On a Family of Achievement and Shortfall Inequality Indices

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Abstract

This paper identifies a family of absolute consistent inequality indices using a weakly decomposable postulate suggested by Ebert (2010). Since one member employs an Atkinson (1970) type aggregation we refer to it as the Atkinson index of consistent inequality. A second member of this family parallels the Kolm (1976) index of inequality. Two innovative features of these indices are that no specific structure is imposed on the form of the index at the outset and no transformation of any existing index is considered to ensure consistency. Each of them regards an achievement distribution as equally unequal as the corresponding shortfall distribution. We apply these indices to study inequality in grip strength among 50+ year-old Europeans.

Key words: Achievement inequality, shortfall inequality, consistency, SHARE.

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1. Introduction

A recent concern in the literature on the measurement of inequality is to look at both achievement and shortfall inequalities and establish relation between them. In measuring inequality in some dimension of human well-being represented by a bounded variable, e.g., nutritional intake or health status, researchers often focus on attainments or shortfalls (Sen, 1992). Achievement inequality concentrates on the attainments of the individuals in the dimension, whereas shortfall inequality focuses on the shortfalls of the attainments from the maximum possible level of attainment. Going back to grip strength as an indicator of health, which is the focus of our empirical exercise, shortfall inequality measures differences in bad health while achievement inequality captures inequality in good health.

When achievement and shortfall inequalities are measured identically, we say that there is consistency between the two notions of inequality and the underlying inequality index is called a consistent index. In particular, under consistency they must always move along the same direction. Using data from Australia and Sweden, Clarke et al. (2002) demonstrated that for the commonly used concentration index, the inequality rankings of achievements and shortfalls are not the same. Working within the generalized Gini and generalized coefficient of variation frameworks, Erreygers (2009) characterized respectively the absolute Gini index and the variance as two consistent indicators of inequality. Both were found to be inversely related to the difference between the bounds of the distributions.

Lambert and Zheng (2011) considered a weaker condition within the Zoli (1999) inequality partial ordering framework and showed that for no documented intermediate inequality orderings and the relative ordering for achievements will coincide with that for shortfalls. In contrast, the absolute inequality partial ordering fulfills this condition unambiguously. They also identified two classes of absolute inequality indices which measure achievement and shortfall inequalities identically and showed that the variance is the only subgroup decomposable consistent absolute inequality index. Lasso de la Vega and Aristondo (2012) devised a procedure that enables conversion of any inequality index into an indicator that measure achievement and shortfall inequalities equally. In particular, they considered relative and absolute indices of inequality. They have also analyzed the Theil (1972) mean logarithmic deviation index, the only subgroup decomposable relative inequality index which uses subgroup population proportions as coefficients of subgroup inequality levels, to determine the within-group component of the total inequality.

In all the above papers the shortfalls were computed in an absolute manner. That is, the shortfall is expressed as the difference from the highest level of attainment. Since it reduces the allowable class of indices to a great extent, there have been criticisms (see Allanson and Petrie, 2012, 2013, Bosmans, 2013 and Kjellsson and Gerdtham, 2013). If we compute shortfalls in a relative sense, then it has been pointed out that consistency rules out any absolute inequality measure. However, it is well known that two different distributions can be ranked differently if we take both relative and absolute inequality measures in our consideration. See Zheng (1994), for an elegant discussion. Thus, it is necessary to clear the position, absolute or relative, at the very outset.

In this paper, we take the absolute route (Erreygers, 2009, Lambert and Zheng, 2011) and identify a family of absolute consistent inequality indices using Ebert’s (2010) weakly decomposable postulate. Ebert (2010) provided numerical examples to show that the definition of the between-group term of a sub-group decomposable inequality index that replaces actual incomes by the average income of the respective subgroup may lead to unintuitive conclusions. He then suggested this weaker decomposability postulate based on pairwise comparisons of income that avoid this shortcoming and that nicely adapts to the structure of Gini-based indices, which are not subgroup decomposable. Given our objective, we wish to examine the properties of shortfall and achievement inequality indices in the Ebert structure.

It is shown that for a general family of inequality indices the consistency condition drops out as an implication of some postulates of an index of inequality. The properties of the derived family are investigated in detail. We also look at the condition under which the family becomes unit consistent, where unit consistency demands invariance of inequality ranking of two distributions under equiproportionate changes in the achievement levels (Zheng, 2007).

Since one member of the general family employs an Atkinson (1970) type aggregation we refer to it as the Atkinson (1970) index of consistent inequality. This parametric index contains positive multiples of the standard deviation and the absolute Gini index as special cases. A second member of this family parallels the Kolm (1976) index of inequality. The maximax index drops out as a special case of both the indices. Since the Atkinson index contains two well-known indices as special cases, we also develop an axiomatic characterization of this index.

Two innovative features of these indices are that no specific structure is imposed on the form of the index at the outset and no transformation of any existing index is considered to ensure consistency. Each of them regards an achievement distribution as equally unequal as the corresponding shortfall distribution.

The paper is organized as follows. The next section builds the formal framework and presents the analysis. An empirical application to the study of inequality in grip strength among 50+ year-old Europeans is presented in Section 3. Finally, Section 4 makes some concluding remarks.

2. The Formal Framework

Assume that for any person i the level of achievement x_i takes on values in the non-degenerate interval $D^1 = [0, a]$ and for any $n \in N/\{1\}$ the achievement distribution is denoted by $x = (x_1, x_2, \cdots, x_n) \in D^n$, where $a > 0$ is finite, $D^n = [0, a]^n$, the $n-$ fold Cartesian product of $[0, a]$ and N is the set of positive integers. Let $D = \bigcup_{n \in N/\{1\}} D^n$. For all $n \in N/\{1\}$ and $x \in D^n$, we write $\lambda(x)$ for the mean achievement. The $n-$coordinated vector of ones is denoted by 1^n, where $n \in N/\{1\}$ is arbitrary. By assumption a is the maximum level of achievement and the shortfall experienced by person i in the distribution x is $s_i = a - x_i$. For any $n \in N/\{1\}$ and $x \in D^n$, the associated shortfall distribution $s(x) = (s_1, s_2, \cdots, s_n)$ is as well an element of D^n. For $x, y \in D$, x is obtained from y by a progressive transfer, if there is a pair (i, j) such that $x_i - y_i = y_j - x_j = \eta > 0$, $y_j - \eta \geq y_i + \eta$ and $x_l = y_l$ for all $l \neq i, j$. That is, there is a transfer of a positive amount of achievement η from y_j to a lower level y_i so that the donor j does not become poorer than the recipient i.

We assume the following postulates for an index of inequality $I : D \rightarrow \Re^1$, where $\Re^1$ is the real line.

Symmetry (SYM): For all $x \in D$, $I(x) = I(y)$, where y is any permutation of x.

Pigou-Dalton Transfers Principle (PDT): For all $y \in D$, if x is obtained from y by a progressive transfer, then $I(x) \leq I(y)$.

Dalton Population Principle (DPP): For all $n \in N/\{1\}$, $x \in D^n$, $I(x) = I(y)$, where y is the $l-$fold replication of x**,** that is**,** each x_i appears l times in y, $l \geq 2$ being any integer.

Continuity (CON): For all $n \in N/\{1\}, I$ is a continuous function.

Translation Invariance (TI): For all $n \in N/\{1\}$, $x \in D^n$, $I(x) = I(x + c1^n)$, where c is a scalar such that $x + c1^n \in D^n$.

SYM demands that inequality should not be sensitive to reordering of the achievements. Thus, for a symmetric index the individuals should not be distinguished by anything other than their levels of achievement in the considered dimension. **SYM** enables us to define the inequality index directly on ordered distributions. **PDT** demands that a progressive transfer of achievement, that is**,** a transfer from a person to anyone who achieved less so that the donor does not become poorer than the recipient, should not increase inequality. Under **SYM** only rank preserving transfers are allowed. Non-increasingness of an inequality index under a rank preserving progressive transfer is equivalent to S-convexity of the index (Dasgupta et al. 1973).[1] According to **DPP**, inequality remains invariant under replications of the population. This postulate, which enables us to view inequality as an average concept, becomes helpful for cross-population comparisons of inequality. **CON** is a condition to guarantee that there will be no abrupt changes because of minor observational errors in incomes. The assumption that the inequality index I is translation invariant, that is, of absolute type, means that inequality does not alter under equal absolute changes in all achievements. This notion of

[1]A real valued function f defined on D^n is called S-convex if $f(xB) \leq f(x)$ for all $x \in D^n$ and for all $n \times n$ bistochastic matrices. An $n \times n$ matrix with non-negative entries is called a bistochastic matrix if each of its columns and rows sums to one. If a function f is S-convex, then - f is S-concave. All S-convex functions are symmetric.

inequality invariance contrasts with a relative concept which requires inequality to remain unaltered when all achievements are scaled equi-proportionally. However, in view of Lambert and Zheng's (2011) finding that orderings involving intermediate and relative inequality concepts cannot produce identical rankings of achievement and shortfall distributions, we rule out this at the outset.

In the current context we also need the following postulates.

Strong Consistency (SC) (Lambert and Zheng, 2011): For all $x, y \in D$, $I(x) \leq I(y)$, if and only if $I(a-x) \leq I(a-y)$.

SC requires that the rankings are preserved whether we measure two distributions in terms of achievement or by shortfall. This is a generalization of the consistency criterion proposed by Erreygers (2009) where the values of shortfall and achievement indices need to be identical.[2]

In order to characterize the family of strongly consistent indices, following Ebert (2010), we consider the following axiom.

Decomposability (DEC): For every $\underline{n} = (n^1, n^2)$, where $n^1 \geq 1$ and $n^2 \geq 1$ are integers, there exist positive weight functions $w^1(\underline{n})$, $w^2(\underline{n})$ and $u(\underline{n})$ such that

$$I(x^1, x^2) = w^1(\underline{n}) I(x^1) + w^2(\underline{n})\ I(x^2) + u(\underline{n}) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(x_i^1, x_j^2) \qquad (1)$$

where x^j is any arbitrary income distribution over the population with size n^j, $j = 1,2$. This axiom enables us to decompose overall inequality of the achievement distribution (x^1, x^2) into a within-group component $w^1(\underline{n}) I(x^1) + w^2(\underline{n}) I(x^2)$ and a between-group component $u(\underline{n}) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(x_i^1, x_j^2)$. While the former corresponds to the usual within-group term used in the literature (e.g., Bourguignon, 1979, Shorrocks, 1980), the latter depends on pairwise comparisons of incomes. The usual between-group term $I(\lambda_1 1^{n_1}, \lambda_2 1^{n_2})$ is level of inequality that would arise if each achievement in a subgroup were replaced by the mean value of the subgroup, where λ_j is the mean of the distribution x^j, $j = 1,2$. In

[2] Bosmans (2013) has considered a weaker form of consistency where the rankings in achievement and shortfall are preserved as in **SC** but with a possibly different but associated shortfall and achievement indices.

the usual case the decomposability postulate is stated for any arbitrary number of subgroups. As Ebert (2010, p.96) stated "the decomposition method is considered for two subgroups. It can, however, be extended to more than two subgroups by repeated (recursive) application of (1)." An inequality index satisfying **DEC** is called weakly decomposable.

Ebert (2010) also assumed that the inequality index is normalized, that is, it takes on the value zero if and only if all the incomes are equal. Formally,

Normalization (NOM): For all $n \in N/\{1\}$, $I(x) = 0$, if and only if $x \in D^n$ is of the form $x = c1^n$, $c \in D^1$ being arbitrary.

The following theorem can now be stated.

Theorem 1: Assume that ***NOM*** holds and that there are two subgroups in the population consisting of *n* persons. Then the *inequality index I satisfies* ***CON***, ***DPP***, ***DEC***, ***SYM*** *and* ***TI*** *if and only if it is of the form*

$$I_{\psi}(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(\left|x_i - x_j\right|\right), \tag{2}$$

where $\psi : D^1 \to R^1$ *is continuous and* $\psi(0) = 0$.

Proof. See Appendix A. □

Since $\left|(a - x_i) - (a - x_j)\right| = \left|x_i - x_j\right|$ and the transformation ψ is same across distributions, the family of indices given by (2) satisfies **SC**.
This is formally stated as follows.

Corollary 1: Assume that ***NOM*** holds and that there are two subgroups in the population consisting of *n* persons. If the *inequality index I satisfies* ***CON***, ***DPP, DEC***, ***SYM*** *and* ***TI,*** *then it also satisfies* ***SC.***

Thus, in view of corollary 1, given **NOM** and a two subgroup society, the postulates **CON**, **DPP, DEC**, **SYM** and **TI** become sufficient for **SC** in the sense that the implied form of indices satisfies **SC.**

For any two persons i and j, if we measure inequality of the achievement distribution (x_i, x_j) by $\left|x_i - x_j\right| = \max(x_i, x_j) - \min(x_i, x_j)$, the excess of the maximum achievement over the minimum achievement, then given the population size, the average of transformed excesses $\psi(|x_i - x_j|)$ for all two-person achievement distributions of the type (x_i, x_j), where $i, j = 1,2,.., n$, leads us to the inequality index in (2). Since inequality of the distribution (x_i, x_j) is not likely to decrease if the gap $|x_i - x_j|$ increases, which may result if, given $x_j > x_i$, there is a regressive transfer from x_i to x_j, we assume that ψ is non-decreasing. Also $\psi(0) = 0$ ensures that if $x_i = x_j$, there is no inequality in the two-person distribution (x_i, x_j) (see Sen, 1973).

In the following theorem we show that non-decreasingness and convexity of ψ are sufficient for I_ψ to satisfy **PDT**.

Theorem 2: *The inequality index I_ψ given by equation (2) satisfies **PDT** if the function ψ identified in Theorem 1 above is non-decreasing and convex.*

Proof. See Appendix A. □

As an illustrative example, let $\psi(t) = t^r$, where $r \geq 1$ is a constant. Then the corresponding index becomes

$$I_r(x) = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|^r \ . \quad (3)$$

This family was suggested by Ebert (2010) as a class of weakly decomposable indices.

It may be noted that I_ψ may not be bounded above. For instance, let $\psi(x_i, x_j) = e^{\theta|x_i - x_j|}, \theta > 0$. Then for any unequal distribution x the resulting index $I_\theta(x) = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta|x_i - x_j|}$ increases as θ increases. As $\theta \to \infty$, for any unequal x, $I_\theta(x) \to \infty$. It may be noted that if the objective is to order the distributions and two distributions can be ranked unambiguously, calculation of any index is not required and boundedness here may not be treated as a desirable condition. As we show later in this section, this situation is clearly brought out in the case of indices satisfying equation (6). For instance, a member of the family given in equation (3), which may not be bounded and the corresponding member of equation (7) given below, which is bounded, will lead to the same ranking of the distributions in view of equation (6). However, in the literature, use of a monotonic bounded transformation of an index which is unbounded is often advised. We may mention here that the unbounded subgroup decomposable translation invariant exponential index of inequality is increasingly related to the Kolm (1976) absolute index which is bounded (see Bosmans and Cowell, 2010, Chakravarty and Tyagarupananda, 2009).[3] However, the use of the latter, although not subgroup decomposable, is more common in empirical applications. This is particularly important when the distributions cannot be ranked by a quasi ordering. In such a case, finite numerical value of an index is useful. This may be regarded as one motivation for a bounded index. Furthermore, since the variables considered in the current context are all bounded, the representative excess, an aggregate representation of their finite differences in terms of inequality should ideally be bounded too. Boundedness also enables us to conclude how far the actual inequality falls below its maximum attainable level.

In order to interpret some relative inequality indices that are not bounded above by 1, as Atkinson-Kolm-Sen indices, Blackorby and Donaldson (1978) adopted the procedure of dividing them by their respective attainable finite upper bounds. One such index is the Theil (1967) entropy index. Thus, existence of a finite upper bound is necessary for this purpose. It may be worthwhile to mention that the relative Gini, the relative Donaldson-Weymark (1980) S-Gini and Atkinson indices are bounded above by 1.

We therefore consider a particular cardinalization of I_ψ that becomes bounded. However, since the definition of this particular cardinalization of I_ψ relies on the inverse of the function ψ, we assume that ψ is increasing.

Let $\Psi = \{\psi : D^1 \to \Re \mid \psi \text{ is increasing, convex, continuous and } \psi(0) = 0\}$. Given any achievement distribution $x \in D^n$ and $\psi \in \Psi$, we define the representative excess $I_\psi^R(x)$ as that level of excess which when arises in all pairwise comparisons will make the existing distribution inequality equivalent. This is similar in spirit to Atkinson's (1970) equally distributed equivalent income. In a two-person achievement distribution (x_1, x_2),

[3] We are grateful to one of the referees for bringing this to our attention.

$I_\psi^R(x)$ is simply the distance between the origin and the foot of the perpendicular drawn on the horizontal axis from the point of intersection of the iso-inequality contour and the 45^0 line passing through the origin. Formally, for any $x \in D^n$ and $\psi \in \Psi$, we have,

$$\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(I_\psi^R(x)\right)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(\left|x_i-x_j\right|\right). \tag{4}$$

From (4) it follows that

$$I_\psi^R(x)=\psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(\left|x_i-x_j\right|\right)\right]. \tag{5}$$

The representative excess is a particular numerical representation of the inequality index identified in (2). That is, for all $n \in N/\{1\}, x, y \in D^n$,

$$I_\psi^R(x) \geq I_\psi^R(y) \leftrightarrow I_\psi(x) \geq I_\psi(y). \tag{6}$$

Given that I_ψ^R is a particular cardinalization of the index in (2), we can as well use I_ψ^R as an index of inequality. Satisfaction of strong consistency by I_ψ^R follows from the observations that $\left|(a-x_i)-(a-x_j)\right|=\left|x_i-x_j\right|$ and the common transformation ψ across distributions is increasing.

The following theorem establishes some important properties of I_ψ^R.

Theorem 3: *For any* $\psi \in \Psi$, I_ψ^R *is continuous and bounded between* 0 *and* a , *where the lower bound is achieved when the achievements are equally distributed. The index* I_ψ^R *also satisfies* ***SYM***, ***DPP***, ***PDT***, ***NOM*** *and* ***SC***.

Proof. See Appendix A. □

In order to illustrate I_ψ^R, let $\psi(t)=t^r$, where $r \geq 1$, so that the resulting index becomes

$$I_r^R(x)=\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i-x_j\right|^r\right)^{1/r}. \tag{7}$$

The index I_r^R is the symmetric mean of order r of the gaps $\left|x_i-x_j\right|$. Since it employs the Atkinson (1970)-type aggregation, we refer to it as the Atkinson strongly consistent inequality index. For a given distribution x, the index increases as r increases.

As the value of r increases, more weight is assigned to higher gaps in the aggregation if $r > 1$. For $r = 1$, I_ψ^R becomes twice the absolute Gini index of inequality, whereas for $r = 2$, it is $\sqrt{2}$ times the standard deviation. As $r \to \infty$, $I_r^R \to \max_{i,j} \{|x_i - x_j|\}$, the maximax index.

Now, as a second example, let us consider $\psi(t) = e^{\theta t} - 1, t \geq 0, \theta > 0,$ which generates the Kolm (1976) strongly consistent inequality index given by

$$I_\theta^R(x) = \frac{1}{\theta} \log\left(\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta |x_i - x_j|} \right). \tag{8}$$

This index does not satisfy **DEC** for any positive value of θ. A transfer of achievement from a person i to a richer person j will increase I_θ^R by a larger amount as θ increases. As $\theta \to 0$, I_θ^R approaches zero, whereas I_θ^R becomes the maximax index as $\theta \to \infty$.

The above examples also show that under the transformations satisfying the conditions which make the index bounded (see theorem 3), the resulting index may or may not satisfy **DEC**. For instance, for $r = 1$, I_r^R is bounded and weakly decomposable. However, for any $r > 1$, this index, although bounded, is not weakly decomposable. Thus, as such, weak decomposability may clash with the requirement of boundedness.

Zheng (2005) investigated the implication of **SYM**, **NOM**, **PDT**, subgroup decomposability and unit consistency on an inequality index I, where subgroup decomposability requires that for any partitioning of the population with respect to some socio-economic characteristic, overall inequality can be written as the sum of within-group and between-group terms (see the discussion after equation (1)). Unit consistency demands that inequality ordering of two distributions remains unaltered when achievements are expressed in different measuring units. Formally, an inequality index I is said to be unit consistent if $I(x) < I(y) \Rightarrow I(cx) < I(cy) \quad \forall c > 0, x, y \in D^n$ (see Zheng, 2007). Although unit consistency is implied by scale invariance, it is an ordinal concept whereas scale invariance is a cardinal condition. Zheng (2005) demonstrated that the implied index has the form $I(cx) = c^\alpha I(x) \, \forall c > 0, x \in D^n$, for some real α.

We can easily see that the index given by (2) is unit consistent if the function $\psi : D^1 \to R^1$ given in (2) has the form $\psi(cx) = c^\alpha \psi(x)$ for some real α. Further, along with continuity and $\psi(0)=0$ if ψ is non-decreasing and convex, then, in addition to satisfying properties mentioned in Theorem 1, the index satisfies **PDT** as well (see Theorem 2). This, therefore, can be regarded as an example of a unit consistent inequality index that satisfies **DEC**.

Given any $\psi \in \Psi$ there exists a corresponding **strongly** consistent inequality index I_{ψ}^{R}. These indices will differ in the way how we specify ψ. However, we can uniquely identify a particular member of the family $I_{\psi \in \Psi}^{R}$, using the following axiom.

LIH (Linear Homogeneity): For all $c > 0$, for all $x \in D^n$, such that $cx \in D^n$, $I_{\psi}^{R}(cx) = cI_{\psi}^{R}(x)$.

LIH implies that equal proportional changes in all achievements changes the index by the same proportion. Several absolute inequality indices, for example, the standard deviation, the absolute Gini index satisfy **LIH.** They have the convenient property of being converted into relative indices when divided by the mean income. Such indices, which are referred to as compromise absolute indices (Blackorby and Donaldson, 1980), also become helpful for measuring economic distances between two income distributions (see Chakravarty and Dutta, 1987).[4] However, since the index given by (7) does not satisfy **DEC**, direct comparison is not feasible.

The following theorem shows that the Atkinson index is the only member of the family $I_{\psi \in \Psi}^{R}$ that satisfies **LIH**.

Theorem 4: *A strongly consistent inequality index* $I_{\psi \in \Psi}^{R}$ *satisfies* ***LIH*** *if and only if* $I_{\psi}^{R}(x)$ *is given by (7).*

Proof. See Appendix A. □

From Theorem 4 it emerges that the only member of (7) that satisfies **DEC**, unit consistency and **LIH** is I_{r}^{R} for $r = 1$.

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[4] Ebert (1988) used **LIH** as an axiom to characterize a class of absolute compromise indices. This property has also been used in closely related contexts. For instance, Mitra and Ok (1998) used **LIH** as an axiom to characterize the symmetric sum of order $\upsilon \geq 1$ of absolute differences between components of two income vectors as a mobility index. Fields and Ok (1999) used this property to axiomatize a measure of income movement.

Theorem 4 also clearly demonstrates that if for empirical purpose one wishes to use a member of the family $I^{R}_{\psi\in\Psi}$ that satisfies **LIH**, then the only choice is the Atkinson index.

3. An Empirical Application

We use data from the Survey of Health, Aging and Retirement (SHARE) in 2011, which is the 4th wave of the dataset. SHARE examines the different ways in which people aged 50+ live in sixteen European countries. Our sample, obtained by dropping missing values, is composed of 53,113 individuals. All indices are estimated using sample weights. The variable we use to assess the health status of Europeans is grip strength (GS) since it has been shown to be a strong predictor of disability, morbidity and mortality (see Andersen-Ranberg et al. 2009, for an analysis based on the same data). GS was assessed using a Smedley dynamometer (S Dynamometer, TTM, Tokyo, 100 Kg), according to the following measurement protocol:

"Participants were instructed to stand (preferably) or sit, with the elbow at 90°, the wrist in neutral position, keeping the upper arm tight against the trunk, and the inner lever of the dynamometer adjusted to suit the hand; and they were then instructed to squeeze as hard as possible for a few seconds". Two values were recorded for each hand. GS can take values in $[0,100]$, indicating no strength and maximum strength respectively. In our dataset measurements were considered valid if the two measurements of one hand differed by less than 20 kg., and values of zero or those above 100 kg were considered invalid. Hence the minimum value we observe in our sample is 1 while the maximum is 99. The mean value is 32.6 and the standard deviation 11.73.

We estimate inequality in GS using one version of the Atkinson consistent index introduced in equation (7), which for $r=1$ satisfies **DEC**, **LIH** and unit consistency, and one version of the Kolm consistent inequality index of equation (8) for $\theta=1$, which does not satisfy **DEC**. We label the Atkinson consistent measure A(1). A(1) is twice the absolute Gini index of inequality. The Kolm consistent index is labeled K(1).

For comparison purposes, we also use the variance, labeled V, which is the only subgroup decomposable consistent absolute inequality index. We plot in Figure 1 the rankings of the three measures ordered by the results of the variance. See Table 1 for all index values and rankings. The Atkinson index replicates the rankings of the variance except for a unique reversal between Sweden and the Czech Republic. The rankings of the countries according to the Kolm index differ considerably. The most notable case is Slovenia, which moves from the third most unequal to the second most equal position. While no change is observed in the position of Sweden. The Spearman’s correlation coefficient among the ranks of K(1) and the other indices is as low as 0.5 with the variance and 0.48 with the Atkinson index.

4. Conclusion

Achievement inequality in a dimension of human well-being looks at interpersonal differences on the attainment levels in the dimension for different individuals in a society. The shortfall inequality in the dimension is concerned with shortfalls of attainments from the maximum possible value of the attainment. An inequality index is said to be consistent if it measures attainment inequality and shortfall inequality equally. This paper develops a general approach to the measurement of consistent inequality. Because of the underlying aggregation procedures, we refer to three members of the general family as the Atkinson (1970), Kolm (1976) and Theil (1972) consistent inequality indices. Positive multiples of the standard deviation and the absolute Gini index turn out to members of the Atkinson family. Essential to our characterization and investigation of properties of different indices is the weakly decomposable postulate suggested by Ebert (2010). Finally, a numerical application of our indices is provided using data on grip strength among Europeans aged 50+.

In section 2 of the paper we have noted that a bounded index may or may not be weakly decomposable. As a general exercise, it may be worthwhile to look into the class of transformations that make an unbounded index satisfying **DEC** bounded while preserving **DEC**. Another general issue of investigation can be characterization of the class of inequality indices that satisfy **DEC** and unit consistency (following Zheng, 2005 and Lasso de la Vega and Urrutia, 2011). We thank a referee for bringing these general issues to our attention and leave them as for future programs.

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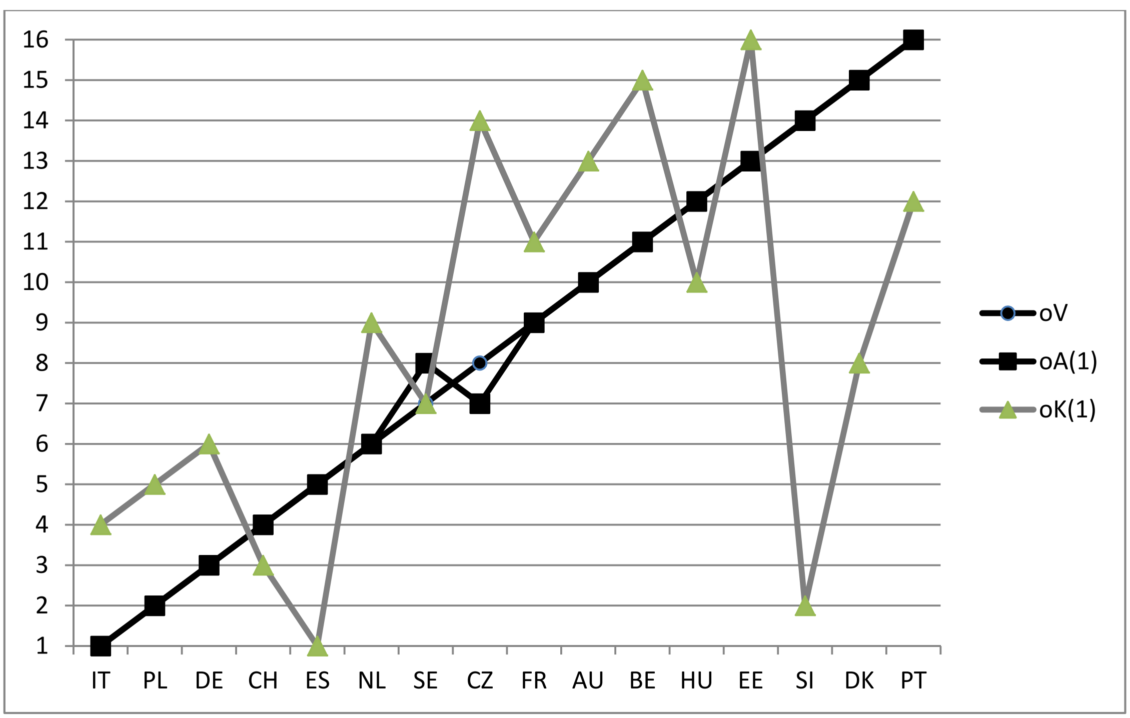
TABLE 1: INEQUALITY IN GS: INDEX VALUES AND RANKINGS

Countries	V	oV	A(1)	oA(1)	K(1)	oK(1)
Italy (IT)	117.73	1	12.314	1	58.510	4
Poland (PL)	127.76	2	12.746	2	58.626	5
Germany (DE)	131.23	3	12.902	3	59.093	6
Switzerland (CH)	132.52	4	12.980	4	56.548	3
Spain (ES)	134.30	5	13.090	5	52.498	1
Netherlands (NL)	135.83	6	13.196	6	61.887	9
Sweden (SE)	137.69	7	13.272	8	60.363	7
Czech Republic (CZ)	138.34	8	13.197	7	78.857	14
France (FR)	140.66	9	13.376	9	66.386	11
Austria (AU)	142.15	10	13.384	10	72.233	13
Belgium (BE)	145.52	11	13.651	11	80.141	15
Hungary (HU)	149.13	12	13.782	12	66.007	10
Estonia (EE)	156.38	13	13.894	13	81.563	16
Slovenia (SI)	157.62	14	14.223	14	56.384	2
Denmark (DK)	161.81	15	14.417	15	60.437	8
Portugal (PT)	196.45	16	15.427	16	70.529	12

FIGURE 1: INEQUALITY IN GRIP STRENGTH



Appendix

Proof of Theorem 1.

Ebert (2010) showed that for a two-subgroup society if an inequality index satisfies **NOM**, then it satisfies **DEC** and **DPP** if and only if $I(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} I(x_i, x_j)$. Now, let $x_i > x_j$. Then by **TI**, $I(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} I(x_i - x_j, 0)$. Likewise, if $x_j > x_i$, $I(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} I(0, x_j - x_i)$ which in view of **SYM** becomes $I(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} I(x_j - x_i, 0)$. This gives $I(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi(|x_i - x_j|)$, where $\psi : D^1 \to \Re^1$.

Continuity of ψ follows from the fact that I is continuous. $\psi(0) = 0$ is an implication of the axiom **NOM**. Conversely, it is easy to check that I satisfies **DPP**, **DEC**, **SYM, CON** and **TI** . □

Proof of Theorem 2.

$D^2 \subset \Re^2$ is convex. Define the real valued function $f : D^2 \to \Re^1$ by $f(x_i, x_j) = |x_i - x_j|$. Now, consider $u = (x_i, x_j), v = (x_k, x_l), u, v \in D^2$ such that $f(u) = |x_i - x_j| \geq |x_l - x_k| = f(v)$. For any $0 < c < 1$, the convex combination $cu + (1-c)v = c(x_i, x_j) + (1-c)(x_k, x_l) \in D^2$, since D^2 is convex. Then, $f(cu + (1-c)v) = f(cx_i + (1-c)x_k, cx_j + (1-c)x_l) =$

$| c(x_j - x_i) + (1-c)(x_l - x_k) | \leq c \, | x_i - x_j | + (1-c) \, | x_l - x_k |$. Given that $u, v \in D^2$ and $c \in (0,1)$ are arbitrary, f is convex. Since a non-decreasing convex transform of a convex function is convex, ψ is convex. Thus, I_ψ , being a finite sum of convex functions is also convex. Note that I_ψ is also symmetric. All symmetric convex functions are S-convex (Marshall et al. 2011, p. 98). Hence I_ψ is an S-convex function, which we know is equivalent to **PDT** under rank preserving transfers.

Proof of Theorem 3.

Since x_i, x_j are drawn from the compact set D^1, the non-negative deviations will also take values in the compact set $[0, a]$. Now, since ψ defined on the compact set $[0, a]$ is increasing and the continuous image of a compact set is compact (Rudin, 1976, p.89), $\psi(|x_i - x_j|)$ takes values in the compact set $[\psi(0), \psi(a)]$, which, in view of the fact that $\psi(0) = 0$, can be rewritten as $[0, \psi(a)]$. Continuity and increasingness of the function ψ implies that the average function $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|)$ is continuous and takes values in $[0, \psi(a)]$. Observe that increasingness of ψ ensures the existence of ψ^{-1}. Continuity and increasingness of ψ^{-1} on $[0, \psi(a)]$ now follows from Theorem 4.53 of Apostol (1974, p.95). This in turn demonstrates continuity of I_{ψ}^{R}.

For boundedness, note that if the achievement distribution x is perfectly equal, each $|x_i - x_j|$ becomes zero which implies that $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|) = \psi(0)$. Hence if x is perfectly equal $I_{\psi}^{R}(x) = \psi^{-1}(\psi(0)) = 0$. Likewise, it can be shown that I_{ψ}^{R} is bounded above by a.

Since an increasing convex function is non-decreasing and convex, by Theorem 2, $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(| x_i - x_j |)$ satisfies **SYM**, **DPP**, **PDT** and **NOM**. Given that ψ^{-1} is also increasing and $\psi^{-1}(0) = 0$, I_{ψ}^{R} satisfies **SYM**, **DPP**, **PDT** and **NOM**. We have already noted that I_{ψ}^{R} fulfils **SC**. (See the discussion after equation (6)). This completes the proof of the theorem. □

Proof of Theorem 4.

The idea of the proof is taken from Chakravarty (2009).

LIH requires that

$$c\psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|)\right] = \psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|cx_i - cx_j|)\right], \quad (1a)$$

where $c > 0$ is a scalar satisfying the condition laid down in the axiom. The only continuous solution to the functional equation given by (1a) is

$$\psi(t) = A + \frac{Bt^r}{r} \quad , \tag{2a}$$

where A, B are constants. (See Aczel, 1966, p.151). The condition $\psi(0) = 0$ along with continuity of ψ requires that $A = 0$ and $r > 0$. Increasingness of ψ demands that $B > 0$. Substituting the form ψ given by (2a) in (5) we get the desired form of the index. This establishes the necessary part of the theorem. The sufficiency is easy to verify. □