The stable GFEM. Convergence, accuracy and Diffpack implementation

Daniel Alves Paladim
Sundararajan Natarajan
Stéphane Bordas
Pierre Kerfriden

Institute of Mechanics and Advanced Materials
Cardiff University

May 12, 2015

Context

Diffpack is a commercial software library used for the development numerical software, with main emphasis on numerical solutions of partial differential equations. It was developed in C++ following the object oriented paradigm.

The library is mostly oriented to the implementation of the finite element method, however it has tools for other methods such as finite volume and finite differences.

Context

The extended/generalized finite element method is usually connected to the following issues:

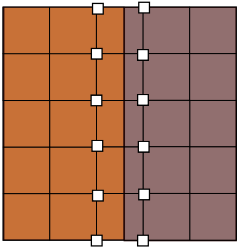
- Blending
- Ill-conditioning of the stiffness matrix
- Numerical integration

Blending

In the extended finite element method, there are 3 types of elements.

- Elements with all its nodes enriched
- Elements that none of its nodes enriched.
- Elements that have both type of nodes (blending elements).

In those elements, there is no partition of unity and the convergence rate is degraded.



Blending solution

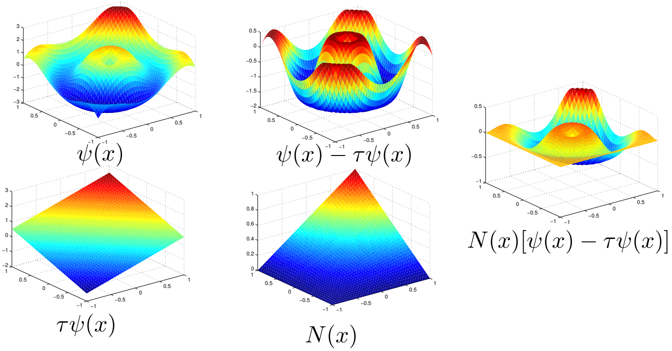
Babuska and Banerjee (2011) proposed the stable generalized finite element method. In the SGFEM, the approximation has the following form

$$u^h(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I*} N_i(x)[\psi_i(x) - \tau\psi_i(x)]a_i \tag{1}$$

where $\tau\psi_i$ is the finite element interpolation of ψ_i

$$\tau\psi_i(x) = \sum_{i \in I} N_i(x)\psi_i(x_i) \tag{2}$$

Enriched basis function



Remarks

- SGFEM has no blending problems.
- Easy to implement, especially when compared to the corrected XFEM.
- The Kronecker delta property is still valid $u(x_i) = u_i$.
- The SGFEM enriched basis function of the absolute value, coincides with the modified absolute value enrichment proposed by Möes.

Ill-conditioning

Consider the following system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

If we consider a small change on right hand side, $\boldsymbol{b}'$, we are interested in determining how this will affect the solution. Defining

$$\boldsymbol{e} = \boldsymbol{b}' - \boldsymbol{b} \qquad \boldsymbol{e_x} = \boldsymbol{x}' - \boldsymbol{x}$$

Then, the relative change of the solution $\boldsymbol{x}$ is

$$\frac{|\boldsymbol{e_x}|/|\boldsymbol{x}|}{|\boldsymbol{e}|/|\boldsymbol{b}|} = \frac{|\boldsymbol{A^{-1}e}|}{|\boldsymbol{e}|} \cdot \frac{|\boldsymbol{b}|}{|\boldsymbol{x}|} = \frac{|\boldsymbol{A^{-1}e}|}{|\boldsymbol{e}|} \cdot \frac{|\boldsymbol{A}\boldsymbol{x}|}{|\boldsymbol{x}|} \leq \|\boldsymbol{A^{-1}}\|\|\boldsymbol{A}\|$$

Therefore, we define

$$Cond(\boldsymbol{A}) = \|\boldsymbol{A^{-1}}\|\|\boldsymbol{A}\|$$

Ill-conditioning solution

- **Scaled condition number** Let $h_{ij} = \frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}$. Then the scaled condition number is defined as $\kappa(\boldsymbol{A}) = \|\boldsymbol{H}^{-1}\|\|\boldsymbol{H}\|$
- The scaled condition number of the FEM stiffness matrix is $O(h^{-2})$.
- The standard GFEM condition number is usually higher than $O(h^{-2})$.
- SGFEM condition number in 1D is $O(h^{-2})$.
- For higher dimensions, if 2 assumptions hold, the condition number of SGFEM also grows at the same rate as FEM.

The two assumptions

Assumption 1 There exist L_1 and $U_1 \in \mathbb{R}$ and indepedent of h (element size) such that

- $0 < L_1 \leq U_1$
- $L_1[\|\alpha\|^2 + \|\beta\|^2] \leq |a(\alpha+\beta, \alpha+\beta)| \leq U_1[\|\alpha\|^2 + \|\beta\|^2]$

where $\alpha = \sum_i u_i N_i$ and $\beta = \sum_j v_j N_j \psi \quad \forall u_i, v_j \in \mathbb{R}$.

The space spanned by the standard FEM shape functions is **almost orthogonal** to the space spanned by the enriched shape functions.

The two assumptions

Let $\mathbf{A}^{(i)}$ be the scaled stiffness matrix of element i.
Assumption 2 There exist L_2 and $U_2 \in \mathbb{R}$ and indepedent of h (element size) such that

- $0 < L_2 \leq U_2$
- $L\|\mathbf{y}\|^2 \leq \mathbf{y}^\mathbf{T}\mathbf{A}^{(i)}\mathbf{y} \leq U\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathbb{R}^k$

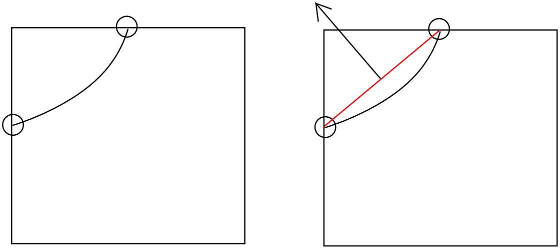
Provided that those 2 assumptions are fulfilled, the scaled condition no. of the stiffness matrix is also $O(h^{-2})$.

Numerical integration

- Numerical integration of discontinuous and singular functions must be perfomed.
- The integration of the branch functions (singular functions) is performed using a parabolic mapping (Béchet et al. 2005).
- Integration of discontinuous and weakly discontinuous functions is performed with subdivision of the elements.

Integration algorithm

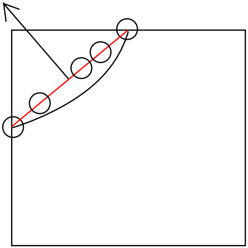
1. The cut points between the interface and the element edges are found.
2. A least squares plane is adjusted to the cut points.



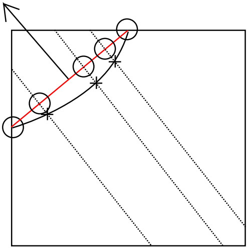
(1) (2)

Integration algorithm

3. Points are placed over the plane.
4. A polynomial interpolation in the normal direction is built and solved.



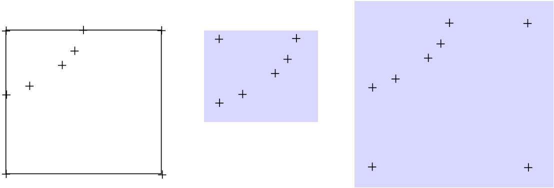
(3)



(4)

Integration algorithm

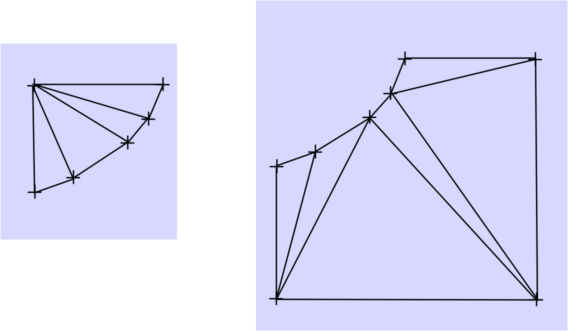
5. Two sets of points are created. $\phi_i \geq 0$ and $\phi_i \leq 0$



(5)

Integration algorithm

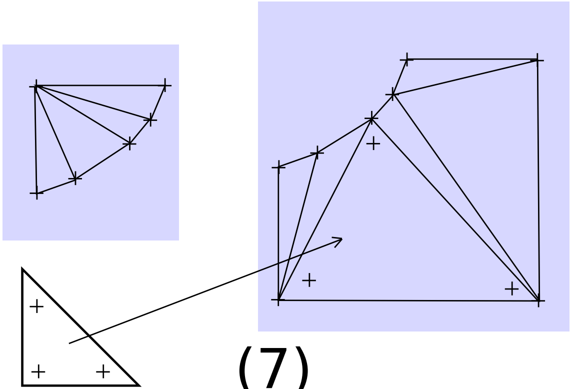
6. Delaunay tetrahedralization is computed for both sets.



(6)

Integration algorithm

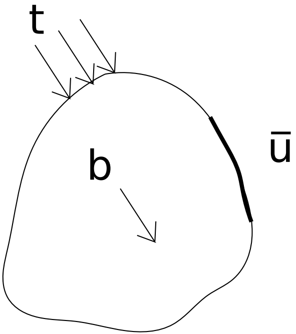
7. Gauss points are mapped into the tetrahedrons



(7)

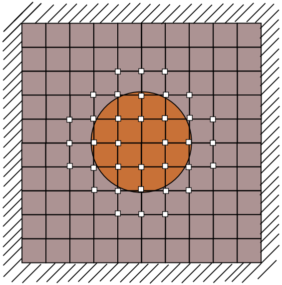
Numerical Examples. Problem definition

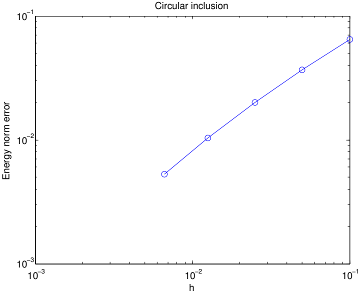
$\nabla \cdot \sigma + \mathbf{b} = 0$ on Ω
$\sigma \cdot \mathbf{n} = \mathbf{t}$ on Γ_t
$\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u
$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$ on Ω
$\sigma = C\epsilon$ on Ω



Circular Inclusion

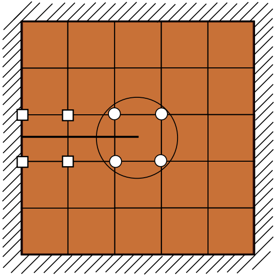
The absolute value of the level set is used as enrichment function.

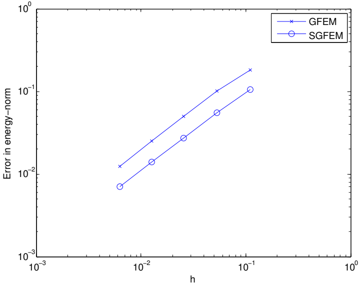




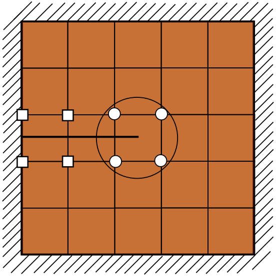
Crack

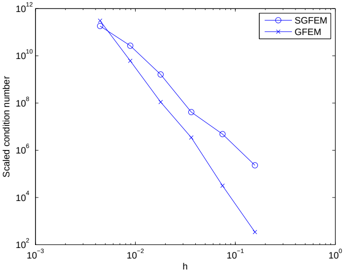
Enriched with the “linear Heaviside” $(H(\boldsymbol{x}),\ H(\boldsymbol{x})x,\ H(\boldsymbol{x})y)$ and the branch enrichemet functions.





Crack




SGFEM
GFEM
Scaled condition number
10^{12}
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{-3}
10^{-2}
10^{-1}
10^{0}
h

Summary

- The stable generalized finite element is an easy to implement solution to blending problems.
- At the moment, SGFEM is not a complete solution against the ill-conditioning.
- Numerical integration perfomed through element subdivision and parabolic mapping.

Literature

- Babuska, I., Banerjee, U. (2012). Stable Generalized Finite Element Method (SGFEM). Computer Methods in Applied Mechanics and Engineering
- Gupta, V., Duarte, C. a., Babuska, I., Banerjee, U. (2013). A Stable and Optimally Convergent Generalized FEM (SGFEM) for Linear Elastic Fracture Mechanics. Computer Methods in Applied Mechanics and Engineering
- Gupta, V., Duarte, C. a., Babuska, I., Banerjee, U. (2015). Stable GFEM (SGFEM): Improved conditioning and accuracy of GFEM/XFEM for three-dimensional fracture mechanics. Computer Methods in Applied Mechanics and Engineering