Generalizations and variants of associativity for aggregation functions

AGOP 2015
Tutorial presentation

Jean-Luc Marichal

in collaboration with Bruno Teheux

University of Luxembourg

Table of contents

Part 0: Introduction and notation

Associative binary operations

Let X be a nonempty set

Consider an operation $F\colon X^2 \to X$
Denote $F(a, b)$ by ab

F is *associative* if

$$(ab)c \;=\; a(bc)$$

$\longrightarrow$ enables us to define the expression abc by setting

$$abc \;=\; (ab)c$$

Question: How can we define $abcd$?

Associative binary operations

By associativity, we have

$$\begin{aligned}(abc)d &= (a(bc))d = a(bc)d = a((bc)d) = a(bcd) \\ &= a(b(cd)) = ab(cd) = \cdots \\ &= \cdots\end{aligned}$$

$\rightarrow$ we can define *abcd* by setting

$$abcd = (abc)d$$

Associativity shows that the expression *abcd* can be computed regardless of how the parentheses are inserted

Associative binary operations

For any $a_1, \ldots, a_n \in X$, we can set

$$a_1 \cdots a_{n-1} a_n \quad = \quad (a_1 \cdots a_{n-1}) a_n$$

... can be computed regardless of how parentheses are inserted

In other words, associativity shows that

$$a_1 \cdots a_j \cdots a_k \cdots a_n \quad = \quad a_1 \cdots (a_j \cdots a_k) \cdots a_n$$

holds for any $1 \leqslant j < k \leqslant n$ with $1 < j$ or $k < n$

(associativity for operations with an indefinite arity)

Associative operations with an indefinite arity

We started with a binary operation $F\colon X^2 \to X$ and we now extend its domain to the set $X^2 \cup X^3 \cup X^4 \cup \cdots$

$$F: \bigcup_{n\geqslant 2} X^n \to X$$

Assume that F satisfies

- $F(F(a_1, a_2), a_3) = F(a_1, F(a_2, a_3))$
- $F(a_1, \ldots, a_{n-1}, a_n) = F(F(a_1, \ldots, a_{n-1}), a_n)\ , \quad n \geqslant 3$

Then (and only then)

$$F(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n) \quad = \quad F(a_1, \ldots, F(a_j, \ldots, a_k), \ldots, a_n)$$

Notation

- X = *alphabet*
- Elements of X: *letters* $(x, y, z, \ldots \in X)$
- The set

$$X^* = \bigcup_{n \geqslant 0} X^n$$

is the set of all tuples on X, called *strings over X*
$(\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots \in X^*)$

Convention: $X^0 = \{\varepsilon\}$, where ε = *the empty string*

Notation

- X^* is endowed with concatenation ($\varepsilon =$ neutral element)

$$\mathbf{x} \in X^n \text{ and } y \in X \quad \Rightarrow \quad \mathbf{x}y\varepsilon = \mathbf{x}y \in X^{n+1}$$

- Repeated strings

$$\mathbf{x}^n = \underbrace{\mathbf{x} \cdots \mathbf{x}}_{n}, \qquad \mathbf{x}^0 = \varepsilon$$

- Length of a string

$$|\mathbf{x}| = n \quad \Longleftrightarrow \quad \mathbf{x} \in X^n$$

$$|\varepsilon| = 0, \qquad |x| = 1$$

Notation

Let Y be a nonempty set

- *n-ary function*

$$F\colon X^n \to Y$$

- *$*$-ary function* or *variadic function*

$$F\colon X^* \to Y$$

n-ary part of F

$$F_n \; = \; F|_{X^n}$$

Default value of F

$$F(\varepsilon) \; = \; F_0(\varepsilon)$$

Part 1: Associativity, generalizations, and variants

Associativity

- The condition

$$F(x_1 \cdots x_n\, z) \;=\; F(F(x_1 \cdots x_n)\, z)\,, \qquad n \geqslant 2$$

can be rewritten as

$$F(\mathbf{x}z) \;=\; F(F(\mathbf{x})z)\,, \qquad |\mathbf{x}z| \geqslant 3$$

(induction condition)

Associativity

- The condition

$$F(x_1 \cdots x_j \cdots x_k \cdots x_n) \;=\; F(x_1 \cdots F(x_j \cdots x_k) \cdots x_n)$$

for any $1 \leqslant j < k \leqslant n$ with $1 < j$ or $k < n$

can be rewritten as

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z})\,, \qquad |\mathbf{y}| \geqslant 2,\; |\mathbf{xz}| \geqslant 1$$

(associativity on $\bigcup_{n\geqslant 2} X^n$)

Associativity

Proposition

For any map $F\colon \bigcup_{n\geqslant 2} X^n \to X$

$$\begin{cases} F_2 = F|_{X^2} \quad \text{is associative} \\ F(\mathbf{x}z) \ = \ F(F(\mathbf{x})z)\,, \quad |\mathbf{x}z| \geqslant 3 \end{cases}$$

$$\Updownarrow$$

$$F(\mathbf{xyz}) \ = \ F(\mathbf{x}F(\mathbf{y})\mathbf{z})\,, \quad |\mathbf{y}| \geqslant 2,\ |\mathbf{xz}| \geqslant 1$$

$\longrightarrow$ Extension to functions F defined on $X^* \ = \ \bigcup_{n\geqslant 0} X^n$?

Associativity

Definitions

- A *variadic operation* on X is a map $F\colon X^* \to X \cup \{\varepsilon\}$
- A variadic operation $F\colon X^* \to X \cup \{\varepsilon\}$ is said to be *associative* if

$$F(\mathbf{xyz}) \ = \ F(\mathbf{x}F(\mathbf{y})\mathbf{z})\,, \qquad \mathbf{x},\mathbf{y},\mathbf{z} \in X^*$$

We often consider the condition

$$F(\mathbf{x}) \ = \ \varepsilon \quad \Leftrightarrow \quad \mathbf{x} \ = \ \varepsilon$$

(F is ε-standard)

Associativity

F is associative on X^*

$+$

ε-standard

↙ ↘

F is associative on $\bigcup_{n\geqslant 2} X^n$

$\Updownarrow$

F_2 associative

$+$

Induction condition

$$
\begin{aligned}
F_1(F_1(x)) &= F_1(x)\\
F_1(F_2(xy)) &= F_2(xy)\\
F_2(x\,F_1(y)) &= F_2(xy)\\
F_2(F_1(x)y) &= F_2(xy)
\end{aligned}
$$

Associativity

Suppose $F_2 \colon X^2 \to X$ associative is given

> How can we extend F_2 to an associative and ε-standard operation $F \colon X^* \to X \cup \{\varepsilon\}$?

Induction condition $\quad\to\quad$ determine $F_3, F_4, \ldots, F_n, \ldots$ uniquely

What about F_1 ?

$$\begin{aligned} F_1(F_1(x)) &= F_1(x) \\ F_1(F_2(xy)) &= F_2(xy) \;=\; F_2(x\,F_1(y)) \;=\; F_2(F_1(x)y) \end{aligned}$$

Note that $F_1 = \mathrm{id}_X$ is a possible solution

Associativity

Example (sum). Take $X = \mathbb{R}$ and

$$\boxed{F_2(x_1 x_2) \;=\; x_1 + x_2}$$

Induction $\rightarrow$ $F_n(\mathbf{x}) \;=\; F(x_1 \cdots x_n) \;=\; x_1 + \cdots + x_n\,, \qquad n \geqslant 2$

$F_1 \;=\; ?$

We have $F_1(x) \;=\; F_1(x+0) \;=\; F_1(F_2(x0)) \;=\; F_2(x0) \;=\; x$

$\Rightarrow$ we get $F_1 = \mathrm{id}_{\mathbb{R}}$.

Associativity

Example (Euclidean norm). Take $X = \mathbb{R}$ and

$$F_2(x_1 x_2) = \sqrt{x_1^2 + x_2^2}$$

Induction $\rightarrow$ $F_n(\mathbf{x}) = \sqrt{x_1^2 + \cdots + x_n^2} = \|\mathbf{x}\|_2 \qquad n \geqslant 2$

F_1 must satisfy

$$\begin{aligned} F_1(F_1(x)) &= F_1(x) \\ F_1(F_2(xy)) &= F_2(xy) = F_2(x\,F_1(y)) = F_2(F_1(x)y) \end{aligned}$$

$\Rightarrow \quad F_1(x) = x$ and $F_1(x) = \sqrt{x^2}$ are possible solutions

Associativity

Example (t-norm). A *t-norm* is an associative binary operation $T\colon [0,1]^2 \to [0,1]$ that is symmetric, nondecreasing in each argument, and satisfies $T(x1) = T(1x) = x$.

$\longrightarrow$ Extension to a variadic t-norm : $T\colon [0,1]^* \to [0,1] \cup \{\varepsilon\}$

We have $T_1(x) \;=\; T_1(T_2(x1)) \;=\; T_2(x1) \;=\; x$

Example. If $T(x_1x_2) = \max(0, x_1 + x_2 - 1)$, then

$$T_n(x_1 \cdots x_n) \;=\; \max(0, \textstyle\sum_{i=1}^{n} x_i - n + 1)$$

Note. Same approach for t-conorms, uninorms,...

Associativity

There are also associative operations $F\colon X^* \to X \cup \{\varepsilon\}$ that are not ε-standard

Example. Let $a \in X$ and define

$$F(\mathbf{x}) \ = \ \begin{cases} a\,, & \text{if } \mathbf{x} = \mathbf{u}a\mathbf{v} \text{ for some } \mathbf{uv} \in X^* \\ \varepsilon\,, & \text{otherwise} \end{cases}$$

Associativity is satisfied

$$F(\mathbf{xyz}) \ = \ F(\mathbf{x}F(\mathbf{y})\mathbf{z})$$

Definition. A *string function* over X is a function $F\colon X^* \to X^*$

$\rightarrow$ not an aggregation procedure

Examples (data processing tasks)

- $F(\mathbf{x})$ = sorting the letters of $\mathbf{x}$ in alphabetic order
- $F(\mathbf{x})$ = transforming a string $\mathbf{x}$ into upper case
- $F(\mathbf{x})$ = removing from $\mathbf{x}$ all occurrences of a given letter
- $F(\mathbf{x})$ = removing from $\mathbf{x}$ all repeated occurrences of letters

$$F(\text{associativity}) = \text{as}\not{\text{s}}\text{oci}\not{\text{a}}\text{t}\not{\text{i}}\text{v}\not{\text{i}}\not{\text{t}}\text{y} = \text{asocitvy}$$

Each of these tasks satisfies

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z})$$

Associativity for string functions

Definition. A string function $F\colon X^* \to X^*$ is said to be *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}), \qquad \mathbf{xyz} \in X^*$$

$\to$ generalizes associativity for operations $F\colon X^* \to X \cup \{\varepsilon\}$

i.e., the properties of associativity for string functions also hold for variadic operations

Proposition

The definition above remains equivalent if we assume $|\mathbf{xz}| \leqslant 1$

Associativity for string functions

Note. Setting $\mathbf{x} = \mathbf{z} = \varepsilon$ in the identity

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z})$$

we obtain $F(\mathbf{y}) \;=\; F(F(\mathbf{y}))$

$$\boxed{F \;=\; F \circ F}$$

Associativity for string functions

Further equivalent forms

Proposition

Assume $F\colon X^* \to X^*$ satisfies $F(\varepsilon) = \varepsilon$

The following conditions are equivalent :

(i) F is associative

(ii) $F(\mathbf{xy}) = F(F(\mathbf{x})F(\mathbf{y}))$

(iii) $F(F(\mathbf{xy})\mathbf{z}) = F(\mathbf{x}F(\mathbf{yz}))$

(iv) $\mathbf{xyz} = \mathbf{uvw} \quad \Rightarrow \quad F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{u}F(\mathbf{v})\mathbf{w})$

Associativity for string functions

Definition. Let $m \geqslant 0$ be an integer. A function $F\colon X^* \to X^*$ is said to be *m-bounded* if $|F(\mathbf{x})| \leqslant m$ for every $\mathbf{x} \in X^*$

Example. The variadic operations $F\colon X^* \to X \cup \{\varepsilon\}$ are exactly the 1-bounded string functions over X

Proposition

Assume that $F\colon X^* \to X^*$ is associative

(a) F is m-bounded if and only if $F_0, \ldots, F_{m+1}$ are m-bounded

(b) If m-bounded, F is uniquely determined by $F_0, \ldots, F_{m+1}$

Example (cont.) An associative function $F\colon X^* \to X^*$ is 1-bounded iff F_0, F_1, and F_2 range in $X \cup \{\varepsilon\}$

In this case, F is completely determined by F_0, F_1, and F_2

Preassociativity

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})$$

(we can assume $|\mathbf{xz}| \leqslant 1$)

Examples. $F\colon \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$

- $F_0 = \varepsilon, \quad F_n(\mathbf{x}) = x_1 + \cdots + x_n$
- $F_0 = \varepsilon, \quad F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 = \|\mathbf{x}\|_2^2$
- $F_0 = \varepsilon, \quad F_n(\mathbf{x}) = g(x_1 + \cdots + x_n), \quad g$ one-to-one

$$\left.\begin{array}{l} F \text{ preassociative} \\ g, h \text{ one-to-one} \end{array}\right\} \Rightarrow \quad g \circ F \text{ and } F \circ (h, \ldots, h) \text{ preassociative}$$

Preassociativity

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Equivalent definition

$$\left.\begin{array}{rcl} F(\mathbf{x}) & = & F(\mathbf{x}') \\ F(\mathbf{y}) & = & F(\mathbf{y}') \end{array}\right\} \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}) \;=\; F(\mathbf{x}'\mathbf{y}').$$

Preassociativity

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy'z})$$

Proposition

Let $F\colon X^* \to X^*$ be a string function

$$F \text{ associative} \quad\Leftrightarrow\quad \begin{cases} F\circ F = F \\ F \text{ preassociative} \end{cases}$$

Proof. ($\Rightarrow$) We have $F\circ F = F$ by associativity.
Assume that $F(\mathbf{y}) = F(\mathbf{y}')$. Then

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \;=\; F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) \;=\; F(\mathbf{xy'z})$$

($\Leftarrow$) We have $F(F(\mathbf{y})) = F(\mathbf{y})$, and hence $F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{xyz})$ by preassociativity. □

Preassociativity

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Aggregation version :

Proposition

Let $F\colon X^* \to X \cup \{\varepsilon\}$ be ε-standard

$$F \text{ associative} \quad \Leftrightarrow \quad \begin{cases} F_1 \circ F^+ = F^+ \\ F \text{ preassociative} \end{cases}$$

where $F^+ = F|_{\cup_{n\geqslant 1} X^n}$

Note. Contrary to associativity, preassociativity does not involve any composition of functions

Preassociativity

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy'z})$$

Various codomains can be considered

Examples $F\colon X^* \to \mathbb{Z}$

- $F(\mathbf{x}) = |\mathbf{x}|$ (number of letters in $\mathbf{x}$)
- $F(\mathbf{x}) =$ number of occurrences in $\mathbf{x}$ of a given letter, say 'z'
- $F(\mathbf{x}) =$ number of letters distinct from z minus the number of occurrences of z

Note. The function that outputs the number of distinct letters in $\mathbf{x}$ is not preassociative:

If $a, b \in X$ are distinct, then $F(a) = F(b) = 1$ but

$$1 = F(aa) \neq F(ab) = 2$$

Preassociativity

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative

(ii) F can be factorized into

$$F \;=\; f \circ H$$

where $H\colon X^* \to X^*$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one

Example. $F(\mathbf{x}) = |\mathbf{x}|$

$$H(\mathbf{x}) \;=\; \mathbf{x}\,, \qquad f(\mathbf{x}) \;=\; |\mathbf{x}| \;? \qquad \text{No !}$$

Preassociativity

Example. $F(\mathbf{x}) = |\mathbf{x}|$

Fix $a \in X$ and define $H(\mathbf{x}) = a^{|\mathbf{x}|}$

$$\mathrm{ran}(H) \;=\; \{a^n \mid n \geqslant 0\}$$

H is associative

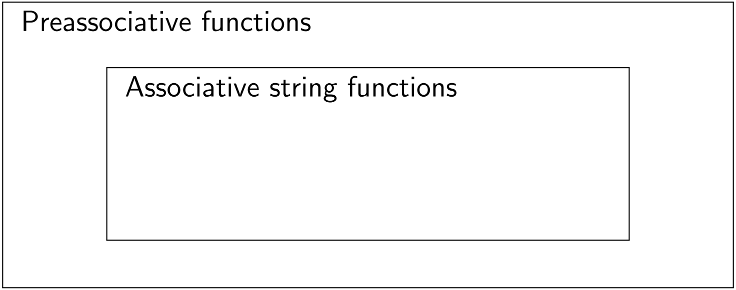
$$H(\mathbf{x}H(\mathbf{y})\mathbf{z}) \;=\; H(\mathbf{x}a^{|\mathbf{y}|}\mathbf{z}) \;=\; a^{|\mathbf{xyz}|} \;=\; H(\mathbf{xyz})$$

Define $f\colon \mathrm{ran}(H) \to \mathbb{N}$ by $f(a^n) = n$

$\longrightarrow$ f is one-to-one and

$$F \;=\; f \circ H$$

Preassociativity



Preassociativity

Recall that a function $F\colon X^* \to X^*$ is said to be *ε-standard* if

$$F(\mathbf{x}) = \varepsilon \quad \Leftrightarrow \quad \mathbf{x} = \varepsilon$$

Definition
A variadic function $F\colon X^* \to Y$ is said to be *standard* if

$$F(\mathbf{x}) = F(\varepsilon) \quad \Leftrightarrow \quad \mathbf{x} = \varepsilon$$

This means that $\operatorname{ran}(F_0) \cap \operatorname{ran}(F^+) = \varnothing$

Preassociativity

Aggregation version :

Theorem

Assume that $F\colon X^* \to Y$ is standard

The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F^+)$

(ii) F can be factorized into

$$F^+ \;=\; f \circ H^+$$

where $H\colon X^* \to X \cup \{\varepsilon\}$ is associative and ε-standard
$f\colon \mathrm{ran}(H^+) \to Y$ is one-to-one

This result enables us to generate preassociative functions from known associative variadic operations

Preassociativity

Example. Let $V = d$-dim. vector space on $\mathbb{R}$ and $B =$ basis for V
$(v)_B =$ coordinate vector for $v \in V$ relative to B
Note. $(\cdot)_B \colon V \to \mathbb{R}^d$ is one-to-one

Consider the operation $H \colon V^* \to V \cup \{\varepsilon\}$ defined by

$$H(\varepsilon) \;=\; \varepsilon \quad \text{and} \quad H(v_1 \cdots v_n) \;=\; \textstyle\sum_{i=1}^{n} v_i$$

Fix $e \notin \mathbb{R}^d$. The function $F \colon V^* \to \mathbb{R}^d \cup \{e\}$ defined by

$$F(\varepsilon) \;=\; e \quad \text{and} \quad F(v_1 \cdots v_n) \;=\; \big(\textstyle\sum_{i=1}^{n} v_i\big)_B$$

is preassociative, standard, and $\mathrm{ran}(F_1) = \mathbb{R}^d = \mathrm{ran}(F^+)$

Preassociativity

This factorization result also enables us to produce axiomatizations of classes of preassociative functions from known axiomatizations of classes of associative functions

A class of associative binary operations $H\colon X^2 \to X$

$\Downarrow$ (extension)

A class of associative variadic operations $H\colon X^* \to X \cup \{\varepsilon\}$

$\Downarrow$ (factorization)

A class of preassociative variadic functions $F\colon X^* \to Y$

Preassociativity

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) \; = \; \varphi^{-1}(\varphi(x) + \varphi(y))$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

$$H(x_1 \cdots x_n) \; = \; \varphi^{-1}\big(\textstyle\sum_{i=1}^{n} \varphi(x_i)\big)$$

Preassociativity

Theorem

Assume that $F\colon \mathbb{R}^* \to \mathbb{R} \cup \{F(\varepsilon)\}$ is standard

The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F^+)$, F_1 and F_2 are continuous and one-to-one in each argument

(ii) we have

$$F(x_1 \cdots x_n) \;=\; \psi\big(\textstyle\sum_{i=1}^n \varphi(x_i)\big)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone

Preassociativity

Extension of a binary t-norm to an (ε-standard) variadic t-norm

$$T\colon [0,1]^2 \to [0,1] \qquad \longrightarrow \qquad T\colon [0,1]^* \to [0,1] \cup \{\varepsilon\}$$

Theorem

Suppose $F\colon [0,1]^* \to \mathbb{R} \cup \{F(\varepsilon)\}$ is standard + F_1 str. $\nearrow$
The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F^+)$,
F_2 is symmetric, nondecreasing in each argument
$F(x1) = F(1x) = F(x)$

(ii) we have

$$F(x_1 \cdots x_n) \ = \ (f \circ T)(x_1 \cdots x_n)$$

where $f\colon [0,1] \to \mathbb{R}$ is strictly increasing
$T\colon [0,1]^* \to [0,1] \cup \{\varepsilon\}$ is a variadic t-norm

Preassociativity

Open questions

1. Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative operations
2. Find interpretations of the preassociativity property in aggregation function theory and/or fuzzy logic

Strong preassociativity

Definition. We say that $F\colon X^* \to Y$ is *strongly preassociative* if

$$F(\mathbf{xz}) = F(\mathbf{x}'\mathbf{z}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{x}'\mathbf{y}\mathbf{z}')$$

We can insert letters anywhere

$$F(abc) = F(abc) \quad \Rightarrow \quad F(axbc) = F(abcx)$$

Proposition

Let $F\colon X^* \to Y$

$$F \text{ strongly preassociative} \quad \Leftrightarrow \quad \begin{cases} F \text{ preassociative} \\ F_n \text{ is symmetric } \forall\, n \geqslant 1 \end{cases}$$

Part 2: Barycentric associativity, generalizations, and variants

Barycentric associativity

Definition. A variadic operation $F\colon X^* \to X \cup \{\varepsilon\}$ is said to be *barycentrically associative* (or *B-associative*) if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

$$F(abcd) \;=\; F(F(ab)^2cd) \;=\; F(F(ab)F(ab)cd)$$

Notes.

- ...first considered for symmetric functions on $\bigcup_{n\geqslant 1}\mathbb{R}^n$ (Schimmack 1909, Kolmogoroff 1930, Nagumo 1930)
- ...can be considered also for string functions $F\colon X^* \to X^*$
  $F(\mathbf{x})$ = removing from $\mathbf{x}$ all repeated occurrences of letters

Barycentric associativity

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

Suppose $F\colon X^* \to X \cup \{\varepsilon\}$ is B-associative and ε-standard
Then F remains B-associative if we modify $F(\varepsilon)$

Proof. Define $G\colon X^* \to X$ by $G(\varepsilon) = e \in X$ and $G^+ = F^+$

$$G(\mathbf{x}G(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \;=\; ?$$

- If $\mathbf{y} = \varepsilon$, then $G(\mathbf{x}G(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) = G(\mathbf{x}\varepsilon\mathbf{z}) = G(\mathbf{xyz})$
- If $\mathbf{y} \neq \varepsilon$, then

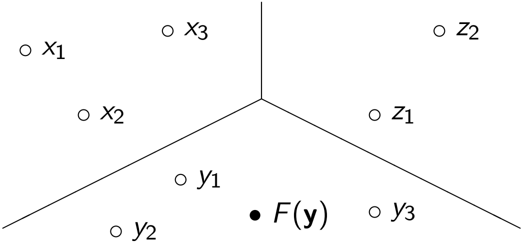
$$G(\mathbf{x}G(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \;=\; F(\mathbf{xyz}) \;=\; G(\mathbf{xyz}) \qquad \square$$

$\Rightarrow$ *The value $F(\varepsilon)$ is unimportant and we can assume* $\mathrm{ran}(F) \subseteq X$

Barycentric associativity

Consider $X = \mathbb{R}^n$ as an infinite set of identical homogeneous balls
i.e., each ball is identified by the coordinates $x \in \mathbb{R}^n$ of its center
Define $F\colon X^* \to X$ as

$$F(x_1 \cdots x_n) = \text{barycenter of the balls } x_1, \ldots, x_n$$



$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

Barycentric associativity

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

Example. Arithmetic mean $F\colon \mathbb{R}^* \to \mathbb{R}$

$$F(x_1 \cdots x_n) \;=\; \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\begin{aligned}
F(x_1\, F(x_2 x_3)^2) &= F\left(x_1\, \tfrac{x_2+x_3}{2}\, \tfrac{x_2+x_3}{2}\right)\\
&= \tfrac{1}{3}\left(x_1 + \tfrac{x_2+x_3}{2} + \tfrac{x_2+x_3}{2}\right)\\
&= \tfrac{1}{3}\left(x_1 + x_2 + x_3\right)\\
&= F(x_1\, x_2\, x_3)
\end{aligned}$$

Barycentric associativity

Definition. *Quasi-arithmetic means*
$\mathbb{I}$ = non-trivial real interval, possibly unbounded
$f : \mathbb{I} \to \mathbb{R}$ continuous and strictly monotonic

$$F : \mathbb{I}^* \to \mathbb{I}$$

$$F(x_1 \cdots x_n) \;=\; f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$

Note. F is B-associative

Barycentric associativity

Theorem (Kolmogoroff-Nagumo, 1930)

$\mathbb{I}$ = non-trivial real interval, possibly unbounded

Let $F\colon \mathbb{I}^* \to \mathbb{I}$

The following assertions are equivalent:

(i)
- F is B-associative
- F_n symmetric
- F_n continuous
- F_n strictly increasing in each argument
- F_n idempotent, i.e., $F_n(x \cdots x) = F(x^n) = x$

(ii) F is a quasi-arithmetic mean

Let us show that idempotence is redundant
i.e., the other assumptions imply that $\delta_{F_n} = \mathrm{id}_{\mathbb{I}}$

$$\delta_{F_n}(x) \; = \; F_n(x^n)$$

Barycentric associativity

Setting $\mathbf{xz} = \varepsilon$ in the identity

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

we obtain

$$\boxed{F(\mathbf{y}) \;=\; F(F(\mathbf{y})^{|\mathbf{y}|})}$$

For $\mathbf{y} = x^n$, we get

$$\delta_{F_n}(x) = F(\delta_{F_n}(x)^n) = \delta_{F_n} \circ \delta_{F_n}(x)$$

that is

$$\delta_{F_n} \;=\; \delta_{F_n} \circ \delta_{F_n}$$

Applying $\delta_{F_n}^{-1}$ to both sides gives (δ_{F_n} is one-to-one)

$$\mathrm{id}_{\mathbb{I}} \;=\; \delta_{F_n}$$

Barycentric associativity

Further examples of real B-associative functions

- $F_n(\mathbf{x}) \;=\; \min(x_1,\ldots,x_n)$
- $F_n(\mathbf{x}) \;=\; \max(x_1,\ldots,x_n)$
- $F_n(\mathbf{x}) \;=\; x_1$
- $F_n(\mathbf{x}) \;=\; x_n$
- $F_n(\mathbf{x}) \;=\; \sum_{i=1}^{n} \frac{2^{i-1}}{2^n-1}\, x_i$

$$F_n^{\alpha}(\mathbf{x}) \;=\; \frac{\sum_{i=1}^{n} \alpha^{n-i}(1-\alpha)^{i-1}\, x_i}{\sum_{i=1}^{n} \alpha^{n-i}(1-\alpha)^{i-1}}\,, \qquad \alpha\in\mathbb{R}$$

Take $\alpha = 1$, $\alpha = 0$, $\alpha = 1/3$, etc.

Barycentric associativity

$$F(\mathbf{xyz}) \ = \ F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$$

Proposition

The definition above remains equivalent if we assume $|\mathbf{xz}| \leqslant 1$

Proposition

The following conditions are equivalent :

(i) F is B-associative

(ii) $F(\mathbf{xy}) = F(F(\mathbf{x})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|})$

(iii) $F(F(\mathbf{xy})^{|\mathbf{xy}|}\mathbf{z}) = F(\mathbf{x}F(\mathbf{yz})^{|\mathbf{yz}|})$

(iv) $\mathbf{xyz} = \mathbf{uvw} \quad \Rightarrow \quad F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) = F(\mathbf{u}F(\mathbf{v})^{|\mathbf{v}|}\mathbf{w})$

Barycentric associativity

Open questions

1. Find new axiomatizations of classes of B-associative operations
2. Prove or disprove: If an operation $F\colon X^* \to X \cup \{\varepsilon\}$ is B-associative, then there exists a B-associative and idempotent operation $G\colon X^* \to X \cup \{\varepsilon\}$ such that $F_n = \delta_{F_n} \circ G_n$ for every $n \geqslant 1$
3. Prove or disprove: Let $F\colon X^* \to X \cup \{\varepsilon\}$ be a B-associative operation. If F_{n+1} is idempotent for some $n \geqslant 1$, then so is F_n

Strong barycentric associativity

Definition. A variadic operation $F\colon X^* \to X \cup \{\varepsilon\}$ is said to be *strongly barycentrically associative* (or *strongly B-associative*) if, for every $\mathbf{x} \in X^*$, the value $F(\mathbf{x})$ does not change if we

1. select a number of letters in $\mathbf{x}$
2. replace each of them by their aggregated value

$$F(abcd) \;=\; F(F(ac)b\,F(ac)d)$$

Notes

- Strong B-associativity $\Rightarrow$ B-associativity
- B-associativity $+$ F_n symmetric $\forall\, n$ $\Rightarrow$ strong B-assoc.
- $F_n(\mathbf{x}) = x_1 \;\; \forall\, n$: strongly B-associative
- $F_n(\mathbf{x}) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n-1}\, x_i \;\; \forall\, n$: B-associative but not strongly

Strong barycentric associativity

Proposition

Assume that $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative.
Then, for every integer $k \geqslant 1$ and every $x, z \in X$, the function $\mathbf{y} \in X^k \mapsto F_{k+2}(x\mathbf{y}z)$ is symmetric

Strong barycentric associativity

Proposition

Let $F\colon X^* \to X \cup \{\varepsilon\}$

The following assertions are equivalent:

(i) F is strongly B-associative

(ii) $F(\mathbf{xyz}) = F(F(\mathbf{xz})^{|\mathbf{x}|}\,\mathbf{y}\,F(\mathbf{xz})^{|\mathbf{z}|})$

(iii) $F(\mathbf{xyz}) = F(F(\mathbf{xz})^{|\mathbf{x}|}\,F(\mathbf{y})^{|\mathbf{y}|}\,F(\mathbf{xz})^{|\mathbf{z}|})$

Moreover, we may assume that $|\mathbf{y}| \leqslant 1$ in assertions (ii) and (iii)

Strong barycentric associativity

In Kolmogoroff-Nagumo's characterization, B-associativity and symmetry can be replaced with strong B-associativity

Theorem

$\mathbb{I}$ = non-trivial real interval, possibly unbounded

Let $F\colon \mathbb{I}^* \to \mathbb{I}$

The following assertions are equivalent:

(i)
- F is strongly B-associative
- F_n continuous
- F_n strictly increasing in each argument

(ii) F is a quasi-arithmetic mean

Barycentric preassociativity

Definition. We say that $F\colon X^* \to Y$ is *barycentrically preassociative* (or *B-preassociative*) if

$$\left.\begin{array}{rcl} F(\mathbf{y}) & = & F(\mathbf{y}') \\ |\mathbf{y}| & = & |\mathbf{y}'| \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{xyz}) \; = \; F(\mathbf{xy}'\mathbf{z})$$

(we can assume $|\mathbf{xz}| = 1$)

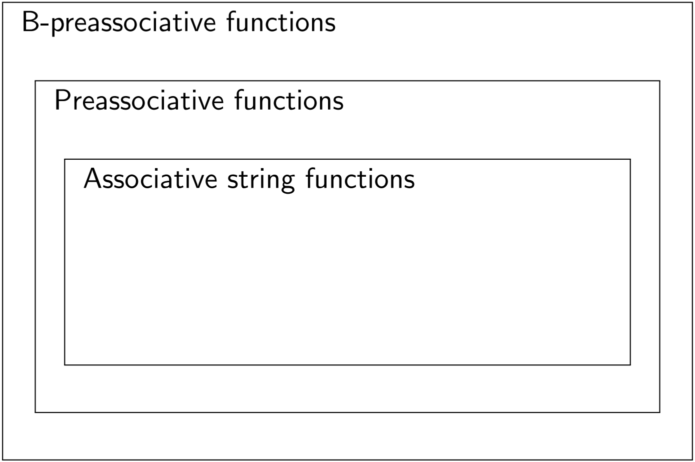
Notes.

- ...inspired from the following property by de Finetti (1931)

$$F(\mathbf{y}) \; = \; F(u^{|\mathbf{y}|}) \quad \Rightarrow \quad F(\mathbf{xyz}) \; = \; F(\mathbf{x}u^{|\mathbf{y}|}\mathbf{z}) \quad (|\mathbf{y}|,\, |\mathbf{xz}| \geqslant 1)$$

- Preassociativity $\Rightarrow$ B-preassociativity
- The value $F(\varepsilon)$ is unimportant

Barycentric preassociativity



Barycentric preassociativity

$$\left.\begin{array}{rcl} F(\mathbf{y}) & = & F(\mathbf{y}') \\ |\mathbf{y}| & = & |\mathbf{y}'| \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Interpretations

- *Decision making* : if we express an indifference when comparing two profiles, then this indifference is preserved when adding identical pieces of information to these profiles
- *Aggregation function theory* : the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation

Barycentric preassociativity

$$\left.\begin{array}{rcl} F(\mathbf{y}) & = & F(\mathbf{y}') \\ |\mathbf{y}| & = & |\mathbf{y}'| \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \; = \; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Equivalent definition

$$\left.\begin{array}{rclcrcl} F(\mathbf{x}) & = & F(\mathbf{x}') & \text{and} & F(\mathbf{y}) & = & F(\mathbf{y}') \\ |\mathbf{x}| & = & |\mathbf{x}'| & \text{and} & |\mathbf{y}| & = & |\mathbf{y}'| \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}) \; = \; F(\mathbf{x}'\mathbf{y}')$$

Barycentric preassociativity

$$\left.\begin{array}{rcl} F(\mathbf{y}) & = & F(\mathbf{y}') \\ |\mathbf{y}| & = & |\mathbf{y}'| \end{array}\right\} \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Let $F\colon X^* \to X^*$

$$F \text{ associative} \quad\Leftrightarrow\quad \begin{cases} F(\mathbf{x}) = F(F(\mathbf{x})) \\ F \text{ preassociative} \end{cases}$$

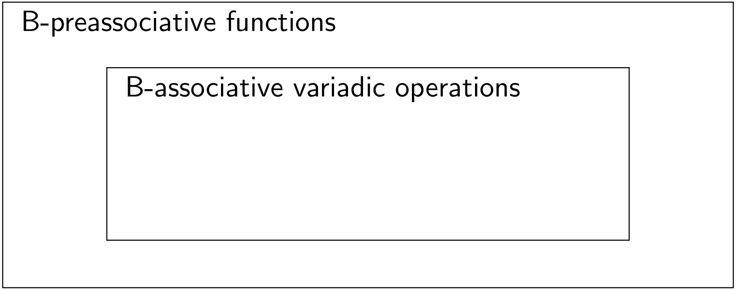
Proposition

Let $F\colon X^* \to X \cup \{\varepsilon\}$

$$F \text{ B-associative} \quad\Rightarrow\quad \begin{cases} F(\mathbf{x}) = F(F(\mathbf{x})^{|\mathbf{x}|}) \\ F \text{ B-preassociative} \end{cases}$$

The converse holds whenever $\mathrm{ran}(F^+) \subseteq X$.

Barycentric preassociativity



Barycentric preassociativity

Examples

- $F(\mathbf{x}) \;=\; |\mathbf{x}|$ (preassociative)
- $\cdots$

Proposition

$$\left.\begin{array}{l} F \quad \text{B-preassociative} \\ g_n \text{ one-to-one } \;\forall\, n \end{array}\right\} \quad\Rightarrow\quad g_n \circ F_n \quad \text{B-preassociative}$$

- F defined by $F_n = g_n\big(\frac{1}{n}\sum_{i=1}^{n} x_i\big)$
- $\cdots$

Barycentric preassociativity

Theorem

Let $F\colon X^* \to Y$

The following assertions are equivalent:

(i) F is B-preassociative and satisfies $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$ $\forall\ n \geqslant 1$

(ii) F can be factorized into

$$F_n \;=\; f_n \circ H_n \qquad \forall\ n \geqslant 1$$

where $H\colon X^* \to X \cup \{\varepsilon\}$ is B-associative
$f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one

... enables us to generalize Kolmogoroff-Nagumo's characterization

Barycentric preassociativity

Quasi-arithmetic means

$\mathbb{I}$ = non-trivial real interval, possibly unbounded

$f\colon \mathbb{I} \to \mathbb{R}$ continuous and strictly monotonic

$$F\colon \mathbb{I}^* \to \mathbb{I}$$

$$F(x_1 \cdots x_n) \;=\; f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$

Definition. *Quasi-arithmetic pre-means*

$f\colon \mathbb{I} \to \mathbb{R}$ and $f_n\colon \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing $(n \geqslant 1)$

$$F\colon \mathbb{I}^* \to \mathbb{R}$$

$$F(x_1 \cdots x_n) \;=\; f_n\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$

Note. F is B-preassociative

Barycentric preassociativity

Quasi-arithmetic pre-means

$$F(x_1 \cdots x_n) \; = \; f_n\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$

$$\left.\begin{array}{c} F \text{ quasi-arithmetic pre-mean} \\ F_n \text{ idempotent } \; \forall n \end{array}\right\} \quad \Leftrightarrow \quad F \text{ quasi-arithmetic mean}$$

Non-idempotent examples

- $f_n(x) = nx$ and $f(x) = x \quad \Rightarrow \quad F(\mathbf{x}) = \sum_{i=1}^{n} x_i$
- $f_n(x) = e^{nx}$ and $f(x) = \ln x \quad \Rightarrow \quad F(\mathbf{x}) = \prod_{i=1}^{n} x_i$

Barycentric preassociativity

Theorem

$\mathbb{I} =$ non-trivial real interval, possibly unbounded

Let $F\colon \mathbb{I}^* \to \mathbb{R}$

The following assertions are equivalent:

(i)
- F is B-preassociative
- F_n symmetric
- F_n continuous
- F_n strictly increasing in each argument

(ii) F is a quasi-arithmetic pre-mean function

Open question. Find a characterization of those quasi-arithmetic pre-mean functions which are preassociative

Barycentric preassociativity

We would like to have...

Theorem

Let $F\colon X^* \to Y$

The following assertions are equivalent:

(i) F is B-preassociative

(ii) F can be factorized into ...

??

Barycentric preassociativity

Definition. A string function $F\colon X^* \to X^*$ is said to be *length-preserving* if $|F(\mathbf{x})| = |\mathbf{x}|$ for every $\mathbf{x} \in X^*$

Examples

- $F = \mathrm{id}_{X^*}$
- $F(\mathbf{x})$ = sorting the letters of $\mathbf{x}$ in alphabetic order
- $F(\mathbf{x})$ = transforming a string $\mathbf{x}$ into upper case
- NOT : $F(\mathbf{x})$ = removing from $\mathbf{x}$ all occurrences of 'z'

Proposition

Let $F\colon X^* \to X^*$ be length-preserving

$$F \quad \text{associative} \quad \Leftrightarrow \quad \begin{cases} F_n \circ F_n = F_n \quad \forall\ n \geqslant 1 \\ F \quad \text{B-preassociative} \end{cases}$$

Barycentric preassociativity

Theorem

Let $F\colon X^* \to Y$

The following assertions are equivalent:

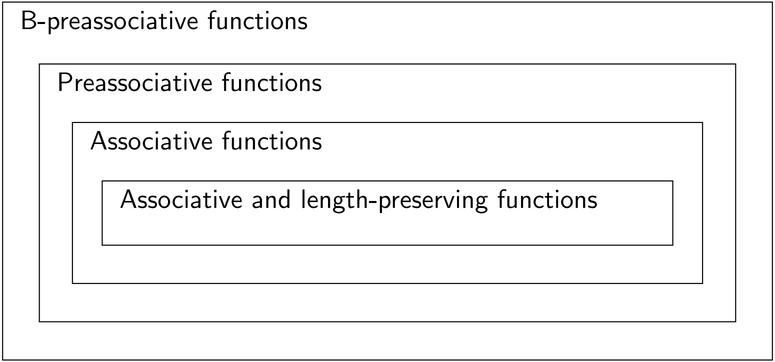
(i) F is B-preassociative

(ii) F can be factorized into

$$F_n \;=\; f_n \circ H_n \qquad \forall\; n \geqslant 1$$

where $H\colon X^* \to X^*$ is associative and length-preserving
$f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one

Barycentric preassociativity



Up to one-to-one unary maps, any of these nested classes can be described in terms of the smallest one

Strong barycentric preassociativity

Definition. A function $F\colon X^* \to Y$ is *strongly B-preassociative* if

$$\left.\begin{array}{rcl} F(\mathbf{x}\mathbf{z}) & = & F(\mathbf{x}'\mathbf{z}') \\ |\mathbf{x}| = |\mathbf{x}'| & \text{and} & |\mathbf{z}| = |\mathbf{z}'| \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}'\mathbf{y}\mathbf{z}')$$

Moreover, we may assume that $|\mathbf{y}| = 1$.

Notes

- Strong B-preassociativity $\Rightarrow$ B-preassociativity
- B-preassoc. + F_n symmetric $\forall\, n$ $\Rightarrow$ strong B-preassoc.
- Factorization results exist ...

Strong barycentric preassociativity

B-preassociativity and symmetry can be replaced with strong B-preassociativity in the axiomatization of the class of quasi-arithmetic pre-mean functions

Theorem

$\mathbb{I} =$ non-trivial real interval, possibly unbounded

Let $F\colon \mathbb{I}^* \to \mathbb{R}$

The following assertions are equivalent:

(i)
- F is strongly B-preassociative
- F_n continuous
- F_n strictly increasing in each argument

(ii) F is a quasi-arithmetic pre-mean function

Thank you for your attention !