Median Preserving Aggregation Functions

Bruno Teheux
joint work with Miguel Couceiro and Jean-Luc Marichal

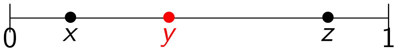
University of Luxembourg

AGOP 2015

Median operation on $[0,1]^n$

The ternary median operation on $[0,1]$:

$$\mathrm{m}(x,y,z) \;=\; (x \wedge y) \vee (x \wedge z) \vee (y \wedge z), \quad x,y,z \in [0,1]$$



Extended to $[0,1]^n$ componentwise: for $\mathbf{x},\mathbf{y},\mathbf{z} \in [0,1]^n$

$$\mathrm{m}(\mathbf{x},\mathbf{y},\mathbf{z})_i \;=\; \mathrm{m}(x_i,y_i,z_i), \quad i \leq n$$

Median preserving aggregation functions

Definition. A function $f\colon [0,1]^n \to [0,1]$ *is* m*-preserving* if

$$f(\mathrm{m}(\mathbf{x},\mathbf{y},\mathbf{z})) = \mathrm{m}(f(\mathbf{x}),f(\mathbf{y}),f(\mathbf{z})), \qquad \mathbf{x},\mathbf{y},\mathbf{z} \in [0,1]^n$$

m-preserving $\iff$ preserves ‘in-betweenness’

How to characterize those $f\colon [0,1]^n \to [0,1]$ which are m-preserving?

A general frame to study median operations

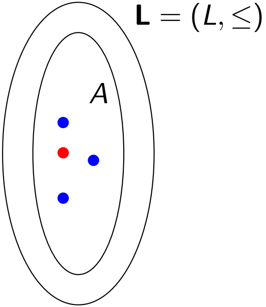
The expression

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

defines an operation $\mathrm{m}_{\leq}(x, y, z)$ on any distributive lattice $(L, \leq)$.

Definition. (Avann, 1948)

median algebra $\mathbf{A} = (A, \mathrm{m}) \iff$ subalgebra of some $(L, \mathrm{m}_{\leq})$



$$\mathrm{m}(\bullet, \bullet, \bullet)$$

$$=$$

$$\bullet$$

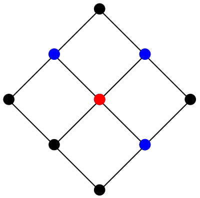
$$=$$

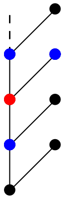
$$\mathrm{m}_{\leq}(\bullet, \bullet, \bullet)$$

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

Examples.

$$\mathrm{m}(\bullet, \bullet, \bullet) = \bullet$$





Median graphs

Some metric spaces

A more general question

How to characterize those $f\colon \mathbf{A}^n \to \mathbf{A}$ which are m-preserving?

An additional assumption: conservativeness

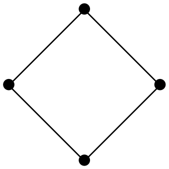
Definition. A median algebra (A, m) is *conservative* if

$$\mathrm{m}(x, y, z) \in \{x, y, z\}, \qquad x, y, z \in A.$$

Examples.

Any chain $\mathbf{C} = (C, \mathrm{m}_{\mathbf{C}})$

$\{0, 1\} \times \{0, 1\}$



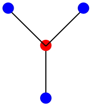
Characterization of conservative median algebras

Theorem. For a median algebra $\mathbf{A} = (A, \mathrm{m})$ with $|A| \geq 5$, the following conditions are equivalent.

(i) $\mathbf{A}$ is conservative

(ii) There is a total order $\leq$ on A such that $(A, \mathrm{m}) = (A, \mathrm{m}_{\leq})$

(iii) $\mathbf{A}$ does not contain any subalgebra isomorphic to



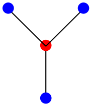
Characterization of conservative median algebras

Theorem. For a median algebra $\mathbf{A} = (A, \mathrm{m})$ with $|A| \geq 5$, the following conditions are equivalent.

(i) $\mathbf{A}$ is conservative

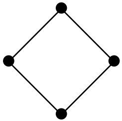
(ii) There is a total order $\leq$ on A such that $(A, \mathrm{m}) = (A, \mathrm{m}_{\leq})$

(iii) $\mathbf{A}$ does not contain any subalgebra isomorphic to



Remark. If $\mathbf{A}$ is conservative with $A \leq 4$, and $\neg$ (ii) then

$\mathbf{A} \cong$



Median preserving aggregation functions are dictatorial

Let $\mathbf{C} = (C, \mathrm{m}_{\leq})$ where $\leq$ is a total order, for instance $C = [0,1]$.

Theorem. For $f\colon \mathbf{C}^n \to \mathbf{C}$, the following conditions are equivalent.

(i) f is m-preserving

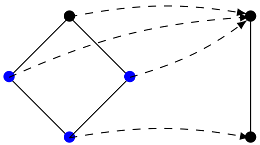
(ii)
$$f(\mathbf{x}) = h(x_i), \qquad \mathbf{x} \in \mathbf{C}^n$$
for some $i \leq n$ and some monotone map $h\colon \mathbf{C} \to \mathbf{C}$.

Median preserving aggregation functions on chains are dictatorial and monotone

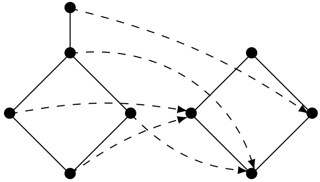
Corollary. A function $f\colon \mathbf{C} \to \mathbf{C}$ is m-preserving if and only if it is monotone.

In general,

$$\mathrm{m}\text{-preserving} \quad \not\Leftrightarrow \quad \text{monotone}$$



monotone, ~~*m*-preserving~~



m-preserving, ~~monotone~~,

Final remarks

Characterization of conservativeness in terms of chains was known:

Bandelt, H.-J., & van de Vel, M. (1999). The Median Stabilization Degree of a Median Algebra. *Journal of Algebraic Combinatorics*, 9, 115–127.

How to relax the condition of conservativeness?