = (1 + a_1) \cdot (1 + a_2) \dots (1 + a_r) = c_0 + c_1 + c_2 + \cdots + c_r.$$

The higher Chern classes $c_i = c_i(E)$, for $i > r$, are defined to be zero.

The total Chern class can be seen as a map

$$\{\text{vector bundles on } X\} \to A, \quad E \mapsto c(E).$$

It turns out that for an exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

the equality $c(E) = c(E') \cdot c(E'')$ holds true.

The first Chern class turns out to be a homomorphism from the Picard group $\operatorname{Pic}(X)$, i. e, the group of the isomorphism classes of the line bundles on X with multiplication given by $[L_1]\cdot[L_2] = [L_1 \otimes L_2]$, to A^1:

$$\operatorname{Pic}(X) \to A^1, \quad [L] \mapsto c_1(L).$$

This means in particular that the first Chern class of a trivial line bundle is zero, which in turn implies that, for a trivial vector bundle $X \times \Bbbk^r$ of arbitrary rank r on X, all Chern classes (in positive degrees) are zeroes.

Definition 1.2. Let $c_1, \dots, c_r$ be the Chern classes (of a vector bundle E of rank r on a manifold (scheme) X of dimension N). A Chern number is by definition a monomial $c_1^{\alpha_1} \cdot c_2^{\alpha_2} \cdot \dots \cdot c_N^{\alpha_N}$ in $\{c_i\}$ of weighted degree

$$\alpha_1 + 2 \cdot \alpha_2 + \cdots + N \cdot \alpha_N = N,$$

i. e., a monomial which lies in A^N.

Remark 1.3. *There is a one-to-one correspondence between the Chern numbers and the partitions of N. For example, the Chern numbers*

$$c_1^3, \quad c_1c_2, \quad c_3$$

on a 3-fold correspond to the partitions

$$1+1+1=3, \quad 1+2=3, \quad 3=3$$

respectively.

Remark 1.4. *As it will be mentioned in Section 6, on the projective space $\mathbb{P}_n$ the degree homomorphism* $\deg : A^N \to \mathbb{Z}$ *is an isomorphism, hence the Chern numbers are indeed integer numbers.*

Definition 1.5. The Segre class

$$s = s(E) = 1 + s_1 + s_2 + \cdots + s_N \in A$$

is defined by the equality $c(E) \cdot s(E) = 1$, hence

$$\begin{aligned}
&s_0 = 1,\\
&s_1 = -c_1,\\
&s_2 = -c_1 \cdot s_1 - c_2 = c_1^2 - c_2,\\
&s_3 = -c_1 \cdot s_2 - c_2 \cdot s_1 - c_3 = -c_1^3 + 2c_1c_2 - c_3\\
&\quad \dots
\end{aligned}$$

$$s_{k+1} = -(c_1 \cdot s_k + c_2 \cdot s_{k-1} + \cdots + c_{k+1}) = -\sum_{i=0}^{k} c_{i+1}s_{k-i},$$

which gives a recursive formula for the Segre classes.

In other words, the Segre classes are up to a sign the complete homogeneous symmetric functions $h_k(a_1, \dots, a_r)$ in the Chern roots:

$$s_k(a_1, \dots, a_r) = (-1)^k \cdot h_k(a_1, \dots, a_r).$$

Definition 1.6. For a given partition $I = (i_1, \dots, i_m)$, $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m$, the Schur polynomial can be defined either in terms of the Serre classes by the first Jacobi-Trudi (first determinantal) formula

$$S_I = S_I(E) = \det \begin{pmatrix} h_{i_1}(a_1,\dots,a_r) & h_{i_2+1}(a_1,\dots,a_r) & \dots & h_{i_m+m-1}(a_1,\dots,a_r) \\ h_{i_1-1}(a_1,\dots,a_r) & h_{i_2}(a_1,\dots,a_r) & \dots & h_{i_m+m-2}(a_1,\dots,a_r) \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_1-m+1}(a_1,\dots,a_r) & h_{i_2-m+2}(a_1,\dots,a_r) & \dots & h_{i_m}(a_1,\dots,a_r) \end{pmatrix} \tag{2}$$

or, if $\tilde{I} = (j_1, \dots, j_n)$ is the partition dual (conjugate) to I, in terms of the Chern classes by the second Jacobi-Trudi (second determinantal) formula

$$S_I = S_I(E) = \det \begin{pmatrix} e_{j_1}(a_1,\dots,a_r) & e_{j_2+1}(a_1,\dots,a_r) & \dots & e_{j_n+n-1}(a_1,\dots,a_r) \\ e_{j_1-1}(a_1,\dots,a_r) & e_{j_2}(a_1,\dots,a_r) & \dots & e_{j_n+n-2}(a_1,\dots,a_r) \\ \vdots & \vdots & \ddots & \vdots \\ e_{j_1-n+1}(a_1,\dots,a_r) & e_{j_2-n+2}(a_1,\dots,a_r) & \dots & e_{j_n}(a_1,\dots,a_r) \end{pmatrix}. \tag{3}$$

Definition 1.7. The Chern character $\mathrm{ch} = \mathrm{ch}(E)$ is defined by $\mathrm{ch} = \sum_{i=1}^{r} \exp(a_i)$. In other words,

$$\mathrm{ch} = r + \mathrm{ch}_0 + \mathrm{ch}_2 + \cdots + \mathrm{ch}_N,$$

where ch_k is the sum of k-th powers of the Chern roots multiplied by $\frac{1}{k!}$

$$\mathrm{ch}_k = \frac{1}{k!} \cdot p_k(a_1, \ldots, a_r), \quad p_k(a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i^k.$$

Since ch_k is a symmetric polynomial of degree k in the Chern roots $a_1, \ldots, a_r$, one can express them as polynomials in the Chern classes with rational coefficients, hence

$$\mathrm{ch}_k = \mathrm{ch}_k(c_1, \ldots, c_k) \in \mathbb{Q}[c_1, \ldots c_r].$$

One can consider the Chern character as a map

$$\{\text{vector bundles on } X\} \to A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad E \mapsto \mathrm{ch}(E). \tag{4}$$

It turns out that $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$ and $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$ for every exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0.$$

This means that the map (4) factors through a homomorphism of rings

$$K(X) \to A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad [E] \mapsto \mathrm{ch}(E),$$

where $K(X)$ (the Grothendieck ring) is the quotient of the free abelian group generated by vector bundles on X by the subgroup generated by the relations $E' - E + E''$ corresponding to a short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0.$$

The multiplication on $K(X)$ is defined by $[E] \cdot [F] = [E \otimes F]$.

Definition 1.8. The Todd class is defined by

$$\mathrm{td} = \mathrm{td}(a_1, \ldots, a_r) = \prod_{i=1}^{r} \frac{a_i}{1 - \exp(-a_i)}.$$

In other words

$$\mathrm{td} = 1 + \mathrm{td}_1 + \mathrm{td}_2 + \cdots + \mathrm{td}_N,$$

where td_k is the k-th degree term of the product

$$\frac{a_1}{1 - \exp(-a_1)} \cdot \frac{a_2}{1 - \exp(-a_2)} \cdots \frac{a_r}{1 - \exp(-a_r)}$$

and

$$\frac{a}{1 - \exp(-a)} = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot B_i a^i$$

is the exponential generating function for the second Bernoulli numbers $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/6$, etc. The Bernoulli numbers can be computed, for example, by the Akiyama-Tanigawa algorithm (cf. [1],[12]).

Since td_k is a symmetric polynomial of degree k in the Chern roots $a_1, \ldots, a_r$, one can express td_k as polynomials with rational coefficients in the Chern classes $c_1, \ldots, c_k$, hence

$$\mathrm{td}_k = \mathrm{td}_k(c_1, \ldots, c_k) \in \mathbb{Q}[c_1, \ldots c_r].$$

The Todd class can be seen as a map

$$\{\text{vector bundles on } X\} \to A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad E \mapsto \operatorname{td}(E).$$

It turns out that $\operatorname{td}(E) = \operatorname{td} E' \cdot \operatorname{td} E$ for every exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

which gives a homomorphism of the additive group of $K(X)$ to the multiplicative group of $A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

$$\operatorname{td} : K(X) \to A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad [E] \mapsto \operatorname{td}(E).$$

Definition 1.9. The Chern classes, Chern numbers, Chern character, Todd class of X are by definition just the Chern classes, Chern numbers, Chern character, Todd class of the tangent bundle on X.

2. Computation of the Chern character

Since the sum of k-th powers of the Chern roots is a symmetric polynomial in $a_1, \dots, a_r$ of degree k, it can be expressed as a polynomial in $c_1, \dots, c_k$.

Though the computation of $\operatorname{ch}_k(c_1, \dots, c_r)$ can be done using elimination as indicated in Appendix A, one can use a faster approach.

We compute the polynomials $\operatorname{ch}_k(c_1, \dots, c_r) = \frac{1}{k!} \cdot p_k(c_1, \dots, c_k)$ using the Newton's identities:

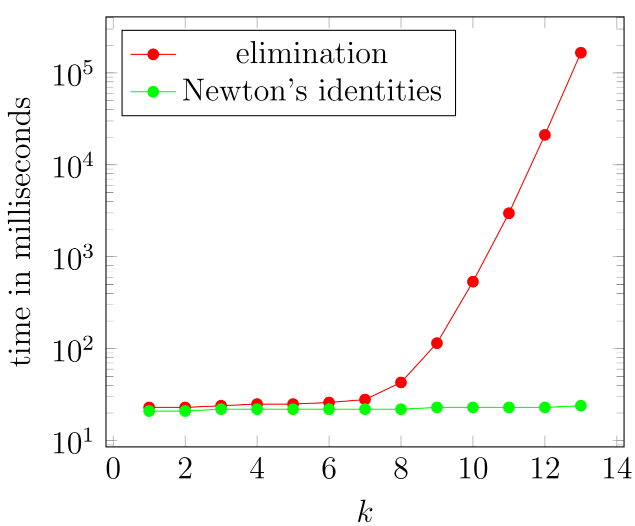
$$p_{k+1} = c_1 \cdot p_k - c_2 \cdot p_{k-1} + \cdots + (-1)^k (k+1) c_{k+1}, \quad k \in \mathbb{Z}_{\geqslant 0}. \tag{5}$$

In particular

$$\begin{aligned}
p_1 &= c_1, \\
p_2 &= c_1 \cdot p_1 - 2c_2 = c_1^2 - 2c_2, \\
p_3 &= c_1 \cdot p_2 - c_2 \cdot p_1 + 3c_3 = c_1^3 - 3c_1c_2 + 3c_3, \\
p_4 &= c_1 \cdot p_3 - c_2 \cdot p_2 + c_3 \cdot p_1 - 4c_4 = c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4.
\end{aligned}$$

Since $p_k(c_1, \dots, c_k) \in \mathbb{Z}[c_1, \dots, c_k]$, one sees that $\operatorname{ch}_k(c_1, \dots, c_k) \in \mathbb{Q}[c_1, \dots, c_k]$.

Computation of the components ch_k of the Chern character



3. Computation of the Todd class

Though the computation of the Todd class can be done using elimination as described in Appendix A, it is faster to do it recursively.

Let $c_{1,i}, \dots, c_{i,i}$ be the Chern classes in case $r = i$. Put $c_{j,i} = 0$ for $j > i$. Let

$$\mathrm{td}_{k,i} = \mathrm{td}_{k,i}(c_{1,i}, \dots, c_{k,i}) \in \mathbb{k}[c_{1,i}, \dots c_{k,i}]$$

be the expression for the k-th degree term of the Todd class corresponding to $r = i$.

For example, since

$$\frac{a}{1-\exp(-a)} = 1 + \frac{1}{2} \cdot a + \frac{1}{12} \cdot a^2 - \frac{1}{720} \cdot a^4 + \dots,$$

taking into account that for $r = 1$ only the first Chern class $c_{1,1} = a$ is non-zero, one obtains

$$\mathrm{td}_{1,1} = \frac{1}{2} \cdot c_{1,1}, \quad \mathrm{td}_{2,1} = \frac{1}{12} \cdot c_{1,1}^2, \quad \mathrm{td}_{3,1} = 0, \quad \mathrm{td}_{4,1} = -\frac{1}{720} \cdot c_{1,1}^4.$$

Assume that for some positive integer i we have all the expressions for

$$\mathrm{td}_{k,i} \in \mathbb{k}[c_{1,i}, \dots, c_{k,i}].$$

In this case there are i Chern roots $a_1, \dots, a_i$ and

$$1 + \mathrm{td}_{1,i} + \dots + \mathrm{td}_{2,i} + \dots = \frac{a_1}{1-\exp(-a_1)} \cdot \frac{a_2}{1-\exp(-a_2)} \cdots \frac{a_i}{1-\exp(-a_i)}$$

Assume now, we have $i+1$ Chern roots $a_1, \dots, a_i, a_{i+1} = a$. Then

$$\begin{aligned} 1 + \mathrm{td}_{1,i+1} + \dots + \mathrm{td}_{2,i+1} + \dots = & \frac{a_1}{1-\exp(-a_1)} \cdots \frac{a_i}{1-\exp(-a_i)} \cdot \frac{a}{1-\exp(-a)} = \\ & (1 + \mathrm{td}_{1,i} + \dots + \mathrm{td}_{2,i} + \dots) \cdot \frac{a}{1-\exp(-a)} = \\ & (1 + \mathrm{td}_{1,i} + \dots + \mathrm{td}_{2,i} + \dots) \cdot (\sum_{i=0}^{\infty} b_i a^i), \quad b_i = \frac{1}{i!} \cdot B_i. \end{aligned}$$

Therefore,

$$\mathrm{td}_{k,i+1} = \sum_{j=0}^{k} b_j \cdot a^j \cdot \mathrm{td}_{k-j,i}(c_{1,i}, \dots, c_{j,i}). \tag{6}$$

However we are interested in getting a formula in $c_{1,i+1}, \dots, c_{k,i+1}$.

Notice that

$$c_{1,i+1} = c_{1,i} + a, \quad c_{2,i+1} = c_{2,i} + a \cdot c_{1,i}, \quad c_{3,i+1} = c_{3,i} + a \cdot c_{2,i}, \dots$$

This implies

$$\begin{aligned} & c_{1,i} = c_{1,i+1} - a, \\ & c_{2,i} = c_{2,i+1} - a \cdot c_{1,i} = c_{2,i+1} - a \cdot c_{1,i+1} + a^2, \\ & c_{3,i} = c_{3,i+1} - a \cdot c_{2,i} = c_{3,i+1} - a \cdot c_{2,i+1} + a^2 \cdot c_{1,i+1} - a^3 \\ & \dots \\ & c_{k,i} = c_{k,i+1} - a \cdot c_{k-1,i+1} + \dots + (-1)^k \cdot a^k \end{aligned} \tag{7}$$

Moreover, since $c_{i+1,i} = 0$, it holds

$$c_{i+1,i+1} = a \cdot c_{i,i+1} - a^2 \cdot c_{i-1,i+1} + \cdots + (-1)^i a^i. \tag{8}$$

Finally substituting in (6) $c_{\nu,i}$ by their expressions from (7) and eliminating if necessary the Chern root a using (8), we obtain the expression for $\mathrm{td}_{k,i+1} \in \mathbb{k}[c_{1,i+1}, \dots, c_{k,i+1}]$.

$$\mathrm{td}_{k,i+1}(c_{1,i+1}, \dots, c_{k,i+1}) = \sum_{j=0}^{k} b_j \cdot a^j \cdot \mathrm{td}_{k-j,i}(c_{1,i+1} - a, c_{2,i+1} - a \cdot c_{1,i+1} + a^2, \dots). \tag{9}$$

In particular

$$\mathrm{td}_{1,i+1}(c_{1,i+1}) = \mathrm{td}_{1,i}(c_{1,i+1} - a) + b_1 a$$
$$\mathrm{td}_{2,i+1}(c_{1,i+1}, c_{2,i+1}) = \mathrm{td}_{2,i}(c_{1,i+1} - a, c_{2,i+1} - ac_{1,i+1} + a^2) + b_1 a\, \mathrm{td}_{1,i}(c_{1,i+1} - a) + b_2 a^2$$

Remark 3.1. *Notice that* (9) *implies that*

$$\mathrm{td}_{k,k}(c_1, \dots, c_k) = \mathrm{td}_{k,i}(c_1, \dots, c_k), \quad i > k.$$

Indeed, it is enough to evaluate both sides of (9) *at* $a = 0$*.*

Example. Let us compute $\mathrm{td}_{3,3}$.

1) One starts with

$$\mathrm{td}_{1,1} = \frac{1}{2} \cdot c_{1,1}, \quad \mathrm{td}_{2,1} = \frac{1}{12} \cdot c_{1,1}^2, \quad \mathrm{td}_{3,1} = 0.$$

2) As mentioned in the Remark above, $\mathrm{td}_{1,2}(c_1) = \mathrm{td}_{1,1}(c_1)$. We compute

$$\mathrm{td}_{2,2}(c_1, c_2) = \mathrm{td}_{2,1}(c_1 - a, c_2 - ac_1 + a^2) + b_1 a \cdot \mathrm{td}_{1,1}(c_1 - a) + b_2 a^2 = \frac{1}{12}(c_1^2 + c_2),$$

and also

$$\mathrm{td}_{3,2}(c_1, c_2, c_3) = \mathrm{td}_{3,1}(c_1 - a, c_2 - ac_1) + \frac{1}{2} a\, \mathrm{td}_{2,1}(c_1, c_2) - \frac{1}{12} a^2\, \mathrm{td}_{1,1}(c_1) + a^3 =$$
$$\frac{1}{24} ac_1^2 - \frac{1}{24} a^2 c_1 = \frac{1}{24} c_1 a(c_1 - a) = \frac{1}{24} c_1 c_2.$$

3) Finally one obtains

$$\mathrm{td}_{1,3}(c_1) = \mathrm{td}_{1,2}(c_1) = \mathrm{td}_{1,1}(c_1) = \frac{1}{2} c_1, \quad \mathrm{td}_{2,3}(c_1, c_2) = \mathrm{td}_{2,2}(c_1, c_2) = \frac{1}{12}(c_1^2 + c_2),$$

and

$$\mathrm{td}_{3,3}(c_1, c_2, c_3) = \frac{1}{24}(c_1 - a)(c_2 - ac_1 + a^2) + \frac{1}{24} a((c_1 - a)^2 + (c_2 - ac_1 + a^2)) + \frac{1}{24} a^2 (c_1 - a) =$$
$$\frac{1}{24}(c_1 c_2 - ac_1^2 + c_1 a^2 - ac_2 + a^2 c_1 - a^3) + \frac{1}{24} a(c_1^2 - 2c_1 a + a^2 + c_2 - ac_1 + a^2) + \frac{1}{24}(a^2 c_1 - a^3) =$$
$$\frac{1}{24} c_1 c_2.$$

Using this approach one gets

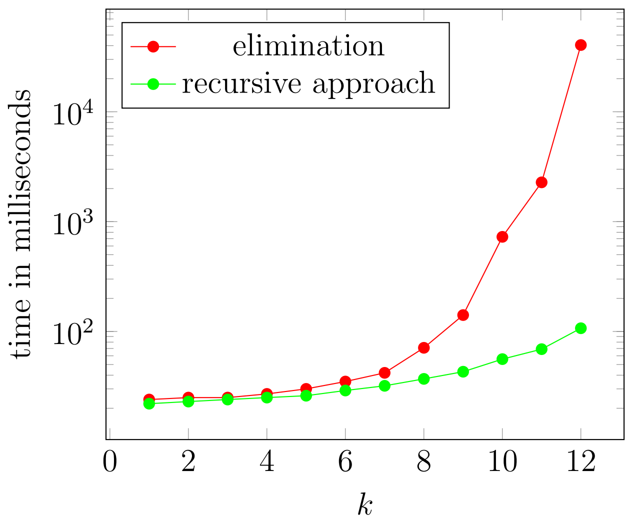
$$\mathrm{td}_1 = \frac{1}{2}\cdot c_1, \quad \mathrm{td}_2 = \frac{1}{12}\cdot(c_1^2+c_2), \quad \mathrm{td}_3 = \frac{1}{24}\cdot c_1c_2, \quad \mathrm{td}_4 = -\frac{1}{720}(c_1^4-4c_1^2c_2-3c_2^2-c_1c_3+c_4),$$

$$\mathrm{td}_5 = -\frac{1}{1440}(c_1^3c_2-3c_1c_2^2-c_1^2c_3+c_1c_4),$$

$$\mathrm{td}_6 = \frac{1}{60480}\cdot(2c_1^6-12c_1^4c_2+11c_1^2c_2^2+5c_1^3c_3+10c_2^3+11c_1c_2c_3-5c_1^2c_4-c_3^2-9c_2c_4-2c_1c_5+2c_6),$$

$$\mathrm{td}_7 = \frac{1}{120960}\cdot(2c_1^5c_2-10c_1^3c_2^2-2c_1^4c_3+10c_1c_2^3+11c_1^2c_2c_3+2c_1^3c_4-c_1c_3^2-9c_1c_2c_4-2c_1^2c_5+2c_1c_6).$$

Computation of the components td_k of the Todd class



4. Chern classes and linear algebra of vector bundles

4.1. Extensions of vector bundles. Let

$$0 \to E' \to E \to E'' \to 0$$

be an extension of vector bundles. Let $a_1, \dots, a_r$ be the Chern roots of E' and let $b_1, \dots, b_R$ be the Chern roots of E''. Then $a_1, \dots, a_r, b_1, \dots, b_R$ are the Chern roots of E.

As mentioned in Definition 1.1, $c(E) = c(E')\cdot c(E'')$ and hence

$$c_k(E) = \sum_{i=0}^{k} c_i(E')\cdot c_{k-i}(E'').$$

4.2. Dual vector bundle. Let E be a vector bundle with Chern roots $a_1, \dots, a_r$. Then $-a_1, \dots, -a_r$ are the Chern roots of the dual vector bundle $E^\vee$ and thus

$$c_i(E^\vee) = (-1)^i \cdot c_i(E). \tag{10}$$

4.3. Tensor products of vector bundles. Let E be a vector bundle with Chern roots $a_1, \dots, a_r$, let F be a vector bundle with Chern roots $b_1, \dots, b_s$.

The set of the Chern roots of the tensor product $E \otimes F$ coincides with the set of sums

$$a_i + b_j, \quad 1 \leqslant i \leqslant r, \quad 1 \leqslant j \leqslant s,$$

and hence the total Chern class of $E \otimes F$ equals

$$c(E \otimes F) = \prod_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s} (1 + a_i + b_j).$$

Eliminating a_i and b_j, one gets the expression for $c(E \otimes F)$ in terms of the Chern classes of E and F (cf. Appendix A).

We compute $c(E \otimes F)$ using the equality $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$, and the Newton's identities (5) rewritten in order to define the Chern classes in terms of the Chern characters. This recursive approach turns out to be faster than elimination of variables.

Rewrite (5) as

$$c_{k+1} = \frac{1}{k+1}(c_k \cdot \mathrm{ch}_1 - c_{k-1} \cdot \mathrm{ch}_2 + \cdots + (-1)^k \cdot \mathrm{ch}_{k+1}). \tag{11}$$

In particular

$$\begin{aligned}
c_1 &= \mathrm{ch}_1, \\
c_2 &= \frac{1}{2}(c_1 \cdot \mathrm{ch}_1 - \mathrm{ch}_2), \\
c_3 &= \frac{1}{3}(c_2 \cdot \mathrm{ch}_1 - c_1 \cdot \mathrm{ch}_2 + \mathrm{ch}_3), \\
c_4 &= \frac{1}{4}(c_3 \cdot \mathrm{ch}_1 - c_2 \cdot \mathrm{ch}_2 + c_1 \cdot \mathrm{ch}_3 - \mathrm{ch}_4), \\
&\ldots
\end{aligned}$$

Now let E be a vector bundle of rank r with Chern classes $c_1, c_2, \ldots$ and let F be a vector bundle of rank R with Chern classes $C_1, C_2, \ldots$.

We obtain the following formulas for the first four Chern classes of $E \otimes F$.

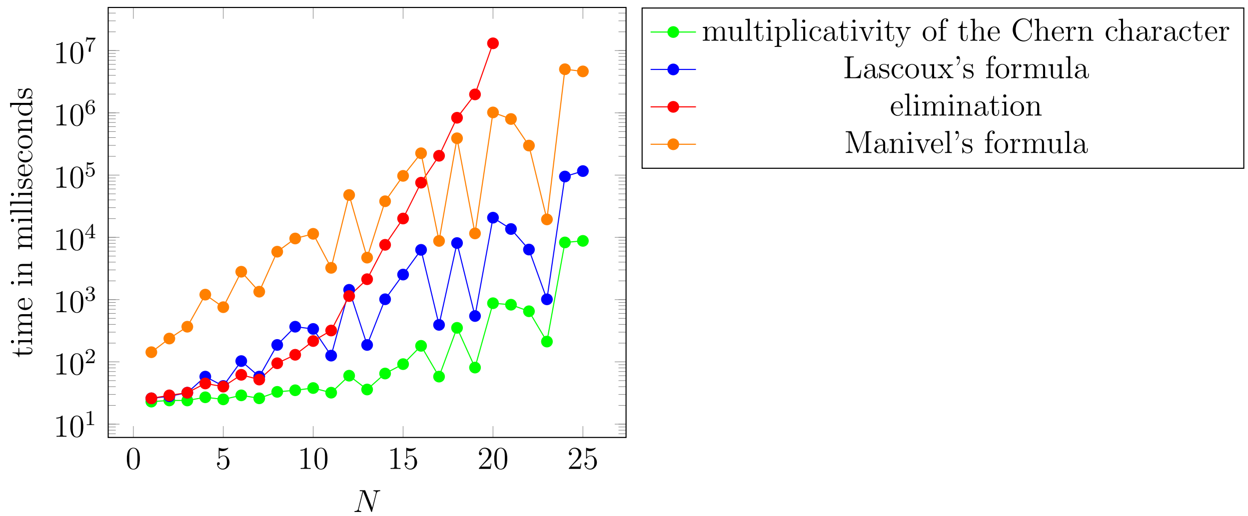
$$
\begin{aligned}
c_1(E\otimes F) =& R\cdot c_1 + r\cdot C_1\\
c_2(E\otimes F) =& \frac{1}{2}R(R-1)\cdot c_1^2 + (rR-1)\cdot c_1C_1 + \frac{1}{2}r(r-1)\cdot C_1^2 + R\cdot c_2 + r\cdot C_2\\
c_3(E\otimes F) =& \frac{1}{6}R(R-1)(R-2)\cdot c_1^3 + \frac{1}{6}r(r-1)(r-2)\cdot C_1^3 +\\
& \frac{1}{2}(r-1)(rR-2)\cdot c_1^2C_1 + \frac{1}{2}(R-1)(rR-2)\cdot c_1C_1^2 +\\
& R(R-1)\cdot c_1c_2 + r(r-1)\cdot C_1C_2 + (rR-2)\cdot c_2C_1 + (rR-2)\cdot c_1C_2 +\\
& R\cdot c_3 + r\cdot C_3\\
c_4(E\otimes F) =& \frac{1}{24}R(R-1)(R-2)(R-3)\cdot c_1^4 + \frac{1}{24}r(r-1)(r-2)(r-3)\cdot C_1^4 +\\
& \frac{1}{6}(R-1)(R-2)(rR-3)\cdot c_1^3C_1 + \frac{1}{6}(r-1)(r-2)(rR-3)\cdot c_1C_1^3 +\\
& \frac{1}{4}(r-1)(R-1)(rR-4)\cdot c_1^2C_1^2 +\\
& \frac{1}{2}R(R-1)(R-2)\cdot c_1^2c_2 + \frac{1}{2}r(r-1)(r-2)\cdot C_1^2C_2 +\\
& (R-1)(rR-3)\cdot c_1c_2C_1 + (r-1)(rR-3)\cdot c_1C_1C_2 +\\
& \frac{1}{2}(r^2R-rR-4r+6)\cdot c_2C_1^2 + \frac{1}{2}(rR^2-rR-4R+6)\cdot c_1^2C_2 +\\
& \frac{1}{2}R(R-1)\cdot c_2^2 + \frac{1}{2}r(r-1)\cdot C_2^2 +\\
& R(R-1)\cdot c_1c_3 + r(r-1)\cdot C_1C_3 + (rR-3)\cdot c_3C_1 + (rR-3)\cdot c_1C_3 + (rR-6)\cdot c_2C_2 +\\
& R\cdot c_4 + r\cdot C_4
\end{aligned}
$$

Remark 4.1. *Notice that A. Lascoux in* [13] *provides an explicit formula for* $c(E\otimes F)$ *in terms of Schur polynomials of* E *and* F*. This formula involves a sum over partitions and, though it turns out to be more efficient than the elimination approach, it is much slower that the approach using the multiplicativity of the Chern character.*

Remark 4.2. *A variation of the Lascoux's formula involving Littlewood-Richardson coefficients (cf.* [14]*) is given by L. Manivel in* [15]*. Our implementation of the Manivel's formula uses the Littlewood-Richardson calculator by Anders Koch* [4] *(via the interface* [11] *for* SINGULAR*) and turns out to be the slowest one for small ranks. However it is more efficient than elimination for big ranks.*

The diagram below shows the times needed to compute the Chern classes of all possible tensor products of vector bundles E and F such that $\mathrm{rank}(E\otimes F) = N$.

Computation of the Chern classes of tensor product



4.4. *Hom*-**vector bundles.** Let E and F be vector bundles as in 4.3. In order to compute the Chern classes of $\mathrm{Hom}(E,F)$ in terms of the Chern classes of E and F, it is enough to consider the canonical isomorphism

$$\mathrm{Hom}(E,F) \cong E^\vee \otimes F$$

and to combine (10) with 4.3.

The set of the Chern roots of $\mathrm{Hom}(E,F) \cong E^\vee \otimes F$ coincides with the set of sums

$$-a_i + b_j, \quad 1 \leqslant i \leqslant r, \quad 1 \leqslant j \leqslant s,$$

and hence the total Chern class of $E \otimes F$ equals

$$c(\mathrm{Hom}(E,F)) = \prod_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s} (1 - a_i + b_j).$$

4.5. **Symmetric and exterior powers.** Assume we deal with a vector bundle with Chern roots $a_1, \dots, a_r$, Chern classes $c_1, \dots, c_r$ and hence with the total Chern class

$$c = (1 + a_1) \cdot (1 + a_2) \dots (1 + a_r) = c_0 + c_1 + c_2 + \dots + c_r.$$

Let k be a positive integer.

4.5.1. *Symmetric powers.* It follows that the set of Chern roots of the symmetric power $S^k E$ coincides with the set of sums

$$a_{i_1} + a_{i_2} + \dots + a_{i_k}, \quad 1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant r.$$

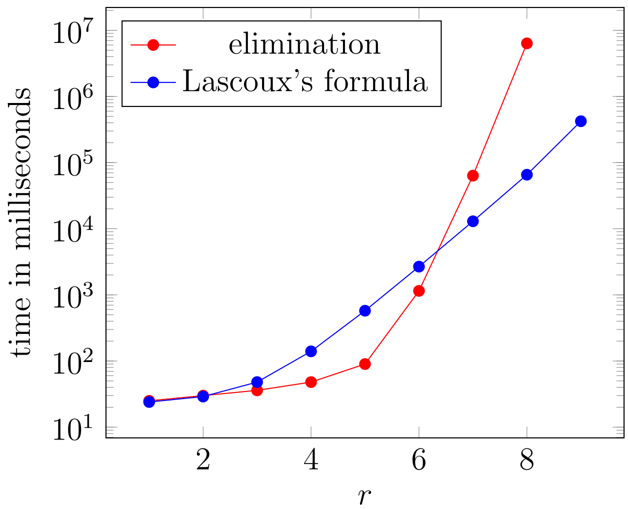
The total Chern class of $S^k E$ is then given by the product

$$c(S^k E) = \prod_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant r} (1 + a_{i_1} + a_{i_2} + \dots + a_{i_k}).$$

Eliminating the Chern roots gives the formula in terms of the Chern classes $c_1, \dots, c_r$ (cf. Appendix A).

Remark 4.3. *Notice that an explicit formula for the second symmetric power in terms of Schur polynomials is given by A. Lascoux in* [13]. *For higher ranks r it is more efficient than the elimination approach.*

Computation of the Chern classes of the second symmetric power



As an example we compute the Chern classes of some symmetric powers of a vector bundle E of rank 2 with Chern classes c_1 and c_2.

$$\begin{aligned}
c_1(S^2E) &= 3c_1,\\
c_2(S^2E) &= 2c_1^2 + 4c_2,\\
c_3(S^2E) &= 4c_1c_2;\\
\\
c_1(S^3E) &= 6c_1,\\
c_2(S^3E) &= 11c_1^2 + 10c_2,\\
c_3(S^3E) &= 6c_1^3 + 30c_1c_2,\\
c_4(S^3E) &= 18c_1^2c_2 + 9c_2^2;\\
\\
c_1(S^4E) &= 10c_1,\\
c_2(S^4E) &= 35c_1^2 + 20c_2,\\
c_3(S^4E) &= 50c_1^3 + 120c_1c_2,\\
c_4(S^4E) &= 24c_1^4 + 208c_1^2c_2 + 64c_2^2,\\
c_5(S^4E) &= 96c_1^3c_2 + 128c_1c_2^2.
\end{aligned}$$

4.5.2. *Exterior powers.* The set of Chern roots of the exterior power $\bigwedge^k E$ coincides with the set of sums

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k}, \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant r.$$

The total Chern class of $\bigwedge^k E$ is then given by the product

$$c(\bigwedge^k E) = \prod_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant r} (1 + a_{i_1} + a_{i_2} + \cdots + a_{i_k}).$$

Eliminating the Chern roots gives the formula in terms of the Chern classes $c_1, \ldots, c_r$ (cf. Appendix A).

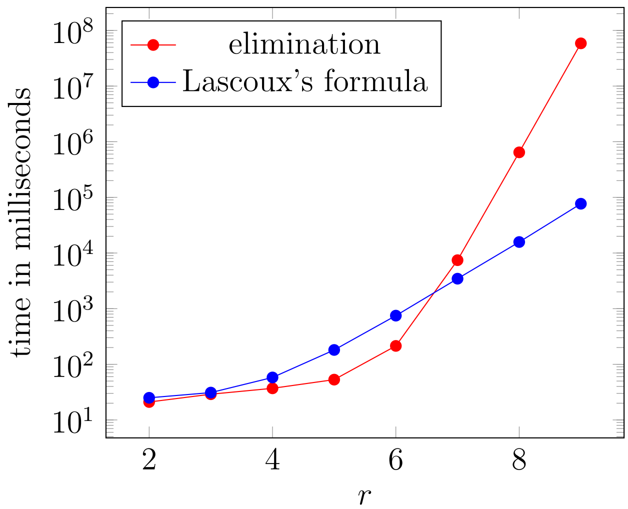
As an example we compute the exterior powers of a vector bundle E of rank 4 with Chern classes c_1, c_2, c_3, c_4.

$$\begin{aligned}
c_1(\bigwedge^2 E) &= 3c_1, \\
c_2(\bigwedge^2 E) &= 3c_1^2 + 2c_2, \\
c_3(\bigwedge^2 E) &= c_1^3 + 4c_1c_2, \\
c_4(\bigwedge^2 E) &= 2c_1^2c_2 + c_2^2 + c_1c_3 - 4c_4, \\
c_5(\bigwedge^2 E) &= c_1c_2^2 + c_1^2c_3 - 4c_1c_4, \\
c_6(\bigwedge^2 E) &= c_1c_2c_3 - c_1^2c_4 - c_3^2;
\end{aligned}$$

$$\begin{aligned}
c_1(\bigwedge^3 E) &= 3c_1, \\
c_2(\bigwedge^3 E) &= 3c_1^2 + c_2, \\
c_3(\bigwedge^3 E) &= c_1^3 + 2c_1c_2 - c_3, \\
c_4(\bigwedge^3 E) &= c_1^2c_2 - c_1c_3 + c_4,
\end{aligned}$$

Remark 4.4. *Notice that an explicit formula for the second exterior power in terms of Schur polynomials is given by A. Lascoux in* [13]*. For higher ranks r it is more efficient than the elimination approach.*

Computation of the Chern classes of the second exterior power



5. Euler Characteristic and the Hirzebruch-Riemann-Roch theorem

For a vector bundle on E on a smooth projective variety (manifold) X, its Euler characteristic $\chi(E)$ is by definition the alternate sum of the dimensions of its cohomology groups

$$\chi(E) = \sum_i (-1)^i \cdot \dim H^i(X, E).$$

Let T be the tangent bundle on X. Let as above N be the dimension of X. The Hirzebruch-Riemann-Roch theorem gives a formula for $\chi(E)$

$$\chi(E) = \deg(\mathrm{ch}(E) \cdot \mathrm{td}(T))_N. \tag{12}$$

Here $(\mathrm{ch}(E)\cdot\mathrm{td}(T))_N \in A^N \otimes_{\mathbb{Z}} \mathbb{Q}$ denotes the component of N-th degree of the product $\mathrm{ch}(E)\cdot\mathrm{td}(T) \in A \otimes_{\mathbb{Z}} \mathbb{Q}$, where

$$\deg : A^N \to \mathbb{Z}$$

is the degree homomorphism mentioned in Section 1.

6. Chern classes on the projective space $\mathbb{P}_n$

On the projective space $\mathbb{P}_n$ we have $A = \mathbb{Z}[h] = \mathbb{Z}[x]/(x^{n+1})$, where $h = \bar{x} \in A^1$ is the Chern class of $S^\vee$, the dual line bundle to the tautological line subbundle $S \subset \mathbb{P}_n \times \Bbbk^{n+1}$. In particular

$$A^k = \mathbb{Z} \cdot h^k \cong \mathbb{Z}, \quad k = 0, \dots, n,$$

the degree homomorphism(1) is in this case an isomorphism and reads as

$$\deg : A^n = \mathbb{Z} \cdot h^n \to \mathbb{Z}, \quad \alpha \cdot h^n \mapsto \alpha.$$

Let T be the tangent vector bundle on $\mathbb{P}_n$. Then there is a short exact sequence of vector bundles (Euler sequence, cf. [9, II, Example 8.20.1])

$$0 \to \mathbb{P}_n \times \Bbbk \to \bigoplus_1^{n+1} S^\vee \to T \to 0.$$

This and the properties of Chern classes imply

$$c(\mathbb{P}_n) = c(T) = (1+h)^{n+1} = 1 + \binom{n+1}{1} \cdot h + \binom{n+1}{2} \cdot h^2 + \dots + \binom{n+1}{n} \cdot h^n,$$
$$\operatorname{ch}(\mathbb{P}_n) = \operatorname{ch}(T) = (n+1)\exp(h) - 1 = n + \frac{n+1}{1!} \cdot h + \frac{n+1}{2!} \cdot h^2 + \dots + \frac{n+1}{n!} \cdot h^n,$$
$$\operatorname{td}(\mathbb{P}_n) = \operatorname{td}(T) = \left(\frac{h}{1-\exp(-h)}\right)^{n+1}.$$

Let E be a vector bundle of rank r with Chern classes $c_i(E) = c_i \cdot h^i$, $c_i \in \mathbb{Z}$.

The Hirzebruch-Riemann-Roch formula (12) reads in this case as

$$\chi(E) = r + c_1 \quad \text{on } \mathbb{P}_1,$$
$$\chi(E) = r + \frac{1}{2} \cdot c_1(c_1+3) + c_2 \quad \text{on } \mathbb{P}_2,$$
$$\chi(E) = r + \frac{1}{6} \cdot c_1(c_1^2 + 6c_1 + 11) - \frac{1}{2} \cdot c_2(c_1+4) + \frac{1}{2}c_3 \quad \text{on } \mathbb{P}_3,$$

and as

$$\chi(E) = r + \frac{1}{24}c_1(c_1+5)(c_1^2+5c_1+10) + \frac{1}{12}c_2(c_2 - 2c_1^2 - 15c_1 - 35) + \frac{1}{12}c_3(2c_1+15) - \frac{1}{6}c_4$$

on $\mathbb{P}_4$.

Appendix A. Symmetric functions and elimination

In this appendix $\Bbbk$ is an arbitrary commutative ring.

Let

$$e_k = e(a_1, \dots, a_r) = \sum_{\mu_1 \leqslant \dots \leqslant \mu_i} a_{\mu_1} a_{\mu_1} \dots a_{\mu_i} \in \Bbbk[a_1, \dots, a_r], \quad k = 1, \dots, r,$$

be the standard elementary symmetric polynomials in variables $a_1, \dots, a_r$. The following is a fundamental fact about the symmetric polynomials.

Theorem A.1 (The fundamental theorem of symmetric functions)**.** *The homomorphism of polynomial rings*

$$\varphi\colon \Bbbk[c_1, \dots, c_r] \to \Bbbk[a_1, \dots, a_r], \quad c_k \mapsto e_k(a_1, \dots, a_r), \quad k = 1, \dots, r, \tag{13}$$

is injective and its image coincides with the subalgebra of all symmetric polynomials

$$\operatorname{Im}\varphi = \Bbbk[a_1, \dots, a_r]^{S_r} \subset \Bbbk[a_1, \dots, a_r].$$

Equivalently, for a symmetric polynomial $f(a_1, \dots, a_r) \in \Bbbk[a_1, \dots, a_r]^{S_r}$ there is a unique polynomial $g(c_1, \dots, c_r) \in \Bbbk[c_1, \dots, c_r]$ such that

$$f(a_1, \dots, a_r) = g(e_1(a_1, \dots, a_r), \dots, e_r(a_1, \dots, a_r)).$$

Proposition A.2. *Let $f \in \Bbbk[a_1, \dots, a_r]^{S_r}$ be a symmetric polynomial and let $g \in \Bbbk[c_1, \dots, c_r]$ be its preimage under the homomorphism φ from Theorem A.1. Consider in the polynomial ring $\Bbbk[a_1, \dots, a_r, x, c_1, \dots, c_r]$ the ideal*

$$I = \langle x - f(a_1, \dots, a_r), c_1 - e_1(a_1, \dots, a_r), \dots, c_r - e_r(a_1, \dots, a_r)\rangle.$$

Then

$$I \cap \Bbbk[x, c_1, \dots, c_r] = \langle x - g(c_1, \dots, c_r)\rangle.$$

Proof. First of all notice that I is the kernel of the homomorphism

$$\psi\colon \Bbbk[a_1,\dots,a_r,x,c_1,\dots,c_r]\to\Bbbk[a_1,\dots,a_r],\quad a_k\mapsto a_i,\quad x\mapsto f(a_1,\dots,a_r),\quad c_k\mapsto e_k(a_1,\dots,a_r).$$

Indeed, clearly I is contained in $\operatorname{Ker}\psi$. On the other hand for $h\in\operatorname{Ker}\psi$ write

$$h=h(a_1,\dots,a_r,x,c_1,\dots,c_r)=h(a_1,\dots,a_r,(x-f)+f,(c_1-e_1)+e_1,\dots,(c_r-e_r)+e_r)$$

and using, say, a Taylor expansion get

$$h=h_0(a_1,\dots,a_r)+h_1,\quad h_1\in I,$$

and hence $0=\psi(h)=\psi(h_0)=h_0$, which shows that h belongs to I.

Now, notice that the homomorphism φ from Theorem A.1 coincides with the composition of the standard embedding of polynomial rings and ψ

$$\Bbbk[c_1,\dots,c_r]\subset\Bbbk[a_1,\dots,a_r,x,c_1,\dots,c_r]\xrightarrow{\psi}\Bbbk[a_1,\dots,a_r].$$

Thus $\psi(x-g)=\psi(x)-\psi(g)=f-\varphi(g)=f-f=0$ and therefore $x-g\in I\cap\Bbbk[x,c_1,\dots,c_r]$.

Consider now an arbitrary $h\in I\cap\Bbbk[x,c_1,\dots,c_r]$. As above, writing

$$h(x,c_1,\dots,c_r)=h((x-g)+g,c_1,\dots,c_r)$$

and expanding into a Taylor series we get

$$h=h_0(c_0,\dots,c_r)+h_1(x,c_1,\dots,c_r)(x-g).$$

Since $0=\psi(h)=\psi(h_0)=\varphi(h_0)$, using Theorem A.1 one concludes $h_0=0$ and finally $h\in\langle x-g\rangle$. $\square$

Proposition A.2 says that one can reduce the computation of a preimage under φ to an elimination problem, i. e., to the computation of the intersection $I\cap\Bbbk[x,c_1,\dots,c_r]$.

This problem can be solved by computing a reduced Gröbner basis of I with respect to an elimination ordering for the variables $a_1,\dots,a_r$. The only element of such a Gröbner basis not involving the variables $a_1,\dots,a_1$ will be (up to a multiplication by a constant) $x-g$ (see [7] for more details on Gröbner bases).

Gröbner bases can be computed using a system of computer algebra. We prefer to use Singular for such purposes (cf. [6]).

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