Asset Pricing Models with Underlying Time-varying Lévy Processes

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Introduction: Time-varying Option Pricing Model with Jumps

- Time-varying volatility are commonly observed in the market. Empirical studies on the statistical properties of realized and/or implied volatilities have given rise to various stochastic volatility models in the literature, such as Heston model, CEV models and also stochastic volatility model with jumps etc.

- Evidence are also found supporting the existence of jumps. e.g. Carr and Wu (2003), Pan (2002). More recent literature has proved the dynamic jump intensity, for example Christoffersen et al (2012).

- In order to get theoretical results in a most general case, we assume a time-varying Lévy process, which has its drift, volatility and jump intensity as general time-varying parameters, to model the jump diffusion economy.

Objectives

- We study an equilibrium asset and option pricing model in an economy given a general form of time-varying Lévy process.

- Then we use Hodrick-Prescott filter and particle filter to decompose S&P500 index into time-varying processes of drift, volatility and jump.

- Based on the theoretical results of equity premium and option pricing formulae, we will further calibrate the models using S&P 500 index and its option data jointly.

Model Build-up

- The transformation of an investment of S_t in the stock market from time t to $t + dt$ in the economy is governed by a stochastic differential equation of the form:

$$\frac{dS_t}{S_{t^-}} = \mu(t)dt + \sigma(t)dB_t + (e^x - 1)dN_t - \lambda(t)E(e^x - 1)dt, \quad (1)$$

where S_{t^-} stands for the value of S_t before a possible jump occurs, $\mu(t)$, $\sigma(t)$ are the *rate of return* and the *volaility* of the investment.
The jump part is assumed to be a Poisson process, with jump intensity of $\lambda(t)$ and jump size of x, which is an arbitrary distribution in our model.

- Formula (1) describes the dynamics of the price of a *single aggregate stock* that is understood as a *stock index* or a *market portfolio*.

- Applying Itô's Lemma with jumps gives a process of the logarithm of S_t,

$$d \ln S_t = \left[\mu(t) - \frac{1}{2}\sigma(t)^2 - \lambda(t)E(e^x - 1) \right] dt + \sigma(t)dB_t + xdN_t.$$

- After an integration, we have the production process in an explicit form,

$$\ln \frac{S_T}{S_t} \equiv Y(t, T)$$
$$= \int_t^T \sigma(s)dB_s + \int_t^T [\mu(s) - \frac{1}{2}\sigma^2(s)]ds - E(e^x - 1) \int_t^T \lambda(s)ds + \sum_{i=1}^{N_{t,T}} x_i$$

where $Y(t, T)$ denotes the *continuously compounded return* of the investment over the period of (t, T)

Money Market Account

- We further assume that there is a market for instantaneous borrowing and lending at a *risk-free rate* $r(t)$. The *money market account*, M_t, follows

$$\frac{dM_t}{M_t} = r(t)dt. \tag{2}$$

The risk-free rate, $r(t)$, will be derived from the general equilibrium later, as a part of the solution.

Representative Investor

- A representative investor seeks to maximize the expected utility function of his life time consumption

$$\max_{c_t} E_t \int_t^T p(t) U(c_t) dt,$$

where c_t is the rate of consumption at time t, $U(c)$ is a utility function with $U' > 0$, $U'' < 0$, and $p(t) \geq 0$, $0 \leq t \leq T$ is a *time preference function*.

- We consider the class of *constant relative risk aversion (CRRA) utility* function

$$U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1, \\ \ln c, & \gamma = 1, \end{cases} \tag{3}$$

where the constant γ is the *relative risk aversion coefficient*, $\gamma = -cU''/U'$.

Total Wealth

The total wealth of the representative investor at time t is written as

$$W_t = W_{1t} + W_{2t}$$

where $W_{1t} = \omega W_t$ is the wealth invested in the stock market (ω being the *wealth ratio*) and $W_{2t} = (1 - \omega)W_t$ is the wealth invested in the money market.

Representative Investor's Optimal Control Problem

- The representative investor's optimal control problem becomes

$$\max_{c_t, \omega} E_t \int_t^T p(t) U(c_t) dt \qquad (4)$$

subject to

$$
\begin{aligned}
\frac{dW_t}{W_t} &= \omega \frac{dS_t}{S_{t^-}} + (1-\omega)\frac{dM_t}{M_t} - \frac{c_t}{W_t} dt \\
&= [r(t) + \omega\mu(t) - \omega r(t) - \omega\lambda(t)E(e^x - 1) - \frac{c_t}{W_t}]dt + \omega\sigma(t)dB_t \\
&\quad + \omega(e^x - 1)dN_t
\end{aligned}
$$

where $\phi(t) = \mu(t) - r(t)$ is the *equity premium*. The consumption rate c_t and the wealth ratio ω are control variables.

- Because there is only one investor in the economy, he has to put all the wealth into the stock market. The general equilibrium occurs at $\omega = 1$, under which the market is cleared.

Equity Premium

Proposition

- In the production economy with jump diffusion and one representative investor with CRRA utility function, the equilibrium equity premium is given by

$$\phi(t) = \phi_\sigma(t) + \phi_J(t), \tag{5}$$

$$\phi_\sigma(t) = \gamma\sigma(t)^2, \ \phi_J(t) = \lambda(t)E[(1 - e^{-\gamma x})(e^x - 1)], \tag{6}$$

$$r(t) = \mu(t) - \phi(t) = \mu(t) - \phi_\sigma(t) - \phi_J(t), \tag{7}$$

where $\phi_\sigma(t)$ is the diffusive risk premium and $\phi_J(t)$ is the rare-event premium.

- The arbitrary deterministic time preference function $p(t)$ affects the investor's optimal consumption rule, but it does not affect the result of the diffusive and jump risk premia.

General Pricing Kernel

Proposition

- *The pricing kernel in the production economy with jump diffusion is given in differential form by*

$$\frac{d\pi_t}{\pi_t} = -r(t)dt - \gamma\sigma(t)dB_t + (e^y - 1)dN_t - \lambda(t)E(e^y - 1)dt, \quad (8)$$

  where the variable y is a random number that models the jump size in the dynamics of the logarithm of the pricing kernel.

- *Integration gives*

$$\frac{\pi_T}{\pi_t} = \exp\{-\int_t^T \gamma\sigma(s)dB_s - \int_t^T [r(s) + \frac{1}{2}\gamma^2\sigma^2(s)]ds - E(e^y - 1)\int_t^T \lambda(s)ds + \sum_{i=1}^{N_{t,T}} y_i\}.$$

- *The martingale condition, $\pi_t S_t = E_t(\pi_T S_T)$, requires that the jump size y satisfies the following restriction $E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$.*

European Call

Proposition

The price of a European call, here denoted as $c(S_t, t)$, in the jump diffusion economy satisfies following integro-differential equation

$$\frac{\partial c(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2(t)S_t^2\frac{\partial^2 c(S_t, t)}{\partial S^2} + [r(t) - \lambda^Q(t)E^Q(e^x - 1)]S_t\frac{\partial c(S_t, t)}{\partial S}$$
$$- r(t)c(S_t, t) + \lambda^Q(t)\{E^Q[c(S_t e^x, t)] - c(S_t, t)\} = 0,$$

with a final condition

$$c(S_T, T) = max(S_T - K, 0),$$

where $\lambda^Q(t) \equiv \lambda(t)E(e^y)$ is the jump intensity in the risk-neutral measure Q, function $f(x)$ is the expectation under risk-neutral measure Q that can be evaluated by $E^Q[f(x)] = \frac{E[e^y f(x)]}{E(e^y)}$ with expectations in the physical measure.

Proposition

The call option price is given by the following formula

$$c(S_t, t) = \sum_{n=0}^{+\infty} e^{-\int_t^T \lambda^Q(s)ds} \frac{(\int_t^T \lambda^Q(s)ds)^n}{n!} E_n^Q[c^{BS}(Se^X e^{-E^Q(e^x-1)\int_t^T \lambda^Q(s)ds}, t)],$$

where $c^{BS}(S, t)$ is the Black-Scholes formula for the European call. The expectation E_n^Q is taken under risk-neutral measure Q against the random number X, which is defined as the sum of n i.i.d. random numbers x, i.e., $X = \sum_{i=1}^n x_i$.

Empirical Investigation

Decomposition of S&P500 Index into time-varying components.

- Hodrick-Prescott Filter
- Particle Filter (Sequential Monte Carlo Methods)

Hodrick-Prescott Filter

- Hodrick-Prescott filters was first proposed in Whittaker (1923), then popularized in economics by Hodrick and Prescott (1997). The method decomposes a time series into a trend component τ_t, which reflects the long term progression of the time series, and c_t, taken as a cyclical component with white noise in economics.

- The mathematical formulas are the following:

$$y_t = \tau_t + c_t, \text{ for } t = 1, \ldots, T$$

where $y_t = ln(S_t)$, S_t is the stock index.
For y_t, given λ, there is a trend τ_t satisfying:

$$\min_\tau \left(\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right)$$

where we take the common setting $\lambda = 129600$ for monthly data.

Time-varying Drift

As a first step, we decompose the stock index into time-varying trend component (drift) and a cyclical component (volatility and jump part):
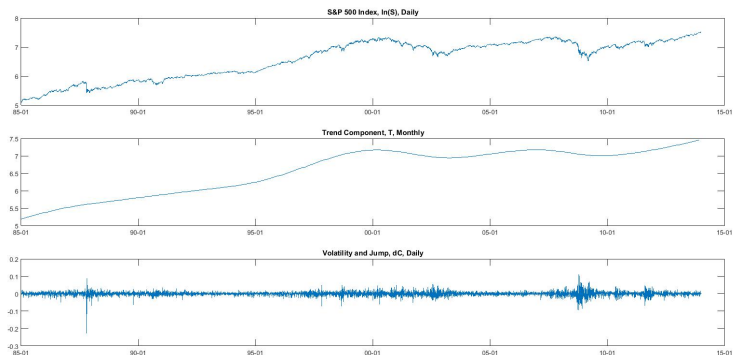

Figure 1.

	mean ($\times 10^{-5}$)	volatility	skewness	kurtosis
$\Delta \ln(S)$	31.9	0.0115	-1.3044	31.8
ΔC	1.91	0.0117	-1.2229	30.1

Table 1.

By taking the difference of the time-varying drift (ΔT), we can observe that:

- Before 2000, the economy was growing. However, after 1997 we cam observe the growth rate fluctuates around zero.
- In the negative return periods, there exists jumps and volatility clustering. In contrast, the positive return period in between, has very small volatlity/jump size.
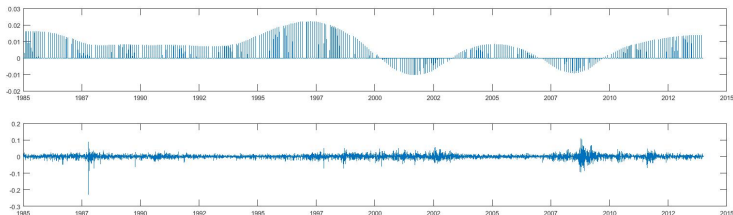

Figure 2.

Particle Filter

- Particle Filters (Johannes, Polson and Stroud 2009, RFS) are simulation-based estimation methods, which include a set of algorithms that estimate the posterior density of the state space by directly implementing the Bayesian recursion equations.

- This method uses sampling approach with a set of particles to represent the posterior density. The state space model can be non-linear and the initial state and noise distributions can take any form required.
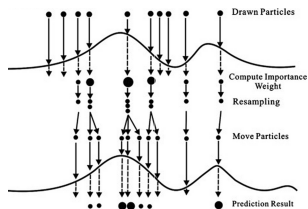
Additional Structure for Volatility

In the following empirical investigation, we assume that the dynamic of the stock market is given by

$$\frac{dS_t}{S_{t^-}} = \mu(t)dt + \sqrt{\nu_t}dB_t^s + (e^x - 1)dN_t - \lambda(t)E(e^x - 1)dt,$$

with stochastic volatility satisfying

$$d\nu_t = k(\theta - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}dB_t^\nu$$

where $\mu(t)$ is the *time-varying drift* that we get from previous part. ν_t is a mean-reverting stochastic volatility process. B_t^s and B_t^ν are Brownian motions with correlation ρ. The other parameters are the same as general model.

Filtered Volatility Processes

- Under this SVJ structure, we apply the particle filter over the period from 1985 to 2014.
- Figure 3 displays the filtered volatility estimates for the SVJ model. Here jump intensity fixed $\lambda = 0.006$, around 1 to 2 jumps per year, and jump size follows a normal distribution $N(-2.562, 4.0720^2)$ (Eraker, Johannes and Polson 2003). M is the number of points we interpolate in between two consecutive days.
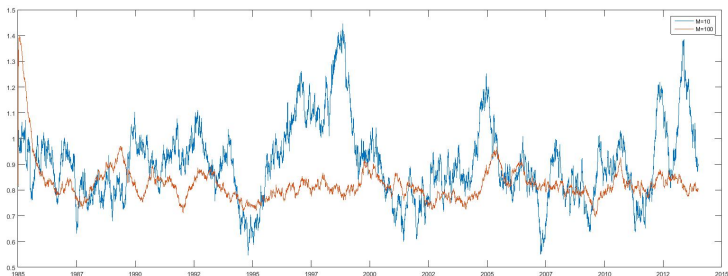

Figure 3.

Summary and Future Research

- In the theoretical part, we develop a general model with time dependent parameter for drift, volatility, and jump intensity. We derive the equity premium and pricing kernel. The formula of European call option is given as well.

- Empirical part, we use HP filter and particle filter methods to decompose the time-varying components from S&P500 index.

- We will further use the filtering methods to get time-varying components from option data jointly with return data. This work is in progress.

Main References

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Thank you!