GENERALIZATIONS AND VARIANTS OF ASSOCIATIVITY FOR AGGREGATION FUNCTIONS

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Summary

Consider an associative operation $G\colon X^2 \to X$ on a set X and denote $G(a,b)$ merely by ab. By definition, we have $(ab)c = a(bc)$ for all $a,b,c \in X$ and this property enables us to define the expression abc unambiguously by setting $abc = (ab)c$. More generally, for any $a_1,a_2,\ldots,a_n \in X$, we can set

$$a_1a_2a_3\cdots a_n \;=\; (\cdots((a_1a_2)a_3)\cdots)a_n$$

and associativity shows that this expression can be computed regardless of how parentheses are inserted. This means that the identity

$$a_1\cdots a_j\cdots a_k\cdots a_n \;=\; a_1\cdots(a_j\cdots a_k)\cdots a_n$$

holds for any integers $1 \leqslant j \leqslant k \leqslant n$.

This latter condition has been considered in aggregation function theory to extend the classical associativity property of binary operations to variadic operations, i.e., those operations that have an indefinite arity. In this note we survey the most recent results obtained not only on this extension of associativity but also on some variants and generalizations of this property, including barycentric associativity and preassociativity.

Keywords: Associativity, Preassociativity, Barycentric associativity, Barycentric preassociativity, Variadic function, String function, Functional equation, Axiomatization.

1 Introduction

Let X denote a nonempty set, called the *alphabet,* and its elements are called *letters*. The symbol X^* stands for the set $\bigcup_{n\geqslant 0} X^n$ of all tuples on X. Its elements are called *strings* and denoted by bold roman letters $\mathbf{x},\mathbf{y},\mathbf{z},\ldots$ If we want to stress that such an element is a letter of X, we use non-bold italic letters $x,y,z,\ldots$ We assume that X^0 has only one element; we denote it by ε and call it the *empty string.* We endow the set X^* with the concatenation operation for which the empty string ε is the neutral element. For instance, if $\mathbf{x} \in X^m$ and $y \in X$, then $\mathbf{x}y\varepsilon = \mathbf{x}y \in X^{m+1}$. For every string $\mathbf{x}$ and every integer $n \geqslant 1$, the power $\mathbf{x}^n$ stands for the string obtained by concatenating n copies of $\mathbf{x}$. By extension, we set $\mathbf{x}^0 = \varepsilon$. The *length* of a string $\mathbf{x}$ is denoted by $|\mathbf{x}|$. For instance, we have $|\varepsilon| = 0$.

Let Y be a nonempty set. Recall that, for every integer $n \geqslant 0$, a function $F\colon X^n \to Y$ is said to be *n-ary.* Also, a function $F\colon X^* \to Y$ is said to have an *indefinite arity* or to be *variadic* or *$\ast$-ary* (pronounced "star-ary"). A unary operation on X^* is a particular variadic function $F\colon X^* \to X^*$ called a *string function* over the alphabet X.

The main functional properties for variadic functions that we present and investigate in this survey are given in the following definition.

Definition 1.1. A string function $F\colon X^* \to X^*$ is said to be

- *associative* if, for every $\mathbf{x},\mathbf{y},\mathbf{z} \in X^*$, we have
$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z})\,;$$
- *barycentrically associative* (or *B-associative*) if, for every $\mathbf{x},\mathbf{y},\mathbf{z} \in X^*$, we have
$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})\,.$$

A variadic function $F\colon X^* \to Y$ is said to be

- *preassociative* if, for every $\mathbf{x},\mathbf{y},\mathbf{y}',\mathbf{z} \in X^*$, we have
$$F(\mathbf{y}) = F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})\,;$$
- *barycentrically preassociative* (or *B-preassociative*) if, for every $\mathbf{x},\mathbf{y},\mathbf{y}',\mathbf{z} \in X^*$, we have
$$\left.\begin{array}{rcl} F(\mathbf{y}) &=& F(\mathbf{y}') \\ |\mathbf{y}| &=& |\mathbf{y}'| \end{array}\right\} \quad\Rightarrow\quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy}'\mathbf{z})\,.$$

For any variadic function $F: X^* \to Y$ and any integer $n \geqslant 0$, we denote by F_n the *n-ary part* of F, i.e., the restriction $F|_{X^n}$ of F to the set X^n. We also let $X^+ = X^* \setminus \{\varepsilon\}$ and denote the restriction $F|_{X^+}$ of F to X^+ by F^+. The range of any function f is denoted by $\mathrm{ran}(f)$.

A variadic function $F: X^* \to Y$ is said to be

- a *variadic operation on* X (or an *operation* for short) if $\mathrm{ran}(F) \subseteq X \cup \{\varepsilon\}$.
- *standard* if $F(\mathbf{x}) = F(\varepsilon)$ if and only if $\mathbf{x} = \varepsilon$.
- *ε-standard* if $\varepsilon \in Y$ and if we have $F(\mathbf{x}) = \varepsilon$ if and only if $\mathbf{x} = \varepsilon$.

2 Associativity and variants

In this main section we investigate the properties given in Definition 1.1.

2.1 Associativity

We first discuss the associativity property for the class of variadic operations, which constitutes an important subclass of string functions. Recall that a binary operation $G: X^2 \to X$ is said to be *associative* if

$$G(G(xy)z) \;=\; G(x\,G(yz))\,, \qquad x,y,z \in X.$$

A huge number of associative binary operations have been discovered and investigated for years. They are at the root of the concepts of group and semigroup. For instance, the set intersection and union over a power set are associative binary operations. Logical connectives "and" and "or" as well as many of their fuzzy counterparts are also associative binary operations.

More recently, associative binary operations have also been studied as real or complex functions within the theories of functional equations and aggregation functions (see, e.g., [15]). Various classes of associative binary operations over real intervals can be found in [2,4,6–9,11,14–17,20–22,24,33,34].

Let us now consider an associative standard operation $F: X^* \to X \cup \{\varepsilon\}$. This operation is necessarily ε-standard and can always be constructed from an associative binary operation $G: X^2 \to X$ simply by setting $F_0 = \varepsilon$, $F_1 = \mathrm{id}_X$, $F_2 = G$, and $F_{n+1}(\mathbf{y}z) = F_2(F_n(\mathbf{y})z)$ for every $n \geqslant 2$. To give an example, from the binary operation $G: \mathbb{R}^2 \to \mathbb{R}$ defined by $G(x,y) = x + y$, we can construct the associative standard operation $F: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined by $F_n(\mathbf{x}) = \sum_{i=1}^n x_i$ for every integer $n \geqslant 1$.

This construction process immediately follows from the following important proposition.

Proposition 2.1 ([19,27,28])**.** *A standard operation $F: X^* \to X \cup \{\varepsilon\}$ is associative if and only if the following conditions hold.*

(a) $F_0(\varepsilon) = \varepsilon$, $F_1 \circ F_1 = F_1$, $F_1 \circ F_2 = F_2$.

(b) $F_2(xy) = F_2(F_1(x)y) = F_2(xF_1(y))$ for all $x, y \in X$.

(c) F_2 is associative.

(d) $F(\mathbf{y}z) = F(F(\mathbf{y})z)$ for all $\mathbf{y} \in X^*$ and all $z \in X$ such that $|\mathbf{y}z| \geqslant 3$.

Proposition 2.1 provides a characterization of associative standard operations $F: X^* \to X \cup \{\varepsilon\}$ in terms of conditions on their constitutive parts F_n $(n \geqslant 0)$. Conditions (a)–(c) are actually necessary and sufficient conditions on F_0, F_1, and F_2 for F to be associative, while condition (d) provides an induction property which shows that every F_n $(n \geqslant 3)$ can be constructed uniquely from F_2.

Corollary 2.2. *Any associative standard variadic operation is completely determined by its unary and binary parts.*

Let us now say some words about associativity for string functions. It is noteworthy that several data processing tasks correspond to associative string functions. For instance, the function which corresponds to sorting the letters of every string in alphabetical order is associative. Similarly, the function which consists in transforming a string of letters into upper case is also associative. In such a context, associativity is a natural property since it enables us to work locally on small pieces of data at a time.

It is to be noted that the definition of associativity remains unchanged if the length of the string $\mathbf{xz}$ is bounded by one. This observation provides an equivalent but weaker form of associativity.

Proposition 2.3 ([19,27,28])**.** *A function $F: X^* \to X^*$ is associative if and only if $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ such that $|\mathbf{xz}| \leqslant 1$.*

2.2 B-associativity

By definition, B-associativity expresses that the function value of a string does not change when replacing every letter of a substring with the value of this substring. For instance, the arithmetic mean over the set of real numbers, regarded as the ε-standard operation $F: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined as $F_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$ for every integer $n \geqslant 1$, is B-associative.

Remark 1. The name B-associativity is justified by the following geometric interpretation. Consider a set of identical homogeneous balls in $X = \mathbb{R}^n$. Each ball

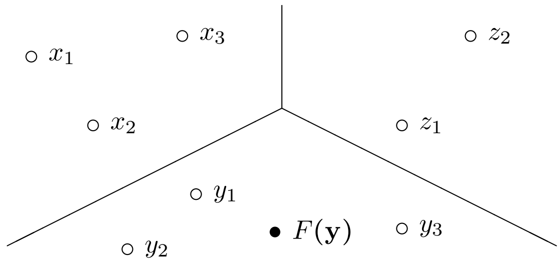

Figure 1: Barycentric associativity

is identified by the coordinates $x \in X$ of its center. Let $F: X^* \to X \cup \{\varepsilon\}$ be the ε-standard variadic operation which carries any set of balls into their barycenter. Because of the associativity-like property of the barycenter, the operation F has to satisfy the functional property of B-associativity (see Fig. 1).

A noteworthy class of B-associative variadic operations is given by the so-called quasi-arithmetic mean functions, axiomatized independently by Kolmogoroff [18] and Nagumo [32].

Definition 2.4. Let $\mathbb{I}$ be a nontrivial real interval (i.e., nonempty and not a singleton), possibly unbounded. A function $F: \mathbb{I}^* \to \mathbb{R}$ is said to be a *quasi-arithmetic mean function* if there is a continuous and strictly monotonic function $f: \mathbb{I} \to \mathbb{R}$ such that

$$F_n(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right), \qquad n \geqslant 1.$$

The following theorem gives the axiomatization by Kolmogoroff. Even though Kolmogoroff considered functions $F: \bigcup_{n\geqslant 1} \mathbb{I}^n \to \mathbb{I}$, here we have extended the domain of these functions to $\mathbb{I}^*$. Also, it has been recently proved [30] that the idempotence property of F_n (i.e., $F_n(x^n) = x$ for every $x \in \mathbb{I}$), originally stated in Kolmogoroff-Nagumo's characterization, is not needed and hence can be removed. Note also that a variant and a relaxation of Kolmogoroff-Nagumo's characterization can also be found in [12, 13, 22].

Theorem 2.5 (Kolmogoroff-Nagumo)**.** *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A function $F: \mathbb{I}^* \to \mathbb{I}$ is B-associative and, for every integer $n \geqslant 1$, the n-ary part F_n is symmetric, continuous, and strictly increasing in each argument if and only if F is a quasi-arithmetic mean function.*

The existence of nonsymmetric B-associative operations can be illustrated by the following example, introduced in [21, p. 81] (see also [26]). For every $z \in \mathbb{R}$, the ε-standard operation $M^z: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined as

$$M_n^z(\mathbf{x}) = \frac{\sum_{i=1}^{n} z^{n-i}(1-z)^{i-1} x_i}{\sum_{i=1}^{n} z^{n-i}(1-z)^{i-1}}, \qquad n \geqslant 1,$$

is B-associative. Actually, one can show [25] that any B-associative ε-standard operation over $\mathbb{R}$ whose n-ary part is a nonconstant linear function for every $n \geqslant 1$ is necessarily one of the operations M^z ($z \in \mathbb{R}$). More generally, the class of B-associative polynomial ε-standard operations (i.e., such that the n-ary part is a polynomial function for every $n \geqslant 1$) over an infinite commutative integral domain D has also been characterized in [25].

2.3 Preassociativity

By definition, a function $F: X^* \to Y$ is preassociative if the function value of any string does not change when modifying any of its substring without changing its value. For instance, any ε-standard operation $F: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined by $F_n(\mathbf{x}) = f(\sum_{i=1}^{n} x_i)$ for every integer $n \geqslant 1$, where $f: \mathbb{R} \to \mathbb{R}$ is a one-to-one function, is preassociative.

The following two results clearly show that preassociativity is a generalization of associativity.

Proposition 2.6 ([19])**.** *A function $F: X^* \to X^*$ is associative if and only if it is preassociative and satisfies $F = F \circ F$.*

Proposition 2.7 ([27, 28])**.** *An ε-standard operation $F: X^* \to X \cup \{\varepsilon\}$ is associative if and only if it is preassociative and satisfies $F^+ = F_1 \circ F^+$.*

Apart from the fact that it constitutes a less stringent form of associativity, preassociativity has the remarkable feature of avoiding functional composition in its definition. Actually, Propositions 2.6 and 2.7 suggest that preassociativity is precisely the property we obtain from associativity when cleared of any functional composition. Due to this feature, preassociativity can be considered within the wider class of functions taking as inputs strings over an alphabet X and valued over a possibly different set Y. A natural and noteworthy example of a preassociative function is the mapping that outputs the length of strings.

We now show that all preassociative functions $F: X^* \to Y$ are actually strongly related to associativity, even if the set Y is different from X^*. More precisely, we give a characterization of the preassociative functions $F: X^* \to Y$ as compositions of the form $F = f \circ H$, where $H: X^* \to X^*$ is associative and $f: \mathrm{ran}(H) \to Y$ is one-to-one.

Theorem 2.8 ([19])**.** *Let $F: X^* \to Y$ be a function. The following conditions are equivalent.*

(i) F is preassociative.

(ii) There exists an associative function $H: X^* \to X^*$ and a one-to-one function $f: \mathrm{ran}(H) \to Y$ such that $F = f \circ H$.

Corollary 2.9 ([27,28]). *Let $F: X^* \to Y$ be a standard function. The following conditions are equivalent.*

(i) *F is preassociative and satisfies* $\mathrm{ran}(F_1) = \mathrm{ran}(F^+)$.

(ii) *There exists an associative ε-standard operation $H: X^* \to X \cup \{\varepsilon\}$ and a one-to-one function $f: \mathrm{ran}(H^+) \to Y$ such that $F^+ = f \circ H^+$.*

Corollary 2.9 enables us to construct preassociative functions very easily from known associative variadic operations. Just take nonempty sets X and Y, an associative ε-standard operation $H: X^* \to X \cup \{\varepsilon\}$, and a one-to-one function $f: \mathrm{ran}(H^+) \to Y$. For any $e \notin Y$, the standard function $F: X^* \to Y \cup \{e\}$ defined by $F(\varepsilon) = e$ and $F^+ = f \circ H^+$ is preassociative.

Example 2.10. Recall that the *multilinear extension* of a pseudo-Boolean function $f: \{0,1\}^n \to \mathbb{R}$ is the unique multilinear polynomial function $\mathrm{MLE}(f)$ obtained from f by linear interpolation with respect to each of the n variables. Its restriction to $\{0,1\}^n$ is the function f. Let X and Y denote the class of n-variable pseudo-Boolean functions and the class of n-variable multilinear polynomial functions, respectively. For any $e \notin Y$ and any ε-standard operation $H: X^* \to X \cup \{\varepsilon\}$, the standard function $F: X^* \to Y \cup \{e\}$ defined by $F(\varepsilon) = e$ and $F^+ = \mathrm{MLE} \circ H^+$ is preassociative.

Corollary 2.9 also enables us to produce axiomatizations of classes of preassociative functions from known axiomatizations of classes of associative functions. Let us illustrate this observation on an example. Further examples can be found in [29].

Let us recall an axiomatization of the Aczélian semigroups due to Aczél [1] (see also [7,8]).

Proposition 2.11 ([1]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. An operation $H: \mathbb{I}^2 \to \mathbb{I}$ is continuous, one-to-one in each argument, and associative if and only if there exists a continuous and strictly monotonic function $\varphi: \mathbb{I} \to \mathbb{J}$ such that*

$$H(x,y) \;=\; \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where $\mathbb{J}$ is a real interval of one of the forms $]-\infty,b[$, $]-\infty,b]$, $]a,\infty[$, $[a,\infty[$ or $\mathbb{R} =]-\infty,\infty[$ ($b \leqslant 0 \leqslant a$). For such an operation H, the interval $\mathbb{I}$ is necessarily open at least on one end. Moreover, φ can be chosen to be strictly increasing.

It is easy to see that there is only one associative ε-standard operation $H: \mathbb{I}^* \to \mathbb{I} \cup \{\varepsilon\}$ whose binary part coincides with the one given in Proposition 2.11. This operation is defined by

$$H_n(\mathbf{x}) \;=\; \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(x_i)\right), \qquad n \geqslant 1.$$

Combining this observation with Corollary 2.9 produces the following characterization result.

Theorem 2.12 ([29]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A standard function $F: \mathbb{I}^* \to \mathbb{R}$ is preassociative and unarily quasi-range-idempotent, and F_1 and F_2 are continuous and one-to-one in each argument if and only if there exist continuous and strictly monotonic functions $\varphi: \mathbb{I} \to \mathbb{J}$ and $\psi: \mathbb{J} \to \mathbb{R}$ such that*

$$F_n(\mathbf{x}) \;=\; \psi\left(\sum_{i=1}^{n} \varphi(x_i)\right), \qquad n \geqslant 1,$$

where $\mathbb{J}$ is a real interval of one of the forms $]-\infty,b[$, $]-\infty,b]$, $]a,\infty[$, $[a,\infty[$ or $\mathbb{R} =]-\infty,\infty[$ ($b \leqslant 0 \leqslant a$). For such a function F, we have $\psi = F_1 \circ \varphi^{-1}$ and $\mathbb{I}$ is necessarily open at least on one end. Moreover, φ can be chosen to be strictly increasing.

2.4 B-preassociativity

Contrary to preassociativity, B-preassociativity recalls the associativity-like property of the barycenter and may be easily interpreted in various areas. In decision making for instance, in a sense it says that if we express an indifference when comparing two profiles, then this indifference is preserved when adding identical pieces of information to these profiles. In descriptive statistics and aggregation function theory, it says that the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation.

The following result is the barycentric version of Proposition 2.6 and shows that B-preassociativity is a generalization of B-associativity.

Proposition 2.13 ([30]). *A function $F: X^* \to X^*$ is B-associative if and only if it is B-preassociative and satisfies $F(\mathbf{x}) = F(F(\mathbf{x})^{|\mathbf{x}|})$ for all $\mathbf{x} \in X^*$.*

The ε-standard sum operation $F: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined as $F_n(\mathbf{x}) = \sum_{i=1}^{n} x_i$ for every $n \geqslant 1$ is an instance of B-preassociative function which is not B-associative.

A string function $F: X^* \to X^*$ is said to be *length-preserving* if $|F(\mathbf{x})| = |\mathbf{x}|$ for every $\mathbf{x} \in X^*$.

Proposition 2.14 ([31]). *Let $F: X^* \to X^*$ be a length-preserving function. Then F is associative if and only if it is B-preassociative and satisfies $F_n = F_n \circ F_n$ for every $n \geqslant 0$.*

We now show that, along with preassociative functions, all B-preassociative functions $F: X^* \to Y$ are strongly related to associativity. More precisely, B-preassociative functions can be factorized as compositions of length-preserving associative string functions with one-to-one unary maps.

Theorem 2.15 ([31]). *Let $F: X^* \to Y$ be a function. The following assertions are equivalent.*

(i) F is B-preassociative.

(ii) There exist an associative and length-preserving function $H: X^* \to X^*$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n: \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.

The following corollary provides an alternative factorization result for B-preassociative functions in which the inner functions are B-associative operations. For every integer $n \geqslant 1$, the diagonal section $\delta_F: X \to Y$ of a function $F: X^n \to Y$ is defined as $\delta_F(x) = F(x^n)$.

Corollary 2.16 ([30]). *Let $F: X^* \to Y$ be a function. The following assertions are equivalent.*

(i) F is B-preassociative and satisfies $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$ for every $n \geqslant 1$.

(ii) There exists a B-associative ε-standard operation $H: X^* \to X \cup \{\varepsilon\}$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n: \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.

Corollary 2.16 enables us to produce axiomatizations of classes of B-preassociative functions from known axiomatizations of classes of B-associative functions. Let us illustrate this observation on the class of *quasi-arithmetic pre-mean functions*.

Definition 2.17 ([30]). Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A function $F: \mathbb{I}^* \to \mathbb{R}$ is said to be a *quasi-arithmetic pre-mean function* if there are continuous and strictly increasing functions $f: \mathbb{I} \to \mathbb{R}$ and $f_n: \mathbb{R} \to \mathbb{R}$ ($n \geqslant 1$) such that

$$F_n(\mathbf{x}) \;=\; f_n\Bigg(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Bigg), \qquad n \geqslant 1.$$

As expected, the class of quasi-arithmetic pre-mean functions includes all the quasi-arithmetic mean functions (just take $f_n = f^{-1}$). Actually the quasi-arithmetic mean functions are exactly those quasi-arithmetic pre-mean functions which are idempotent (i.e., such that $f_n \circ f = \mathrm{id}_{\mathbb{I}}$ for every integer $n \geqslant 1$). However, there are also many non-idempotent quasi-arithmetic pre-mean functions. Taking for instance $f_n(x) = nx$ and $f(x) = x$ over the reals $\mathbb{I} = \mathbb{R}$, we obtain the sum function. Taking $f_n(x) = \exp(nx)$ and $f(x) = \ln(x)$ over $\mathbb{I} =]0, \infty[$, we obtain the product function.

We have the following characterization of the quasi-arithmetic pre-mean functions, which generalizes Kolmogoroff-Nagumo's axiomatization of the quasi-arithmetic mean functions.

Theorem 2.18 ([30]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A function $F: \mathbb{I}^* \to \mathbb{R}$ is B-preassociative and, for every $n \geqslant 1$, the function F_n is symmetric, continuous, and strictly increasing in each argument if and only if F is a quasi-arithmetic pre-mean function.*

3 Historical notes

In the framework of aggregation function theory, the associativity property for functions having an indefinite arity was introduced first for functions $F: \bigcup_{n\geqslant 1} X^n \to X$ satisfying $F_1 = \mathrm{id}_X$ (see [21, p. 24]; see also [5, p. 16], [15, p. 32], [17, p. 216] for alternative forms). Then it was introduced for ε-standard variadic operations $F: X^* \to X \cup \{\varepsilon\}$ (see [6,27,28]), and finally for string functions (see [19]).

A basic form of B-associativity was first proposed for symmetric real functions $F: \bigcup_{n\geqslant 1} \mathbb{R}^n \to \mathbb{R}$ independently by Schimmack [35], Kolmogoroff [18], and Nagumo [32]. More precisely, Schimmack introduced the condition $F(\mathbf{y}z) = F(F(\mathbf{y})^{|\mathbf{y}|}z)$ while Kolmogoroff and Nagumo considered the condition $F(\mathbf{yz}) = F(F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$ with $|\mathbf{z}| \geqslant 1$. A more general definition appeared more recently in [3] and [21] and has then been used to characterize various classes of functions; see, e.g., [12,13,23,25,26]. The general definition of B-associativity given in Definition 1.1 appeared in [31]. For general background on B-associativity and its links with associativity, see [15, Sect. 2.3] and [30]. The B-associativity property and its different versions are known under at least three different names: *associativity of means* [10], *decomposability* [14, Sect. 5.3], and *barycentric associativity* [3, 30].

Preassociativity was introduced in [27,28] to generalize the associativity property. B-preassociativity was introduced in [30] to generalize the B-associativity property. The basic idea behind this latter definition goes back to 1931 when de Finetti [10] introduced the following associativity-like property for mean functions: for any $u \in X$ and any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ such that $|\mathbf{xz}| \geqslant 1$ and $|\mathbf{y}| \geqslant 1$, we have $F(\mathbf{xyz}) = F(\mathbf{x}u^{|\mathbf{y}|}\mathbf{z})$ whenever $F(\mathbf{y}) = F(u^{|\mathbf{y}|})$.

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