# Strongly B-associative and preassociative functions

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ISFE 2015

# Strongly B-associative functions

**Definition.** $F\colon X^* \to X \cup \{\varepsilon\}$ is *strongly B-associative* if

$$F(\mathbf{xyz}) \;=\; F(F(\mathbf{xz})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{xz})^{|\mathbf{z}|}) \qquad \forall\ \mathbf{xyz} \in X^*$$

**Example.**
$F$ defined by $F(\varepsilon) = \varepsilon$ and $F(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ for $n \geq 1$ and $\mathbf{x} \in \mathbb{R}^n$ is strongly B-associative and symmetric.

$F$ defined by $F(\varepsilon) = \varepsilon$ and $F(\mathbf{x}) = x_1$ for every $n \geq 1$ and $\mathbf{x} \in \mathbb{R}^n$ is strongly B-associative but not symmetric.

# A strong version of B-associativity

$$F(\mathbf{xyz}) \;=\; F(F(\mathbf{xz})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{xz})^{|\mathbf{z}|}) \qquad \forall\ \mathbf{xyz} \in X^* \qquad (1)$$

**Proposition.** $F$ is strongly B-associative if and only if its value on $\mathbf{x}$ does not change when replacing each letter of a substring $\mathbf{y}$ of not necessarily consecutive letters of $\mathbf{x}$ by $F(\mathbf{y})$.

For instance,

$$\begin{aligned} F(x_1x_2x_3x_4x_5) &= F(F(x_1x_3)x_2F(x_1x_3)x_4x_5), \\ &= F(F(x_1x_3)x_2F(x_1x_3)F(x_4x_5)F(x_4x_5)). \end{aligned}$$

**Remark.** We can assume $|\mathbf{y}| = 1$ in (1).

**Corollary.** Any strongly $B$-associative function is B-associative.

**Example.** $F\colon \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined by $F(\varepsilon) = \varepsilon$ and

$$F(\mathbf{x}) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^{n-1}}\, x_i, \qquad n \geq 1, \mathbf{x} \in \mathbb{R}^n,$$

is B-associative but not strongly B-associative.

**Proposition.** If $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then $\mathbf{y} \mapsto F(x\mathbf{y}z)$ is symmetric for every $xz \in X^2$.

# A composition-free version of strong B-associativity

**Definition.** $F\colon X^* \to Y$ is *strongly B-preassociative* if for all $\mathbf{x}\mathbf{x}'\mathbf{z}\mathbf{z}'\mathbf{y} \in X^*$ such that $|\mathbf{x}| = |\mathbf{x}'|$ and $|\mathbf{z}| = |\mathbf{z}'|$

$$F(\mathbf{x}\mathbf{z}) \;=\; F(\mathbf{x}'\mathbf{z}') \implies F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}'\mathbf{y}\mathbf{z}').$$

**Example.** The length function $F\colon X^* \to \mathbb{R}\colon \mathbf{x} \mapsto |\mathbf{x}|$ is strongly B-preassociative.

**Proposition.** Let $F\colon X^* \to X \cup \{\varepsilon\}$. The following conditions are equivalent.

(i) $F$ is strongly B-associative.

(ii) $F$ is strongly B-preassociative and satisfies $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$.

# Factorization of strongly B-preassociative functions with strongly B-associative ones

$F\colon X^* \to X \cup \{\varepsilon\}$ is *$\varepsilon$-standard* if $\quad F(\mathbf{x}) = \varepsilon \iff \mathbf{x} = \varepsilon$.

$$\delta_{F_n}(x) := F_n(x \cdots x)$$

**Theorem.** (AC) Let $F\colon X^* \to Y$. The following conditions are equivalent.

(i) $F$ is strongly B-preassociative & $\mathrm{ran}(F_n) = \mathrm{ran}(\delta_{F_n})$ for all $n$;

(ii) $F_n = f_n \circ H_n \quad$ for every $n \geq 1$ where

- $H\colon X^* \to X \cup \{\varepsilon\}$ is $\varepsilon$-standard and strongly B-associative,
- $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one for every $n \geq 1$.

# Factorization of strongly B-preassociative functions with associative ones

$H\colon X^* \to X^*$ is *length-preserving* if $|H(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in X^*$.

**Theorem.** (AC) Let $F\colon X^* \to Y$. The following conditions are equivalent.

(i) $F$ is strongly B-preassociative.

(ii) $F_n = f_n \circ H_n$ for every $n \geq 1$ where

- $H\colon X^* \to X^*$ is associative, length-preserving and strongly B-preassociative,
- $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one for every $n \geq 1$.

# Invariance by replication

$F\colon X^* \to Y$ is *invariant by replication* if $F(\mathbf{x}^k) = F(\mathbf{x})$ for all $\mathbf{x} \in X^*$ and $k \geq 1$.

**Proposition.** If $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then the following conditions are equivalent.

(i) $F$ is invariant by replication.

(ii) $\mathrm{ran}(F_n) \subseteq \mathrm{ran}(F_{kn})$ for every $n \geq 0$ and $k \geq 1$.

# Quasi-arithmetic pre-mean functions and Kolmogoroff - Nagumo's characterization

# Quasi-arithmetic pre-mean functions

$\mathbb{I} \equiv$ non-trivial real interval.

**Definition.** $F\colon \mathbb{I}^* \to \mathbb{R}$ is a *quasi-arithmetic pre-mean function* if there are continuous and strictly increasing functions $f\colon \mathbb{I} \to \mathbb{R}$ and $f_n\colon \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$ such that

$$F(\mathbf{x}) \;=\; f_n\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big), \qquad n \geq 1, \mathbf{x} \in X^n.$$

If $f_n = f^{-1}$ for every $n \geq 1$ then $F$ is a *quasi-arithmetic mean function*.

**Example.** The product function is a quasi-arithmetic pre-mean function over $\mathbb{I} = ]0, +\infty[$ (take $f_n(x) = \exp(nx)$ and $f(x) = \ln(x)$) which is not a quasi-arithmetic mean function.

# Kolmogoroff - Nagumo's characterization of quasi-arithmetic mean functions

**Theorem** (Kolmogoroff - Nagumo). Let $F\colon \mathbb{I}^* \to \mathbb{I}$. The following conditions are equivalent.

(i) $F$ is a quasi-arithmetic mean function.

(ii) $F$ is B-associative, and for every $n \geq 1$, $F_n$ is

symmetric,

continuous,

strictly increasing in each argument,

reflexive.

**Theorem.** B-associativity and symmetry can be replaced by strong B-associativity. Moreover, reflexivity can be removed.

# Characterization of quasi-arithmetic pre-mean functions

**Theorem.** Let $F\colon \mathbb{I}^* \to \mathbb{R}$. The following conditions are equivalent.

(i) $F$ is a quasi-arithmetic pre-mean function.

(ii) $F$ is strongly B-preassociative, and for every $n \geq 1$, $F_n$ is

symmetric,

continuous,

strictly increasing in each argument.

## Open problems

Characterization of the class of $F\colon X^* \to X^*$ which are associative, length-preserving and strongly B-preassociative?

Which of those B-associative functions that satisfy

$$F(xyz) = F(F(xz)yF(xz))$$

are strongly B-associative?

**Reference.** J.-L. Marichal and B. Teheux. Strongly barycentrically associative and preassociative functions. arXiv:1411.5897