FEniCS '15

29 June - 1 July 2015
Imperial College London

M **BRUNETTI**[*], J S **HALE**[⋆], S **BORDAS**[⋆], C **MAURINI**[*]

∗ Institut Jean Le Rond d'Alembert · UNIVERSITÉ PIERRE ET MARIE CURIE
⋆ Research Unit in Engineering Science · UNIVERSITÉ DU LUXEMBOURG

Imperial College London, 01/07/2015

FEniCS-shells

a UFL-based library for simulating thin structures

Outline

Outline

M. BRUNETTI, J. S. HALE, S. BORDAS, C. MAURINI

Motivations

Why thin-structures?

- Shell, plate and beam (thin) structures are widely used in civil, mechanical and aeronautical engineering because they are capable of carrying high loads with a minimal amount of structural mass.
- To our knowledge a unified open-source implementation of a wide range of thin structural models is not yet available.

Motivations

Some applications

Multistable shells [Coburn et al., 2013]

Stress focusing in elastic sheets

Objectives

Why FEniCS-shells?

The UFL language provides an excellent framework for writing extensible, reusable and pedagogical numerical models of thin structures.

FEniCS-shells is (will be) a library consisting of various thin structural models and associated numerical techiniques expressed in the UFL.

- To have a solid and extensible open-source platform of quality numerical methods for thin structures.
- To link in a clear and direct way the continuous mathematical model and its finite element solution.

Outline

Remarks on shell theories

M. Brunetti, J. S. Hale, S. Bordas, C. Maurini

- Shells are three-dimensional elastic bodies which occupy a thin region around a two-dimensional manifold situated in three-dimensional space.
- A three-dimensional problem is reduced to a two-dimensional problem. Quantities of engineering relevance are computed directly.
- Non-trivial numerical problems arise also for flat shells (plates) and linear models.

- Shearable theories (thick plates)

FLEXURE = BENDING + SHEARING

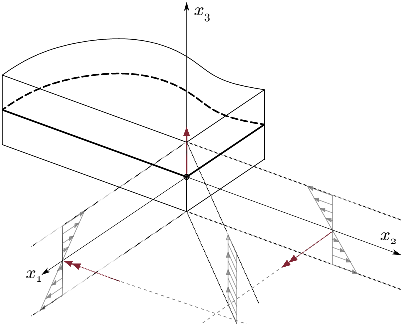
e.g. for the Reissner-Mindlin (RM) plate model

$$E_{\mathrm{RM}} = \frac{1}{2}\int_\Omega \mathcal{D}\nabla^{\mathrm{s}}\boldsymbol{\theta} : \nabla^{\mathrm{s}}\boldsymbol{\theta} + \frac{t^{-2}}{2}\int_\Omega F|\nabla w - \boldsymbol{\theta}|^2 - L_e$$

Kinematic descriptors

w : transverse displacement

$\boldsymbol{\theta} = \{\theta_1, \theta_2\}$: rotation



- Shearable theories (thick plates)

FLEXURE = BENDING + SHEARING

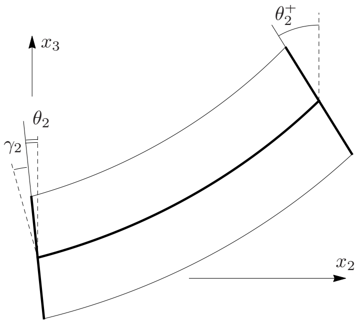
e.g. for the Reissner-Mindlin (RM) plate model

$$E_{\mathsf{RM}} = \frac{1}{2}\int_\Omega \mathcal{D}\nabla^{\mathsf{s}}\boldsymbol{\theta} : \nabla^{\mathsf{s}}\boldsymbol{\theta} + \frac{t^{-2}}{2}\int_\Omega F|\nabla w - \boldsymbol{\theta}|^2 - L_e$$

Strain measures

$\boldsymbol{K} = \nabla^{\mathsf{s}}\boldsymbol{\theta}$: curvature

$\boldsymbol{\gamma} = \nabla w - \boldsymbol{\theta}$: shearing



- Shearable theories (thick plates)

FLEXURE = BENDING + SHEARING

e.g. for the Reissner-Mindlin (RM) plate model

$$E_{\mathsf{RM}} = \frac{1}{2}\int_\Omega \mathcal{D}\nabla^{\mathsf{s}}\boldsymbol{\theta} : \nabla^{\mathsf{s}}\boldsymbol{\theta} + \frac{t^{-2}}{2}\int_\Omega F|\nabla w - \boldsymbol{\theta}|^2 - L_e$$

- Bending theories (thin plates)

$$\nabla w = \boldsymbol{\theta} \quad \text{and} \quad \text{FLEXURE} = \text{BENDING}$$

e.g. for the Kirchhoff-Love (KL) plate model

$$E_{\mathsf{KL}} = \frac{1}{2}\int_\Omega \mathcal{D}\nabla\nabla w : \nabla\nabla w - L_e$$

Remark: when $t \to 0$ the RM-model asymptotically converges to the KL-model with $\nabla w = \boldsymbol{\theta}$.

FEniCS-shells. Models

		Shearable	Bending	Linear	Nonlinear
✓	Kirchhoff-Love plate model		●	●	
✓	Reissner-Mindlin plate model	●		●	
✓	Hierarchical plate models			●	
✓	von Kármán plate model		●		●
✓	Marguerre shallow shell model		●		●
	Koiter shell model		●		●
	Naghdi shell model	●			●

- Bending models have been implemented by employing the C/D Galerkin formulation [Engel et al., 2002] in order to avoid $H^2(\Omega)$-finite elements
- Shearable models have been implemented by employing the MITC formulation [Dvorkin and Bathe, 1986] in order to avoid numerical locking.

Outline

Motivations and Objectives

Remarks on shell theories

MITC implementation

Results

Future Perspectives and References

Numerical locking

We consider the sequence of problems in the thickness parameter t

$$\min_{\boldsymbol{\theta}, w \in U} \left\{ \frac{1}{2} a(\boldsymbol{\theta}, \boldsymbol{\theta}) + \frac{t^{-2}}{2} \int_\Omega |\nabla w - \boldsymbol{\theta}|^2 - L_e \right\} \; U = \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$$

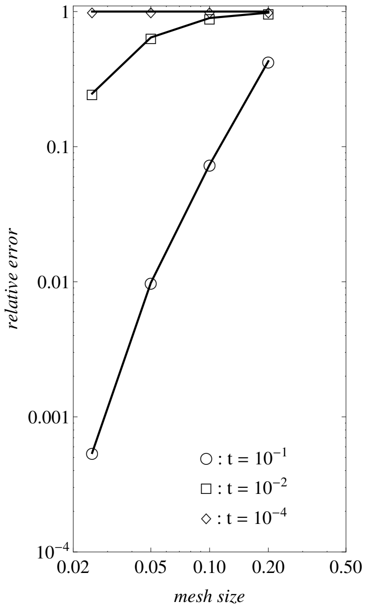
For $t \to 0$ we have the limit problem

$$\min_{\boldsymbol{\theta}, w \in K} \left\{ \frac{1}{2} a(\boldsymbol{\theta}, \boldsymbol{\theta}) - L_e \right\} \qquad K = \{ (\boldsymbol{\theta}, w) \in U \; : \; \nabla w = \boldsymbol{\theta} \}$$

while the corresponding discrete problem has limit problem

$$\min_{\boldsymbol{\theta}_h, w_h \in K_h} \left\{ \frac{1}{2} a(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) - L_e \right\} \qquad K_h = K \cap (\boldsymbol{\Theta}_h \times W_h)$$

If K_h is not large enough the basis functions cannot properly represent the Kirchhoff's constraint $\nabla w = \boldsymbol{\theta}$. The shear term doesn't vanish.


1
0.1
0.01
0.001
10^{-4}
0.02
0.05
0.10
0.20
0.50
relative error
mesh size
○ : t = 10^{-1}
□ : t = 10^{-2}
◇ : t = 10^{-4}

Locking cure

Since the problem relies in the shear term $\boldsymbol{\gamma} = t^{-2}(\nabla w - \boldsymbol{\theta})$ and for $\boldsymbol{\theta} \in \boldsymbol{H}^1(\Omega), w \in H^1(\Omega)$

$$\nabla w - \boldsymbol{\theta} \in \boldsymbol{H}(\text{curl}\,, \Omega)$$

it is possible to use the mixed formulation with penalty term

Find $(\boldsymbol{\theta}, w) \in U$ and $\boldsymbol{\gamma} \in \boldsymbol{H}(\text{curl}\,, \Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}) + (\nabla \tilde{w} - \tilde{\boldsymbol{\theta}}, \boldsymbol{\gamma}) &= (f, \tilde{w}) \qquad && \forall (\tilde{\boldsymbol{\theta}}, \tilde{w}) \in U \\ (\nabla w - \boldsymbol{\theta}, \tilde{\gamma}) - t^2(\boldsymbol{\gamma}, \tilde{\gamma}) &= 0 && \forall \tilde{\gamma} \in \boldsymbol{H}(\text{curl}\,, \Omega) \end{aligned}$$

MITC formulation

The idea: the discretization of the mixed formulation can be transformed into a displacement form

$$\min_{\boldsymbol{\theta}_h, w_h \in U_h} \left\{ \frac{1}{2} a(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) + \frac{t^{-2}}{2} \int_\Omega |\nabla w_h - R_h \boldsymbol{\theta}_h|^2 - L_e \right\}$$

where the reduction operator

$$R_h \;:\; \boldsymbol{H}^1(\Omega) \to \boldsymbol{\Gamma}_h \subset \boldsymbol{H}(\mathsf{curl}\,, \Omega)$$

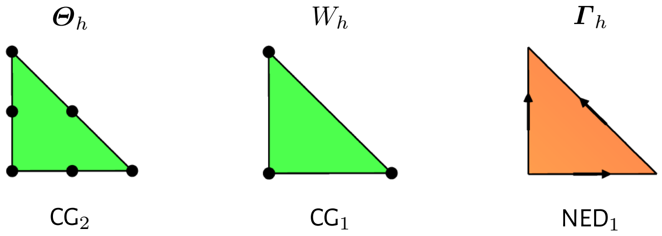
interpolates piecewise smooth functions into the shear space $\boldsymbol{\Gamma}_h$.

Main advantages

- It leads to systems of equations with positive definite matrices and fewer unknowns
- The action of R_h is local and the system matrix can be assembled locally

Elements in MITC family are different for the choice of $\boldsymbol{\Theta}_h, W_h, \boldsymbol{\Gamma}_h$ and the tying, as expressed by R_h, between the interpolation in Γ_h and the shear strain as evaluated from $\boldsymbol{\Theta}_h, W_h, \boldsymbol{\Gamma}_h$.

- Duran-Liberman [Duran, Liberman, 1992]



$$R_h \; : \; \int_e (\boldsymbol{\theta} - R_h \boldsymbol{\theta}) \cdot \boldsymbol{t} = 0 \quad \forall e \in T$$

FEniCS implementation

First, we define the full mixed space $U_h^F = \boldsymbol{\Theta}_h \times W_h \times \boldsymbol{\Gamma}_h$

```
V_3 = FunctionSpace(mesh, "CG", 1)
R = VectorFunctionSpace(mesh, "CG", 2)
RR = FunctionSpace(mesh, "N1curl", 1)
U_F = MixedFunctionSpace([R, V_3, RR])
```

and the reduction/tying operator

$$\int_e (\boldsymbol{\gamma}_h \cdot \boldsymbol{t})(\tilde{\boldsymbol{\theta}}_h^R \cdot \boldsymbol{t})$$

```
def R_e(gam, R_th_t, U):
    dSp = Measure('dS', metadata={'quadrature_degree': 1})
    dsp = Measure('ds', metadata={'quadrature_degree': 1})
    n = FacetNormal(U.mesh())
    t = as_vector((-n[1], n[0]))
    area = FacetArea(U.mesh())
    return (area*inner(gam, t)*inner(R_th_t, t))("+")*dSp + \
           (area*inner(gam, t)*inner(R_th_t, t))*dsp
```

We define the projection form a_{10}

```
ga = lambda z1, z2: grad(z2) - z1
g = ga(r, w)
g_t = ga(r_t, w_t)
a_10 = R_e(g, rr_t, U_F) + R_e(g_t, rr, U_F)
```

the shear form a_{01}

```
Pi_she = (eps**-2)*0.5*inner(T(ga(rr_, w_)), ga(rr_, w_))*dx
F_she = derivative(Pi_she, u_, u_t)
a_01 = derivative(F_she, u_, u)
```

the primal mixed space $U_h = \boldsymbol{\Theta}_h \times W_h$

```
U = MixedFunctionSpace([R, V_3])
```

and the unprojected bending form a_{00}

```
Pi_ben = 0.5*inner(M(ep(r_)), ep(r_))*dx
dPi_ben = derivative(Pi_ben, u_, u_t)
a_00 = derivative(dPi_ben, u_, u)
```

Finally, we use a custom assembler to perform the projection at the local linear algebra level

```
A = mitc_assemble([a_00, a_01, a_10])
```

- C++ code to assemble the stiffness matrix

$$A = A_{00} + A_{10} A_{01} A_{10}$$

A_{00} : unprojected bending term

A_{01} : projected shearing term

A_{10} : projection matrix

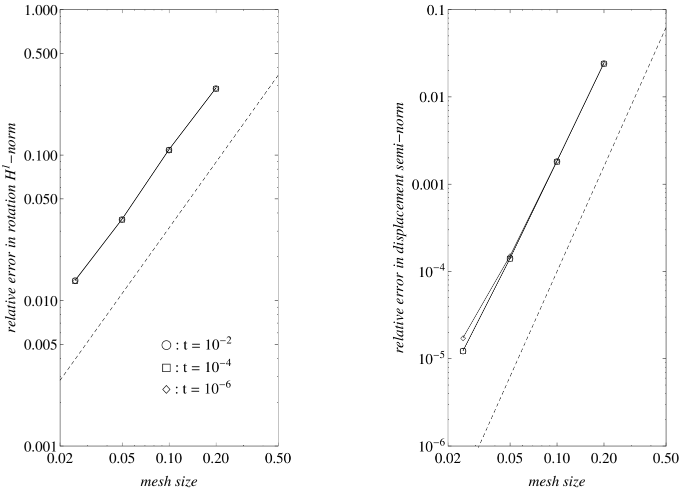
- JIT compilation with Instant

Outline

M. Brunetti, J. S. Hale, S. Bordas, C. Maurini

RM plate. Convergence

Comparison with the analytical solution provided by [Lovadina, 1995].



von Kármán plate model

Starting with the scalings

$$\boldsymbol{u} = O(\epsilon^2) \qquad w = O(\epsilon) \qquad \epsilon = \|\boldsymbol{K}\|$$

the vK plate model retains the minimal geometrical nonlinearities able to catch the coupling between bending and membrane strains

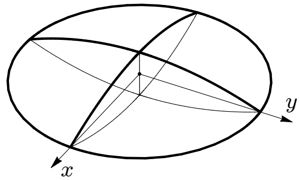
$$\boldsymbol{E} = \nabla^{\mathsf{s}}\boldsymbol{u} + \frac{\nabla w \otimes \nabla w}{2} \qquad \boldsymbol{K} = \nabla\nabla w$$

whose integrability conditions correspond to the linearization of the Gauss theorema egregium

$$\text{curl curl}\,\boldsymbol{E} = \det \boldsymbol{K}$$

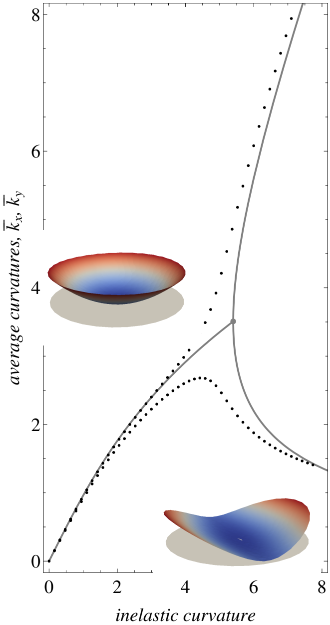
Since $E_{\mathsf{vK}} = E_{\mathsf{KL}} + t^{-2}E_{\mathsf{m}}$, whenever possibile a plate tends to bend in a developable surface ($E_{\mathsf{m}} = 0$ for $\det \boldsymbol{K} = 0$).

Circular plate with lenticular cross section subjected to a temperature gradient through its thickness



Average curvatures bifurcation at critical value [Mansfield, 1962]

$$\kappa_T = \frac{8}{(1+\nu)^{3/2}}$$



Outline

Future Perspectives

- to implement membrane locking-free element for the nonlinear Koiter shell model
- to implement membrane and shear locking-free element for the nonlinear Naghdi shell model

Main References

- Jeon, H.-M., Lee, P.-S. and Bathe, K. J., *The MITC3 shell finite element enriched by interpolation covers*, Computers and Structures, 2014
- Hale, J. S. and Baiz P., M., *Towards effective shell modelling with the FEniCS project*, FEniCS Workshop 2013
- Chapelle, D. and Bathe, K. J., *The finite element analysis of shells*, 2010.
- Lovadina, C., *A Brief Overview of Plate Finite Element Methods*, Integral Methods in Science and Engineering, 2010
- Engel, G., et al. *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Computer Methods in Applied Mechanics and Engineering, 2002
- Durán, R., and Liberman, E., *On mixed finite element methods for the Reissner-Mindlin plate model*, Mathematics of computation, 1992
- Brezzi, F., Bathe, K .J., and Fortin, M., *Mixed-interpolated elements for Reissner-Mindlin plates*, International Journal for Numerical Methods in Engineering, 1989
- Mansfield, E. H., *Bending, buckling and curling of a heated thin plate*, Proceedings of the Royal Society of London A, 1962

Thanks for listening