A generalized finite mixture model

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Outline

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1. Nagin's Finite Mixture Model
2. Generalizations of Nagin's model

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- mixture : population composed of a mixture of unobserved groups
- finite : sums across a finite number of groups

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Aim of the analysis: Find r groups of trajectories of a given kind (for instance polynomials of degree 4, $P(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4$.

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Statistical Model:

$$y_{i_t} = \beta_0^j + \beta_1^j t + \beta_2^j t^2 + \beta_3^j t^3 + \beta_4^j t^4 + \varepsilon_{i_t}, \qquad (2)$$

where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma)$, σ being a constant standard deviation.

We try to estimate a set of parameters $\Omega = \left\{ \beta_0^j, \beta_1^j, \beta_2^j, \beta_3^j, \beta_4^j, \pi_j, \sigma \right\}$ which allow to maximize the probability of the measured data.

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- censored data $\Rightarrow$ Censored normal distribution

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$$L = \frac{1}{\sigma} \prod_{i=1}^{N} \sum_{j=1}^{r} \pi_j \prod_{t=1}^{T} \phi \left(\frac{y_{i_t} - \beta^j t}{\sigma} \right). \qquad (3)$$

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It is too complicated to get closed-forms equations.

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- age in the first year of professional activity

Result for 9 groups (dataset 1)

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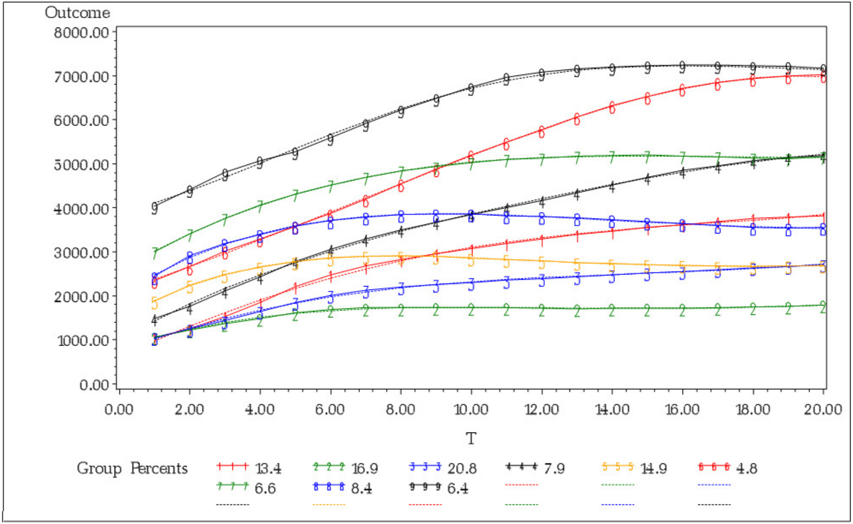

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Unfortunately the influence of the covariates in this model is limited to the intercept of the trajectory.

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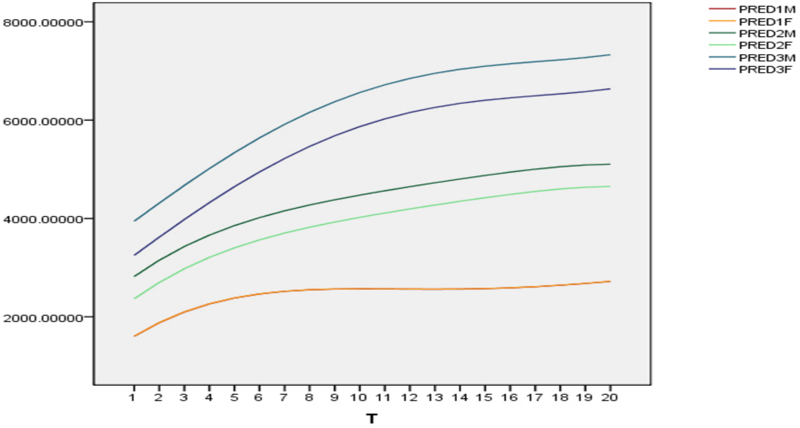

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where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma^j)$, σ^j being the standard deviation, constant in group j.

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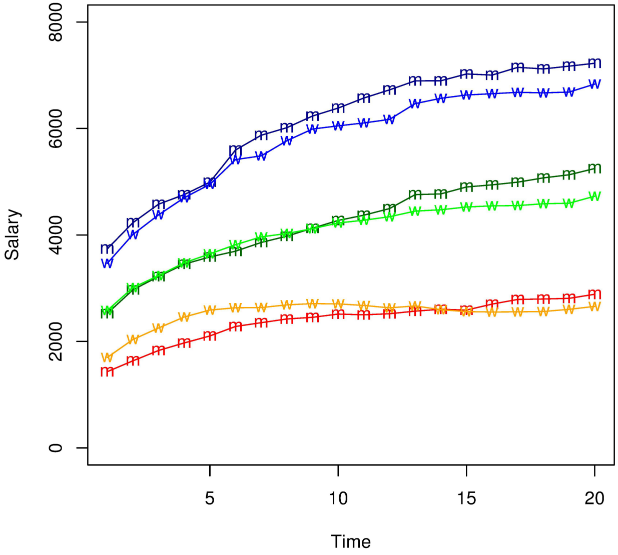
Men versus women



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$$CI_\alpha(\sigma_j) = \left[\sqrt{\frac{(N-(2+M)s-1)\hat{\sigma}_j{}^2}{\chi^2_{1-\alpha/2;N-(2+M)s-1}}}; \sqrt{\frac{(N-(2+M)s-1)\hat{\sigma}_j{}^2}{\chi^2_{\alpha/2;N-(2+M)s-1}}}\right]. \quad (8)$$

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Hence a model like

$$S_{it} = (\beta_0^j + \gamma_0^j z_t) + (\beta_1^j + \gamma_1^j z_t)t + (\beta_2^j + \gamma_2^j z_t)t^2 + (\beta_3^j + \gamma_3^j z_t)t^3, \quad (9)$$

where S denotes the salary and z_t is Luxembourg's CPI in year t of the study, makes no sense.

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Therefore, we simplify the model and calibrate

$$S_{it} = (\beta_0^j + \gamma_0^j z_t) + \gamma_1^j z_t t + \gamma_2^j z_t t^2 + \gamma_3^j z_t t^3. \quad (10)$$

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Results for group 1

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	321.381	1189.430	-2213.502	2856.093
γ_0	1689.492	277.834	-4.232	7.611
γ_1	**0.400**	0.120	0.143	0.656
γ_2	**-0.034**	0.007	-0.049	-0.019
γ_3	**0.0008**	0.0002	0.0005	0.0013

Results for group 2

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	**7688.158**	951.103	5660.197	9714.832
γ_0	**-13.095**	2.222	-17.822	-8.350
γ_1	**1.260**	0.096	1.055	1.465
γ_2	**-0.097**	0.006	-0.109	-0.085
γ_3	**0.0025**	0.0002	0.0022	0.0028

Results for group 3

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	**682.638**	196.327	141.924	1101.045
γ_0	**-11.367**	4.586	-21.135	-1.586
γ_1	**0.983**	0.199	0.559	1.406
γ_2	**-0.048**	0.012	-0.073	-0.023
γ_3	**0.0010**	0.0003	0.0003	0.0017

Results for group 4

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	**8473.081**	1859.349	4511.016	12434.892
γ_0	**-13.083**	4.342	-22.335	-3.825
γ_1	**0.927**	0.188	0.527	1.328
γ_2	-0.013	0.011	-0.036	0.010
γ_3	-0.0003	0.0003	-0.0009	0.0004

Results for group 5

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	4798.276	3205.141	-2034.302	11630.238
γ_0	-2.846	7.488	-18.806	13.115
γ_1	**1.315**	0.324	0.0624	2.006
γ_2	**-0.081**	0.019	-0.122	-0.040
γ_3	**0.0016**	0.0005	0.0005	0.0027

Results for group 6

Parameter	Estimate	Standard error	95% confidence Lower	intervals Upper
β_0	**8332.439**	1139.127	5903.348	10759.713
γ_0	**-12.472**	2.661	-18.145	-6.800
γ_1	**1.378**	0.015	1.132	1.623
γ_2	**-0.094**	0.007	-0.108	-0.079
γ_3	**0.0022**	0.0002	0.0018	0.0026

Disturbance terms

The disturbance terms for the six groups are $\sigma_1 = 41.49$, $\sigma_2 = 33.18$, $\sigma_3 = 68.48$, $\sigma_4 = 64.84$, $\sigma_5 = 111.83$ and $\sigma_6 = 39.74$

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